A NEW FORM OF THE CHARACTERISTIC EQUATION OF A DENSITY MATRIX

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In order to put the characteristic polynomial of a density matrix in a new form we use a result of A. J. Mountain. The coefficients of the characteristic polynomial in this form are functions of a new class of invariants introduced by K. Lendi. These invariants are expressed using the symmetric and scalar product for the Bloch vector associated with the density matrix. The conversion relations from the Bloch vectors to the Fano parameters allow us the calculation of Lendi's invariants as functions of the Fano parameters. We obtain also the beautiful result of Horodeckis [16] about the factorization of the characteristic equation of a two-qubits density matrix.

I. INTRODUCTION

The characteristic polynomial $P(\lambda)$ of a density matrix ρ acting on a d-dimensional Hilbert space H_d is given by:

$$P(\lambda) = \det(\lambda E - \rho) \tag{1.1}$$

In the equation (1.1) we have denoted by E the identity operator on H_d. The roots of the characteristic equation:

$$P(\lambda) = \det(\lambda E - \rho) = \prod_{j=1}^{j=d} (\lambda - \lambda_j) \tag{1.2}$$

are the eigenvalues of the density matrix ρ and are invariant with respect to the unitary group action $SU(d)$ on the density matrices given by $U\rho U^{\bullet}$ for any $U \in SU(d)$. It follows that the coefficients of the characteristic polynomial are also invariant with respect to this action. Another set of invariants is associated with the density matrix ρ using the unitary invariance of the trace:

$$K_m = Tr(\rho^m) = \sum_{j=1}^{j=d} \lambda_j^{\ m} \; ; m = 1,2,...,d \tag{1.3}$$

The coefficients of the characteristic polynomial are algebraic functions of these invariants. These invariants can be computed in a an easy way. In the paper [9], S. Weigert developed a non-commutative calculus for the density matrices expressed in the Bloch parametrization which. This allows us to calculate the invariants K_m and other traces of the similar powers. In the following we shall consider the Bloch vector parametrization [1-15] and the Fano parametrization [2-5] and the relations between them [15].

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2. THE BLOCH PARAMETRIZATIONS

Let H be a finite-dimensional Hilbert space with dimension equal to d. We denote by $End(H)$ the vector space of the linear operators on H and define on this space the Hilbert-Schmidt inner product by the formula: $(A,B) = Tr(A^*B)$ for any $A, B \in End(H)$ (the operator A^* is the adjoint of the operator A). The Lie algebra $su(d)$ of all selfadjoint operators $A \in End(H)$ with $TrA = 0$ is a real subspace of $End(H)$, with dimension equal to $D = d^2 - 1$. We shall take a basis $\{\tau_j\}_{j=1}^{D}$ of this subspace such that the following relations are valid $(\tau_j, \tau_k) = 2\delta_{jk}$. Any density matrix ρ i.e. any linear self-adjoint and positive definite operator with $Tr\rho = 1$ can be decomposed in the following form:

$$\rho(v) = \frac{1}{d}E + \frac{1}{2}\sum_{j=1}^{D}v_j\tau_j \tag{2.1}$$

with some constraints on the real vector $v = (v_1, v_2, ..., v_D) \in R^D$, called the generalized Bloch vector [1-15]. For any density matrix ρ we have a unique Bloch vector with the following components: $v_j = Tr\rho\tau_j = (\rho, \tau_j)$. The fact that the density matrix ρ is positive definite imposes severe restrictions on the Bloch vectors [7, 11]. We denote by $<v, u> = \sum_{j=1}^{j=D} v_j u_j$ the scalar product in R^D. The Lie brackets of the generators $\{\tau_j\}_{j=1}^{D}$ of the Lie algebra $su(d)$ are given by the structure constants $\{f_{jkl}\}_{j,k,l=1}^{D}$:

$$[\tau_j, \tau_k] = 2i\sum_{l=1}^{D}f_{jkl}\tau_l \tag{2.2}$$

These structure constants are the components of an anti-symmetric tensor and fulfill the Jacoby identity:

$$\sum_{m=1}^{D}(f_{klm}f_{mpq} + f_{plm}f_{mkq} + f_{kpm}f_{mlq}) = 0 \tag{2.3}$$

