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SEARCH FOR A SOLUTION OF SEELIGER'S GRAVITATIONAL PARADOX IN THE FRAMEWORK OF GENERAL RELATIVITY THEORY

Nicholas IONESCU-PALLAS[*], Ion SIMACIU[**], Marius-Ioan PISO[***]

[*] Former Senior Scientific Research, Institute of Atomic Physics, 76900 Bucharest, Măgurele, Romania
[**] Physics Department, University Petroleum - Gas of Ploiesti, Romania
[***] ROSA Research Center, 70168 Bucharest, Romania
Corresponding author: Ion Simaciu, University Petroleum - Gas of Ploiesti,
Bd. Bucuresti 39, 100680 Ploiesti, phone: +40 244 573 171, fax: +40 244 575 847, email: isimaciu@yahoo.com

A new integration condition for Einstein's field equations is found in the static case of a spherical body with mass density $\rho(r)$ and radius R. Taking the metric as $\mathrm{d}s^2 = e^{\nu}c^2\,\mathrm{d}t^2 - e^{\lambda}\,\mathrm{d}r^2 - e^{\sigma}r^2\,\mathrm{d}\mathbf{\Omega}^2$, with $\mathrm{d}\mathbf{\Omega} = (\mathbf{r}\times\mathrm{d}\mathbf{r})/\mathbf{r}^2$, the condition is $\nu+\sigma=0$. It allows the metric functions (ν,λ,σ) to be expressed in terms of a single potential function, Φ, which satisfies a Poisson-type equation with Euclidian coefficients. The source of the field is entirely located inside the body and contains explicitly the gravitational energy. The integration condition may be interpreted in terms of a gauge condition of a generalized De Donder-Rosen-Fock type. Among the characteristic features of the potential Φ, we point out the compliance with the point-like black hole and with the existence of a finite pressure for $\rho = const.$ and $R\to\infty$, namely $p=\rho c^2/2$. Thus, a solution of the famous Seeliger's Paradox, of the divergent cosmical pressure, is given in the framework of General Relativity Theory without introducing neither a global curvature space nor a cosmological constant. A point-like black hole, surrounded by a spherical horizon of gauge invariant radius, is the natural equivalent of a classical point-like body, in the framework of General Relativity Theory. In its turn, such an object may by interpreted, in terms of a flat space-time geometry, as due to the existence of a repulsive gravitation acting at interbody distance of the magnitude order of GM/c^2.

Key words: general relativity; Seeliger's paradox; point-like black hole

1. ISTORICAL SURVEY REGARDING THE PARADOX OF COSMICAL PRESSURE. FAILURE OF THE ATTEMPT TO EXPLAIN THE PARADOX WITH THE AID OF INTERIOR SCHWARZSCHILD METRIC. THE LEAVING OF THE HYDRODINAMIC MODEL OF MATTER.

The aim of this work is to take again the problem of mechanical equilibrium of an infinite physical Universe, in view of its complete solving in the framework of Einstein's General Relativity Theory. As it is known, in 1895 Hugo von Seeliger formulated the "gravitational paradox", named equally the "paradox of the cosmical pressure" [1]. The content of the paradox is the denying of the possibility to build up a model of Universe, in the framework of Newtonian Gravitational Theory, in which smoothed out matter should be represented by an ideal fluid filling up the infinite Euclidian space and being characterized by the finite values of pressure and mass density. For ensuring the pressure finiteness, Seeliger put forward the conjecture about the exponential decrease of the Newtonian potential in terms of distance, without being able to predict the magnitude order of the cosmical equilibrium pressure [2].

The advent in 1905 of the Special Theory of the Relativity [3] created the possibility to foresee not only the finitude of the pressure, but equally its magnitude order $p\approx\rho c^2/3\approx 10^{-16}\,\mathrm{atm}$ [2]. The advent in 1915 of the General Relativity Theory allowed the reappraisal of the Seeliger's gravitational paradox in expectation of a convenient solving, without being necessary to modify just the Newtonian potential, since

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the new theory predicted no such modification [4]. But, since about 1922 [5} it was ascertained that using the inner solution of the Schwarzschild one body metric, worked out in 1916 [6], it is not possible to prevent the appearance of an infinite pressure. For a spherical body of constant mass density, this happens when $R=(9/8)(2GM/c^2)$ [7]. Albert Einstein and William de Sitter, believing that the possibilities of the theory put forward in 1915 were already exhausted, without being able to avoid the infinite pressure, decided to abandon the hydrodynamic model and to add to the field equations a term containing the controversed Λ constant [8]. Perhaps, without this hurried leaving of the hydrodynamic model, which induced suspicion on possibility of edifying an infinite Universe, according to the philosophical line of reasoning of the XIX - th century, the subsequent development of the Relativistic Cosmology would had a different course [9].

