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HYSTERETIC DAMPING MODELLING BY NONLINEAR KELVIN-VOIGT MODEL

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ABSTRACT. This paper present a nonlinear Kelvin-Voigt model (NKV model) with the stiffness and damping characteristics as function in term of displacements. The behaviour of this model for harmonic imput was verified by means of the resonant column experimental data.

Key word: nonlinear dynamics, soils dynamics, damping modelling

1. INTRODUCTION

The strong dependence of the soils dynamic properties on strain or stress level produced by external loads is very well known. In the previous author's papers [1], [2], [3] this nonlinear behaviour was modeled assuming that the geological materials are nonlinear viscoelastic materials. The dynamic model obtained was built upon two dynamic nonlinear functions – one for material strength modeling and another including material damping, both in terms of strain level caused by external loading conditions and both functions being completely determined from resonant column test data.

The resonant column system can be considered as a one degree-of-freedom system that is made up of a single mass (the vibration device) supported by a spring and a damper represented by the specimen [3]. But, due to the mechanical properties of the specimen materials both spring and damper have non-linear characteristics and thus the entire system is a non-linear one [4].

In linear dynamics a usual description of a solid single-degree-of-freedom behaviour is given by the Kelvin-Voigt model consisting of a spring (with a stiffness k) and a dashpot (with a viscosity c) connected in parallel. The governing equation of this system for torsional harmonic vibrations (usually resonant column system excitation) is:

$$J_0\ddot{\theta} + c \cdot \dot{\theta} + k \cdot \theta = M_0 \cdot \sin \omega t \qquad (1.1)$$

where θ is the system's displacement (rotation, in this case), J_0 is the moment of inertia of the vibrator, M_0 are the amplitude and ω the pulsation of the harmonic external imput.

In the non-linear case, due to the mechanical properties of the specimen materials both spring and dashpot characteristics become non-linear functions in terms of deformation (or rotation) level [4], [5]. The most expected form of the governing equation for non-linear behaviour of a single- degree-of-freedom system is:

$$J_0\ddot{\theta} + c(\theta) \cdot \dot{\theta} + k(\theta) \cdot \theta = M_0 \cdot \sin \omega t \qquad (1.2)$$

with the analogic model from fig.1.1.

The purpose of this paper is to verify this non-linear forms of the Kelvin-Voigt model and the capabilities of this model to modelling the hysteresis loops.

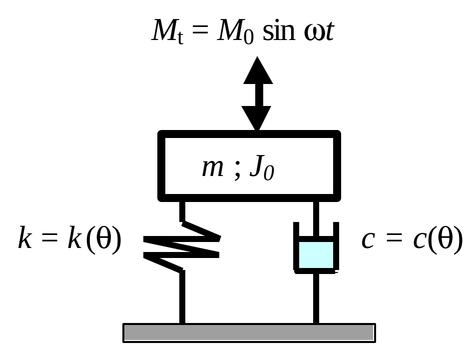

Fig.1.1 Non-linear Kelvin-Voigt model

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2. NONLINEAR MATERIAL FUNCTION

In order to model the non-linear material behaviour, in [1], [2], and [3] a non-linear viscoelastic constitutive law for dynamic response of soils was presented. This model describes the nonlinearity by the dependence of the material mechanical parameters: shear modulus G and damping ratio D in terms of shear strain invariant γ: $G = G(\gamma)$, $D = D(\gamma)$, or twisting angle θ: $G = G(\theta)$, $D = D(\theta)$.

As an example, in fig.2.1 such non-linear material functions obtained from resonant column test performed upon clay sample are given.