A remarkable fact, specific to the Lie algebra $su(d)$, is the existence of a symmetric bracket:

$$\tau_j\tau_k + \tau_k\tau_l = \frac{4}{d}\delta_{jk}I + 2\sum_{l=1}^{D}d_{jkl}\tau_l \tag{2.4}$$

Here d_{jkl} are the components of a symmetric tensor. With the aid of anti-symmetric and symmetric tensors we define an anti-symmetric and a symmetric product on the Euclidean space R^D. The anti-symmetric product is given by:

$$(x \bigcap y)_j = \sum_{k=1}^{D}\sum_{l=1}^{D}f_{jkl}x_k\, y_l \tag{2.5}$$

The symmetric product is given by:

$$(x \bigcup y)_j = \sum_{k=1}^{D}\sum_{l=1}^{D}d_{jkl}x_k\, y_l \tag{2.6}$$

Then we have:

$$<x, \tau><y, \tau> = \frac{2}{d}<x, y>E + <[(x \bigcup y) + i(x \bigcap y)], \tau> \tag{2.7}$$

It is evident from the definition of the symmetric and scalar products that they are invariants to the unitary group action on the density matrices. This justifies the definition of K. Lendi invariants [6]:

$$L_1 = 1$$

$$L_2 = \frac{1}{2} < v, v >$$

$$...$$

$$L_k = \frac{1}{2^{k-1}} < v \bigcup v \bigcup v \bigcup v ... \bigcup v, v >$$

(2.8)

In the definition of, L_k we have $k - 1$ factors in the multiple symmetric product.

3. A NEW FORM OF THE CHARACTERISTIC EQUATION

We shall use the following formula from A. J. Mountain [10]:

$$\det(E + \alpha F) = 1 - \frac{\alpha^2}{2} TrF^2 + \frac{\alpha^3}{3} TrF^3 - \frac{\alpha^4}{4} [TrF^4 - \frac{1}{2}(TrF^2)^2] +$$

$$\frac{\alpha^5}{5} [TrF^5 - \frac{5}{6}(TrF^2)(TrF^3)] - \frac{\alpha^6}{6} [TrF^6 - \frac{1}{3}(TrF^3)^2 - \frac{3}{4}(TrF^2)(TrF^4) + \frac{1}{8}(TrF^2)^3] +$$

$$\frac{\alpha^7}{7} [TrF^7 - \frac{7}{10}(TrF^2)(TrF^5) - \frac{7}{12}(TrF^3)(TrF^4) + \frac{7}{24}(TrF^2)^2(TrF^3)] -$$

(3.1)

$$\frac{\alpha^8}{8} [TrF^8 - \frac{2}{3}(TrF^2)(TrF^6) - \frac{8}{15}(TrF^3)(TrF^5) - \frac{1}{4}(TrF^4)^2 + \frac{1}{4}(TrF^4)(TrF^2)^2$$

$$+ \frac{2}{9}(TrF^3)^2(TrF^2) - \frac{1}{48}(TrF^2)^4] + ...$$

We remark that for the traceless density operator $F = \rho - \frac{1}{d} E = \frac{1}{2} < v, \tau >$ and with notations $\xi = \lambda - \frac{1}{d}$ and $\alpha = -\xi^{-1}$ we have:

$$\det(\lambda E - \rho) = (\xi)^d \det(E + \alpha F)$$

(3.2)