2. THE TRUE CAUSES OF THE FAILURE OF THE HYDRODYNAMICAL MODEL. THE CONCEPT OF ACCESIBLE SPACE. ITS TOPOLOGICAL ORIGINE. THE CONCEPT OF POINT LIKE BLACK HOLE SURROUNDED BY A SPHERICAL GAUGE INVARIANT HORIZON AND ITS COMPLIANCE WITH “ZEROTH LAW OF THERMODYNAMICS”

Leaving aside the historical aspect, we focused our attention in view of identifying the true reason which encumbered the reaching of the cosmical equilibrium at a finite pressure, by a cancellation in every point of the space between the attractive gravitational force (of the Newton type) and the repulsive hydrodynamic force (of the Euler type). In a reliable manner, we established that the pressure finiteness is obtained provided that, by a special gauge, the forbidden zone of the position space, due to the existence of a black hole, is excluded from physical calculation. As a black hole is a topologic singularity, it cannot be avoided altogether, but the inaccessible space may be reduced to a single point ($r=0$). So, we reached the concept of point-like black hole. Black hole is an equivalent, in the framework of General Relativity Theory (GRT) of the Newtonian concept of point-like body [10]. It has two kinds of physical properties: 1) invariant under a gauge transformation (horizon surface, mass, electric charge, intrinsic angular momentum) and 2) gauge dependent (radial size, defined as $g_{00}(R_1)=0$). A point-like black hole is therefore characterized by $R_1=0$ (radial size) and $A=16\pi(GM/c^2)^2$ (horizon surface) [11, 12].

Further on, our aim is to find the integrability condition in compliance with a point-like black hole. We assume the static metric, yielded by a spherical body placed in the origin of an inertial frame, to have the form [13]

$$\mathrm{d}s^2=e^{\nu}c^2\mathrm{d}t^2\pm 2\eta c\,\mathrm{d}t\mathrm{d}r-\left\{e^{\lambda}\mathrm{d}r^2+e^{\sigma}r^2\left[\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right]\right\} \tag{1}$$

where $(\nu,\eta,\lambda,\sigma)$ are function of the radial coordinate r.

We determine the four metric functions $(\nu,\eta,\lambda,\sigma)$ by resorting to the field equations of GRT, the field source being the matter tensor of the hydrodynamic model (the so called perfect fluid scheme). The equations to be solved are the following ones [14]

$$\begin{gathered}E_{\alpha\beta}\equiv-8\pi\frac{G}{c^4}T_{\alpha\beta},\quad E_{\alpha\beta}\equiv R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R;\\ T_{\alpha\beta}=\left(c^2+H\right)\rho U_{\alpha}U_{\beta}-pg_{\alpha\beta},\quad H=\int_0^{p(\rho)}\frac{\mathrm{d}p}{\rho(p)}.\end{gathered} \tag{2}$$

Here $E_{\alpha\beta}$ is the conservative field tensor of Einstein, $T_{\alpha\beta}$ is the matter tensor of the fluid, (ρ,p) are the invariant mass density and the invariant pressure, respectively, and H is the potential of Helmholtz. $\rho=\rho(r),\ p=p(r)$ for $0<r<R$ and $\rho=0,\ p=0$ for $R<r<\infty$ (Do not confound R - radius of the spherical source with R - the scalar of $R_{\alpha\beta}$ in $E_{\alpha\beta}$ expression). For $r>R$, the solution we look for is

$$e^{\nu}=1-2\frac{\mu}{r}\exp\left(-\sigma/2\right),\quad \exp\left(\lambda+\nu\right)=\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left[r\exp\left(\sigma/2\right)\right]\right\}^{2}-\eta^{2},\tag{3}$$

σ and η being arbitrary functions of their argument r, as a result of spherical symmetry and of the $E_{\alpha\beta}$-conservativity, $\mu=GM/c^{2}$, M stands for the mass of the source. Now, we may use the solutions (1) and (3) for defining the parameter R_1 and calculate the parameter A (not depending on R_1 !):

$$1-2\left(\mu/R_1\right)\exp\left[-\frac{\sigma(R_1)}{2}\right]=0,\quad A=4\pi R_1^{2}\exp\left[\sigma(R_1)\right]=4\pi R_1^{2}\left(2\mu/R_1\right)^{2}=16\pi\mu^{2}.\tag{4}$$

Thus the parameter A (the horizon surface) depends neither on the function η, nor on the function σ. In other words it is independent on gauge[1] Accordingly, we may adopt a special gauge which delivers us $R_1=0$.