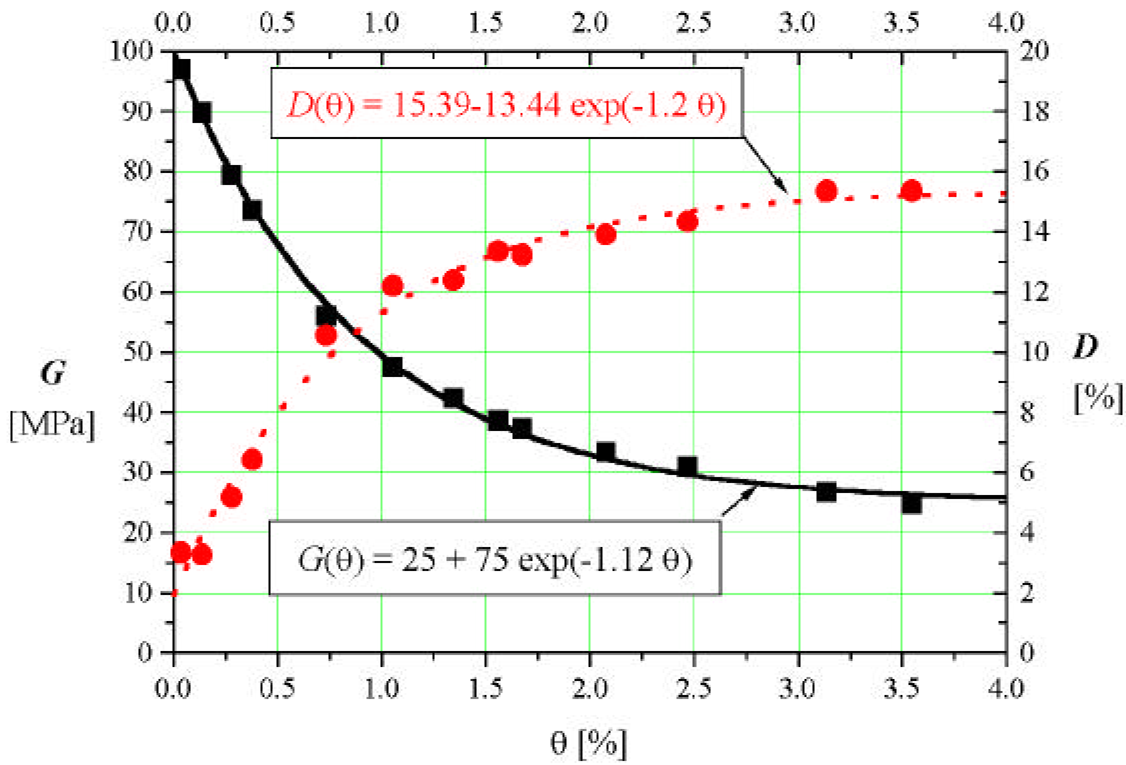

Fig.2.1 Dynamic material functions

Using the same method that describes the nonlinearity by strain dependence of the material parameters, we assume that the damper viscosity c and the spring stiffness k are functions in terms of rotation θ [4], [5] (fig.2.2):

$$c(\theta) = 2J_0\omega_0 \cdot D(\theta) \quad [\text{Nms}]$$
$$k(\theta) = \frac{I_p}{h} \cdot G(\theta) \qquad [\text{Nm}] \tag{2.3}$$

where ω_0 is the system undamped natural pulsation, $I_p = \pi\phi^4/32$ is the polar moment of the specimen and (ϕ, h) are the diameter and height of the cylindrical specimen. For the linear systems, the undamped natural pulsation ω_0 is defined in terms of spring stiffness k: $\omega_0 = \sqrt{k / J_0}$. In this case, the spring stiffness is a function, and we define the undamped natural pulsation in terms of initial value of stiffness function $k(0): \omega_0 = \sqrt{k(0) / J_0}$.

3. VALIDATION OF THE NON-LINEAR FORM

Eq. (1.1) can be numerical solved [4], [11], [12] and the computed results can be compared with the measured resonant test results. Thus, by using the change of variable $\tau = \omega_0 t$ and by introducing a new "time" function [4]:

$$\varphi(\tau) = \theta(t) = \theta(\tau / \omega_0) \tag{3.1}$$

one obtains from eq.(1.2) a dimensionless form of the non-linear equation of motion:

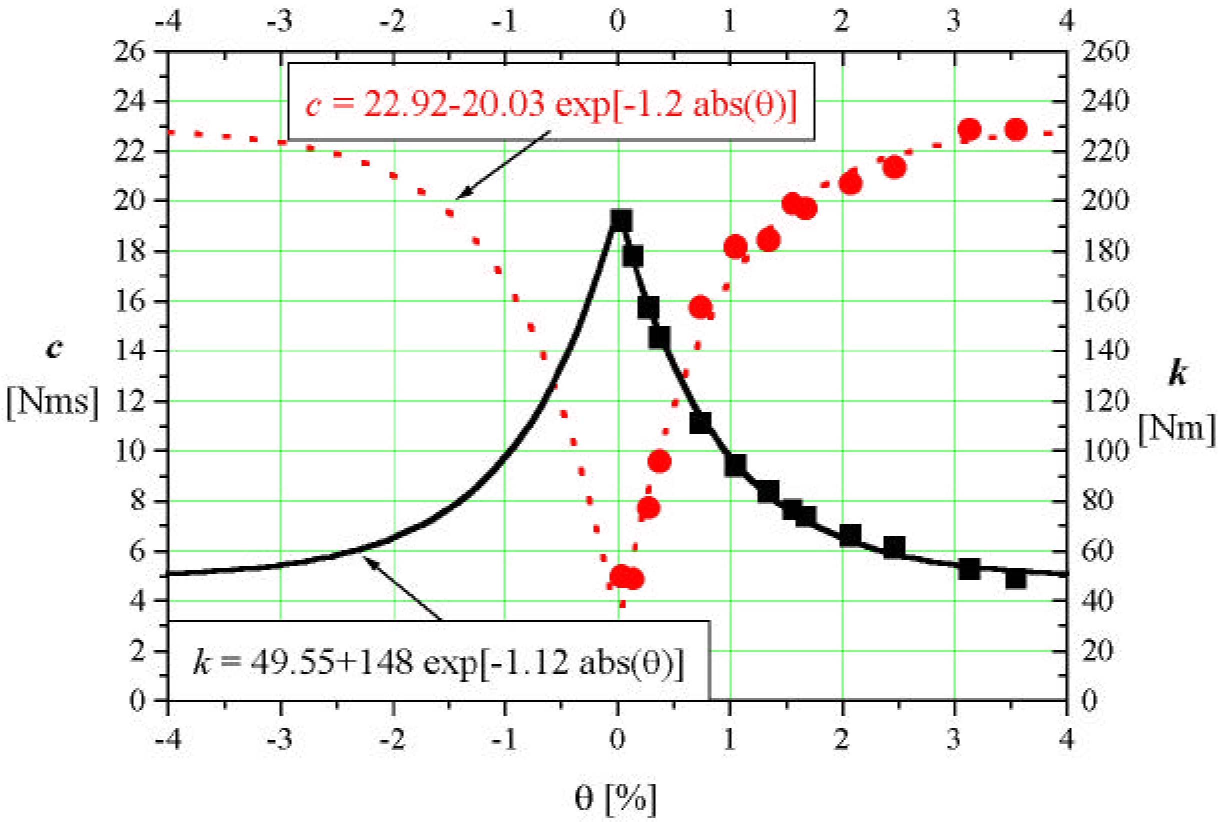

Fig.2.2 Dynamic non-linear characteristics

$$\varphi'' + C(\varphi)\cdot\varphi' + K(\varphi)\cdot\varphi = \mu\cdot\sin\upsilon\tau \qquad (3.2)$$

where the superscript accent denotes the time derivative with respect to τ, and:

$$C(\varphi) = C(\theta) = \frac{c(\theta)}{J_0\omega_0} = 2D(\theta)$$

$$K(\varphi) = K(\theta) = \frac{k(\theta)}{J_0\omega_0^2} = \frac{k(\theta)}{k(0)} = \frac{G(\theta)}{G(0)} = G_n(\theta) \qquad (3.3)$$

$$\mu = \frac{M_0}{J_0\omega_0^2} = \frac{M_0}{k(0)} = \theta_{st} \quad ; \quad \upsilon = \frac{\omega}{\omega_0}$$