Because we have:

$$TrF^2 = L_2$$

$$TrF^3 = L_3$$

$$TrF^4 = L_4 + \frac{1}{d}(L_2)^2$$

(3.3)

and

$$TrF^5 = L_5 + \frac{2}{d}L_2L_3$$

$$TrF^6 = L_6 + \frac{2}{d}L_2L_4 + \frac{1}{d}L_3{}^2 + \frac{1}{d^2}L_2{}^3$$

$$TrF^7 = L_7 + \frac{2}{d}L_2L_5 + \frac{2}{d}L_3L_4 + \frac{3}{d^2}L_2{}^2L_3 \qquad (3.3)$$

$$TrF^8 = L_8 + \frac{2}{d}L_2L_6 + \frac{2}{d}L_3L_5 + \frac{1}{d}L_4{}^2 +$$

$$\frac{3}{d^2}L_2{}^2L_4 + \frac{3}{d^2}L_2L_3{}^2 + \frac{1}{d^3}L_2{}^4$$

Then we obtain the following form for the characteristic equation:

$$\det(\lambda E - \rho) = (\xi)^d - \frac{1}{2}L_2\xi^{d-2} - \frac{1}{3}L_3\xi^{d-3} - \frac{1}{4}[L_4 + (\frac{1}{d} - \frac{1}{2})L_2{}^2]\xi^{d-4} -$$

$$\frac{1}{5}[L_5 + (\frac{2}{d} - \frac{5}{6})L_2L_3]\xi^{d-5} -$$

$$\frac{1}{6}[L_6 + (\frac{2}{d} - \frac{1}{8})L_2L_4 + (\frac{1}{d} - \frac{1}{3})L_3{}^2 + (\frac{1}{d} - \frac{3}{4d} + \frac{1}{8})L_2{}^3]\xi^{d-6} -$$

$$\frac{1}{7}[L_7 + (\frac{2}{d} - \frac{7}{10})L_2L_5 + (\frac{2}{d} - \frac{7}{12})L_3L_4 + (\frac{3}{d^2} - \frac{119}{60d} + \frac{7}{24})L_2{}^2L_3]\xi^{d-7} - \qquad (3.4)$$

$$\frac{1}{8}[L_8 + (\frac{2}{d} - \frac{2}{3})L_2L_6 + (\frac{2}{d} - \frac{8}{15})L_3L_5 + (\frac{1}{d} - \frac{1}{4})L_4{}^2 + (\frac{3}{d^2} - \frac{11}{6d} + \frac{1}{4})L_2{}^2L_4 +$$

$$(\frac{3}{d^2} - \frac{26}{15d} + \frac{2}{9})L_3{}^2L_2 + (\frac{1}{d^3} - \frac{11}{2d^2} + \frac{1}{4d} - \frac{1}{48})L_2{}^4]\xi^{d-8}$$

Two interesting cases with $d \le 8$ are the followings:

a) $d = 4 = 2 \times 2$, corresponding to a system composed from two spins $\frac{1}{2}$ when the characteristic equation is given by:

$$\xi^4 - \frac{1}{2}L_2\xi^2 - \frac{1}{3}L_3\xi - \frac{1}{4}[L_4 - \frac{1}{4}L_2{}^2] = 0 \qquad (3.5)$$

b) $d = 6 = 2 \times 3$, corresponding to a system composed from a spin $\frac{1}{2}$ system and a spin 1 system with the characteristic equation given by:

$$\xi^6 - \frac{1}{2}L_2\xi^4 - \frac{1}{3}L_3\xi^3 - \frac{1}{4}[L_4 - \frac{1}{3}L_2{}^2]\xi^2 - \frac{1}{5}[L_5 - \frac{1}{2}L_2L_3]\xi -$$

$$\frac{1}{6}[L_6 - \frac{5}{12}L_2L_4 - \frac{1}{36}L_3{}^2 + \frac{1}{36}L_2{}^3] = 0 \qquad (3.6)$$

4. THE FANO PARAMETRIZATION

The density matrix, which corresponds to a state of a bipartite quantum system, composed from two subsystems of dimensions d_1 and d_2, can be parametrized by the Fano parameters [2]:

$$\rho = \frac{1}{d_1 d_2}(I_1 \otimes I_2) + \frac{1}{2d_2} <x,\tau> \otimes I_2 + \frac{1}{2d_1} I_1 \otimes <y,\tau> + \frac{1}{4}\sum_{k=1}^{d_1^2-1}\sum_{l=1}^{d_2^2-1} K_{kl}(\tau_k \otimes \tau_l) \qquad (4.1)$$

In the particular case when $d_1 = d_2 = 2$ we have [15]:

$$
\begin{aligned}
&x_1 = v_4 + v_{11} &\quad &y_1 = v_1 + v_{13} &\quad &K_{11} = v_6 + v_9 \\
&x_2 = v_5 + v_{12} &\quad &y_2 = v_2 + v_{14} &\quad &K_{12} = -v_7 + v_{10} \\
&x_3 = \frac{2}{\sqrt{3}}v_8 + \sqrt{\frac{2}{3}}v_{15} &\quad &y_3 = v_3 - \frac{1}{\sqrt{3}}v_8 + \sqrt{\frac{2}{3}}v_{15} &\quad &K_{13} = v_4 - v_{11}
\end{aligned}
\qquad (4.2)
$$

and

$$
\begin{aligned}
&K_{21} = v_7 + v_{10} &\quad &K_{31} = v_1 - v_{13} \\
&K_{22} = v_6 - v_9 &\quad &K_{32} = v_2 - v_{14} \\
&K_{23} = v_5 - v_{12} &\quad &K_{33} = v_3 + \frac{1}{\sqrt{3}}v_8 - \sqrt{\frac{2}{3}}v_{15}
\end{aligned}
\qquad (4.3)
$$

The converse relations are in an easy way obtained from these relations [15].

5. THE CALCULATION OF THE SYMMETRIC PRODUCT WITH FANO PARAMETERS

In order to obtain the Lendi invariants in the Fano parametrization we must have the values of the symmetric product as functions of the Bloch vector components [15]:

$$(v \bigcup v)_1 = \frac{2}{\sqrt{3}}v_1 v_8 + v_4 v_6 + v_5 v_7 + v_9 v_{11} + v_{10} v_{12} + \sqrt{\frac{2}{3}}v_1 v_{15}$$

$$(v \bigcup v)_2 = \frac{2}{\sqrt{3}}v_2 v_8 - v_4 v_7 + v_5 v_6 + v_{10} v_{11} - v_9 v_{12} + \sqrt{\frac{2}{3}}v_2 v_{15} \qquad (5.1)$$

$$(v \bigcup v)_3 = \frac{2}{\sqrt{3}}v_3 v_8 + \frac{1}{2}(v_4^2 + v_5^2 - v_6^2 - v_7^2 + v_9^2 + v_{10}^2 - v_{11}^2 - v_{12}^2) + \sqrt{\frac{2}{3}}v_3 v_{15}$$

and

$$(v \bigcup v)_4 = v_1 v_6 - v_2 v_7 + v_3 v_4 - \frac{1}{\sqrt{3}}v_4 v_8 + v_9 v_{13} + v_{10} v_{14} + \sqrt{\frac{2}{3}}v_4 v_{15}$$

$$(v \bigcup v)_5 = v_1 v_7 + v_2 v_6 + v_3 v_5 - \frac{1}{\sqrt{3}}v_5 v_8 + v_{10} v_{13} - v_9 v_{14} + \sqrt{\frac{2}{3}}v_5 v_{15}$$

$$(v \bigcup v)_6 = v_1 v_4 + v_2 v_5 - v_3 v_6 - \frac{1}{\sqrt{3}}v_6 v_8 + v_{11} v_{13} + v_{12} v_{14} + \sqrt{\frac{2}{3}}v_6 v_{15} \qquad (5.2)$$

$$(v \bigcup v)_7 = v_1 v_5 - v_2 v_4 - v_3 v_7 - \frac{1}{\sqrt{3}}v_7 v_8 + v_{12} v_{13} - v_{11} v_{14} + \sqrt{\frac{2}{3}}v_7 v_{15}$$