For a given normalized amplitude μ and relative pulsation υ, the non-linear equation (3.2) can be numerically solved and a solution of the form $\varphi = \varphi(\tau;\mu,\upsilon)$ can be obtained in n points [4], [11]. After that, dropping the transitory part of the solution and keeping only the stationary part, the amplitude of rotation θ_0 becomes:

$$\theta_0 = |\theta(t)| = |\varphi(\tau)| \qquad (3.4)$$

The same rotation θ_0 can be obtained directly form resonant column output:

$$\theta_0 = \frac{A}{r_a\omega^2} \qquad (3.5)$$

where A is the measured accelerometer value, r_a is the distance from the axis of rotation to the accelerometer axis (r_a = 0.03175m for Drnevich resonant column) and $\omega = 2\pi f_r$ is the pulsation of the vibrator device under resonant frequency f_r.

The comparison of the θ values obtained from the steady-state solution of the non-linear single degree-of-freedom system with experimental data can give a pertinent information about the model validity. The results of such evaluation, given in fig.3.1, show a good behaviour for the NKV model.

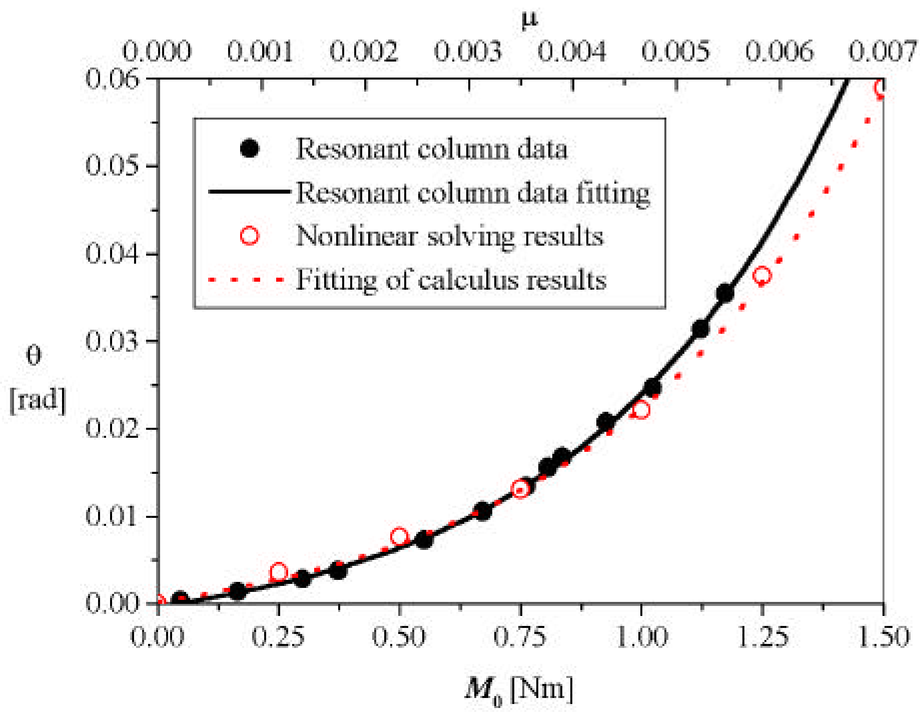

Fig.3.1 Model validation

4. HYSTERETIC DAMPING MODELLING

A great number of laboratory tests on soils shows that the cyclic stress-strain curves are high nonlinear and constitute a closed hysteresis loops. These testing results seem to indicate that damping properties are especially of hysteretic type, and not viscous as those corresponding to the Kelvin-Voigt model.

All of the resonant column determinations of the damping capacity for a certain strain level are based on the equivalence between the hysteretic damping of the soil specimen and the viscous damping for a uniform viscoelastic specimen of the same mass, density and dimensions as the soil specimen [2]. As a result of this *rheo-hysteretic* hypothesis, the Kelvin-Voigt model is able to describe the dissipated energy of the specimen-vibrator system and the verifying method presented in capther 3 demonstrated a good agreement between model and experimental data.

Several methods use for damping evaluation of the experimental registered hysteretic loops and determine the damping ratio as:

$$D = \frac{1}{4\pi}\frac{\Delta W}{W} \tag{4.1}$$

where W is the maximum stored energy and ΔW is the energy loose per cycle represented by the area enclosed inside the hysteresis loop (fig.4.1). By another methods the hysteresis loop is obtained from the skeleton curve by applying the Masing rule (the superior and inferior branches are obtained from the skeleton curve by multiplying by a factor two in both directions). [9], [10].

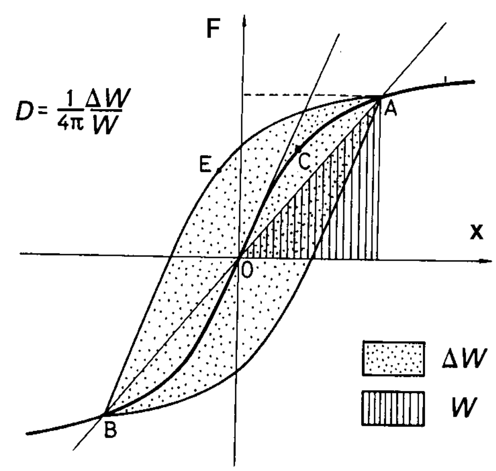

Fig.4.1 Hysteretic damping definition

Using the same single degree-of-freedom system from fig.1.1 there is a possibility to compare the resonant column experimental damping values with the values obtained from non-linear Kelvin-Voigt model and Masing model.

We mention that in the ordinary resonant column test the damping evaluation uses a different method – the magnification factor method based on measuring both current and acceleration at two different frequencies - from resonant frequency f_r and at $\sqrt{2} f_r$.

To verify the capabilities of the non-linear Kelvin-Voigt equation (1.2) to modelling a hysteresis loop one can use a inverse strategy – starting from the given dynamic material functions $c = c(\theta)$ and $k = k(\theta)$ included in the restoring force $Q = Q(\theta, \dot{\theta})$ one can built the hysteresis loops for a certain levels θ_0 and then the damping ratio value D for level θ_0 can be obtained from eq. (4.1). This value can be compared with the experimental values at level θ_0: $D = D(\theta)\big|_{\theta=\theta_0}$

Thus, for nonlinear Kelvin-Voigt model from fig.1.1 the restoring force is:

$$\begin{aligned} Q(\theta, \dot{\theta}) &= Q_{el}(\theta) + Q_{dam}(\theta, \dot{\theta}) = \\ &= k(\theta) \cdot \theta + c(\theta) \cdot \dot{\theta} \end{aligned} \tag{4.2}$$

where $Q_{el}(\theta) = k(\theta) \cdot \theta$ is the *backbone curve* or *skeleton curve*.

For a certain amount of the excitation $M = M_0 \sin \omega t$ the response rotation θ (after the dropping the transitory part) has the form:

$$\theta = \theta_0 \cos \omega t \tag{4.3}$$

and, then:

$$\dot{\theta} = \theta_0 \omega \sin \omega t \tag{4.4}$$

Therefore, by eliminating the time t between this two equations, (4.3) and (4.4), result:

$$\dot{\theta} = \pm\omega\sqrt{\theta_0^2 - \theta} \tag{4.5}$$

and the restoring force (4.2) becomes:

$$Q(\theta) = k(\theta) \cdot \theta \pm c(\theta) \cdot \omega\sqrt{\theta_0^2 - \theta} \tag{4.6}$$

where the sign "+" is for the superior branch of the hysteresis loop and the sign "-" for the inferior branch (fig.4.2).