and

$$(v \bigcup v)_8 = \frac{1}{2\sqrt{3}}(v_9{}^2 + v_{10}{}^2 + v_{11}{}^2 + v_{12}{}^2 - v_4{}^2 - v_5{}^2 - v_6{}^2 - v_7{}^2) +$$

$$\frac{1}{\sqrt{3}}(v_1{}^2 + v_2{}^2 + v_3{}^2 - v_8{}^2 - v_{13}{}^2 - v_{14}{}^2) + \sqrt{\frac{2}{3}}v_8 v_{15}$$

(5.3)

and

$$(v \bigcup v)_9 = v_1 v_{11} - v_2 v_{12} + v_3 v_9 + v_4 v_{13} - v_5 v_{14} + \frac{1}{\sqrt{3}}v_9 v_8 - \sqrt{\frac{2}{3}}v_9 v_{15}$$

$$(v \bigcup v)_{10} = v_1 v_{12} + v_2 v_{11} + v_3 v_{10} + v_4 v_{14} + v_5 v_{13} + \frac{1}{\sqrt{3}}v_{10} v_8 - \sqrt{\frac{2}{3}}v_{10} v_{15}$$

(5.4)

$$(v \bigcup v)_{11} = v_1 v_9 + v_2 v_{10} - v_3 v_{11} + v_6 v_{13} - v_7 v_{14} + \frac{1}{\sqrt{3}}v_{11} v_8 - \sqrt{\frac{2}{3}}v_{11} v_{15}$$

$$(v \bigcup v)_{12} = v_1 v_{10} - v_2 v_9 - v_3 v_{12} + v_6 v_{14} + v_7 v_{13} + \frac{1}{\sqrt{3}}v_{12} v_8 - \sqrt{\frac{2}{3}}v_{12} v_{15}$$

and

$$(v \bigcup v)_{13} = -\frac{2}{\sqrt{3}}v_{13} v_8 + v_4 v_9 + v_5 v_{10} + v_6 v_{11} + v_7 v_{12} - \sqrt{\frac{2}{3}}v_{13} v_{15}$$

$$(v \bigcup v)_{14} = -\frac{2}{\sqrt{3}}v_{14} v_8 + v_4 v_{10} - v_5 v_9 + v_6 v_{12} - v_7 v_{11} - \sqrt{\frac{2}{3}}v_{14} v_{15}$$

(5.5)

$$(v \bigcup v)_{15} = \frac{1}{\sqrt{6}}(v_1{}^2 + v_2{}^2 + v_3{}^2 + v_4{}^2 + v_5{}^2 + v_6{}^2 + v_7{}^2 + v_8{}^2) -$$

$$\frac{1}{\sqrt{6}}(v_9{}^2 + v_{10}{}^2 + v_{11}{}^2 + v_{12}{}^2 + v_{13}{}^2 + v_{14}{}^2) - \sqrt{\frac{2}{3}}v_{15}{}^2$$

The remarkable fact discovered by Kummer in [13] and [14] is the very simple and tractable form of these equations when they are written for the Fano parameters.

We shall find these equations directly from the equations for the Bloch vector using the relations between the Bloch and Fano parametrizations given above. We shall define the Fano parameters (denoted by the same symbols with a hat) associated with the vector $v \bigcup v$ as functions of the Fano parameters of the Bloch vector v. Or in a more short way

$$\hat{x} = Ky$$
$$\hat{y} = K^T x$$
$$\hat{K} = xy^T - (adjK)^T$$

(5.6)

Here $adjK = K^{-1} \det K$ and K^T denote the transpose of the matrix K. For the scalar product of a two Bloch vectors we have:

$$<v, \tilde{v}> = \frac{1}{2}[<x, \tilde{x}> + <y, \tilde{y}> + Tr(K^T \tilde{K})]$$

(5.7)