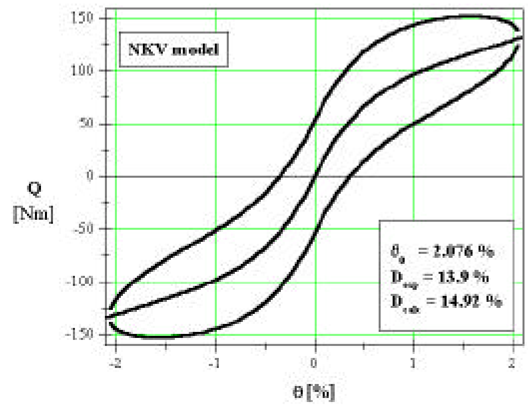

Fig.4.2 NKV hysteresis loop

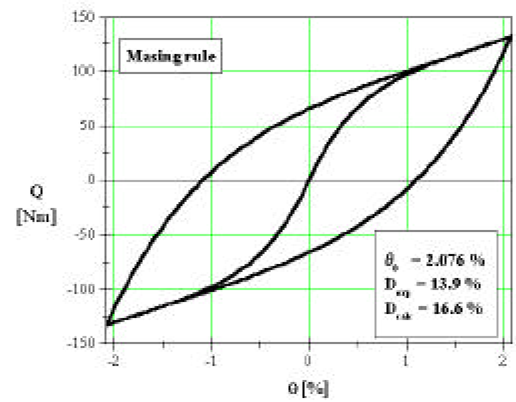

Fig.4..3 Masing hystersis loop

For comparison, the Masing hysteresis loop is given in fig.4.3 for the same tested clay, at the same amplitude level $\theta_0 = 2.076\%$ built using the same skeleton curve $Q_{el}(\theta) = k(\theta) \cdot \theta$. The superior and inferior branches are obtained usig the Masing rule – starting from the skeleton curve and multiplying by a factor two in both Q and θ directions.

As can see in these figures the geometrical aspect of these hysteresis loops are different. But, the damping value is directly connected with the loop area and not with its form. Fortunately, the loop area differences are not so obvious. This can be proved by computing the damping ratio for different amplitude θ_0 and for each kind of hysteresis loop. The results of such calculus together with the corresponding D experimental values are given in fig.4.4.

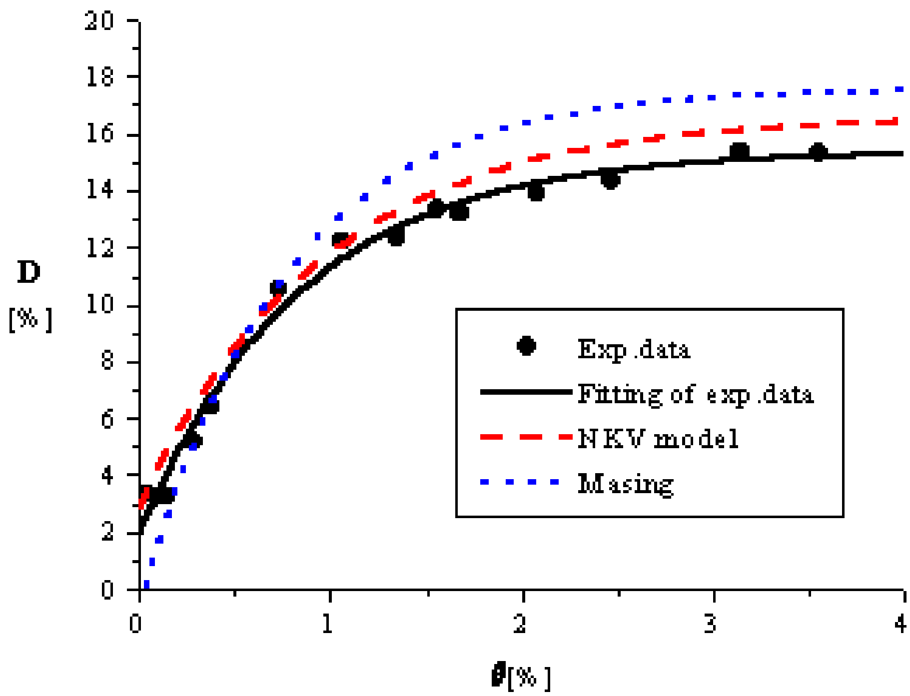

Fig.4.4 NKV and Masing damping modelling

4. CONCLUDING REMARKS

- The non-linear viscoelasticity can be used as starting point for building a dynamic model for soils behaviour.
- The non-linear dynamic characteristics for damping and stiffness can be obtained as an extension in the non-linear domain of the corresponding linear constants.
- The non-linear Kelvin-Voigt model provides a good agreement with the experimental resonant column data.
- The non-linear Kelvin-Voigt model is able to model the damping characteristics of the hysteretic type materials.

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