Ten we obtain:

$$L_2 = \frac{1}{2} <v,v> = \frac{1}{4} [<x,x> + <y,y> + TrK^T K]$$

$$L_3 = \frac{1}{4} <v \bigcup v, v> = \frac{3}{8} [<x,Ky> - \det(K)]$$

$$L_4 = \frac{1}{8} <v \bigcup v, v \bigcup v> = \frac{1}{8} [<K^T x, K^T x> + <Ky, Ky> + <x,x><y,y>] +$$

$$\frac{1}{8} [-2 <(adjK)x, y> + Tr((adjK)^T (adjK))]$$

(5.8)

6. THE CHARACTERISTIC EQUATION OF A SYSTEM COMPOSED FROM TWO SPIN $\frac{1}{2}$

As we have seen in the section 3, in the case of a system composed from two spins $\frac{1}{2}$ the characteristic equation is given by:

$$\xi^4 - \frac{1}{2} L_2 \xi^2 - \frac{1}{3} L_3 \xi - \frac{1}{4} [L_4 - \frac{1}{4} L_2^2] = 0$$

(6.1)

This equation is exactly the incomplete form of the initial characteristic equation. The Decartes-Euler solution of the incomplete equation are given by

$$\xi_1 = \frac{1}{4} (-\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} - \sqrt{\varepsilon_3})$$

$$\xi_2 = \frac{1}{4} (-\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} + \sqrt{\varepsilon_3})$$

(6.2)

$$\xi_3 = \frac{1}{4} (\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2} + \sqrt{\varepsilon_3})$$

$$\xi_4 = \frac{1}{4} (\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} - \sqrt{\varepsilon_3})$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the roots of the cubic resolvent equation:

$$\varepsilon^3 - 4L_2 \varepsilon^2 + 16L_4 \varepsilon - \frac{64}{9} L_3^2 = 0$$

(6.3)

or

$$\varepsilon^3 - 2 <v,v> \varepsilon^2 + 2 <v \bigcup v, v \bigcup v> \varepsilon - \frac{4}{9} <v \bigcup v, v>^2 = 0$$

(6.4)

It is evident that we have real roots. In the particular case when $x = y = 0$ we have:

$$\varepsilon^3 - Tr(K^T K)\varepsilon^2 + Tr((adjK)^T (adjK))\varepsilon - (\det K)^2 = 0$$

(6.5)

When $K^T K = diag(t_1^2, t_2^2, t_3^2)$ as in the Horodeckis 's case [16] the equation (6.5) is the Cayley – Hamilton of the matrix $K^T K = diag(t_1^2, t_2^2, t_3^2)$ and have the roots $\varepsilon_1 = t_1^2, \varepsilon_2 = t_2^2, \varepsilon_3 = t_3^2$. This is the solution of the characteristic equation obtained in [16] by physical arguments.

Let us consider another case in which the solutions of the equation (6.5) can be obtained directly. This is the case of pseudo-pure states treated in [15]. In this case we can have $x \neq 0$, $y \neq 0$ but for the Bloch vector we have the restriction $v \bigcup v = \mu v$ which for the corresponding Fano parameters becomes:

$Ky = \mu x$, $K^T x = \mu y$, $xy^T - (adjK)^T = \mu K$. In this case we have $Tr(K^T K) = 3\mu^2 - 2 <x, x>$, $Tr((adjK)^T (adjK)) = 3\mu^4 - 4\mu^2 <x, x> + <x, x>^2$, and $\det(K) = \mu(\mu^2 - <x, x>)$ i.e. the equation (6.5) becomes:

$$\varepsilon^3 - (3\mu^2 - 2 <x, x>)\varepsilon^2 + (3\mu^4 - 4\mu^2 <x, x> + <x, x>^2)\varepsilon - \mu^2(\mu^2 - <x, x>)^2 = 0 \tag{6.6}$$

i.e. the equation (6.5) factorizes in the following way:

$$(\varepsilon - \mu^2)(\varepsilon - (\mu^2 - <x, x>))^2 = 0 \tag{6.7}$$

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