3. JUSTIFYING THE CHOICE OF THE METRICAL FUNCTION η (TIME OBLIQUENESS) AND σ (RADIAL SIZE GRADUATION) IN AN INERTIAL FRAME OF COORDINATES

Concerning the function η, we may adopt the value $\eta=0$. This may be argued in the following way. Out of $\mathrm{d}s^{2}$, we may define first a Lagrange function of motion through the formula $L=-m_0c(\mathrm{d}s/\mathrm{d}t)$. Thereafter, we define a momentum as $\vec{p}=\partial L/\mathrm{d}\vec{\upsilon}$ and ask the condition $\vec{p}\to 0$ for $\vec{\upsilon}\to 0$. This is possible provided that $\eta=0$ or, equivalently, time is orthogonal onto position space (no obliqueness of time). Further on, we go to determine the function σ. For this purpose, we consider a gravitational plane rotator. It delivers the metric $\mathrm{d}s^{2}=e^{\nu}c^{2}\mathrm{d}t^{2}-e^{\sigma}\mathrm{d}l_{\perp}^{2},\mathrm{d}l_{\perp}\equiv r_0\mathrm{d}\theta$, and the bi-dimensional elementary invariant $\delta I_{(2)}=\exp\left[(\nu+\sigma)/2\right]c\delta t\delta l_{\perp}$. If $\nu+\sigma=0$, then $\delta I_{(2)}$ is an universal invariant no longer depending on the mass of the source body. In this case, $\delta I_{(2)}$ may be put in compliance with the quantizing of space and time under the form $\delta I_{(2)}=l_P^{2}\delta n=(\hbar G/c^{3})\delta n$, were l_P is the Planck length and δn the number of discrete states confined in the elementary bi-dimensional Universe. Now, we obtain the explicit expression [15] of σ: $\nu+\sigma=0\to 1-(2\mu/r)\exp\left(-\sigma/2\right)=\exp\left(-\sigma\right)$ whence $\nu+\sigma=\mathrm{sh}\left(\sigma/2\right)=\mu/r$ id est

$$\sigma=\ln\left(\sqrt{1+\mu^{2}/r^{2}}+\mu/r\right)^{2},\ r_s=r\exp\left(\sigma/2\right)=\sqrt{r^{2}+\mu^{2}}+\mu.\tag{5}$$

Here r_s is „radial coordinate" used by Schwarzschild [16].

4. COMPLIANCE OF THE POINT LIKE BLACK HOLE CONCEPT WITH THE BEKENSTEIN – HAWKING MASS FORMULA

Denoting $r_s=f(r)$, $f(r)=\sqrt{r^{2}+\mu^{2}}+\mu$, we obtain the transition from Schwarzschild metric, for a point-like body, to the natural metric, for both point-like source and point-like black hole, under the form

[1] This result is known in the literature of the subject as „ Zeroth law of black hole Thermodynamics" (R. M. Wald, General Relativity, Chicago, University Press, 1984)

$$\begin{gathered}\mathrm{d}s^2=\left(1-2\frac{\mu}{f}\right)c^2\mathrm{d}t^2-\left\{\frac{f'^2}{1-2\mu/f}\mathrm{d}r^2+\left(f/r\right)^2\frac{\left(\vec{r}\times\mathrm{d}\vec{r}\right)^2}{r^2}\right\}=\\ \frac{c^2\mathrm{d}t^2}{\left(\sqrt{1+\mu^2/r^2}+\mu/r\right)^2}-\left(\sqrt{1+\mu^2/r^2}+\mu/r\right)^2\left\{\frac{\mathrm{d}r^2}{1+\mu^2/r^2}+\frac{\left(\vec{r}\times\mathrm{d}\vec{r}\right)^2}{r^2}\right\}\end{gathered}. \tag{6}$$

This change of gauge is motivated, on a side by invoking the equivalence principle and, on another side, by recovering the inertia principle in view of avoiding the infiniteness of the cosmical pressure.

The key for ensuring the finiteness of the cosmical pressure is, as we already pointed out, the avoiding from calculations of the inaccessible zone of the position space, yielded by the inevitable presence of the black hole. The "inevitability" is closely connected to the Einstein's theory and has a topological origin. In other words, only by changing the field equations of relativistic gravitation it is possible to avoid altogether the black hole from the physical theories. Coming back to the problem of avoiding the inaccessible zone, this avoidance mast be made in compliance with the equivalence principle, that is through the agency of a gauge changing. It is possible to perform the gauge change in a trivial way, as a purely geometric transformation $r_s = r + 2\mu$, but, in this way, the metric is still written in a non inertial frame and this feature alters the value of the cosmical pressure, without altering its order of magnitude. We consider that the transformation $f(r)=\sqrt{r^2+\mu^2}+\mu$ is just the physical one, to be taken into account, as far as it not only avoids from calculation the inaccessible zone, placed beyond the time spending barrier, but equally ensures the restoring of the inertia frame, because the position space of the transformed metric is almost conformally Euclidian.

Further on, we shall give an example about the full compliance of the concept of point-like black hole with the thermodynamic theory of black hole, worked out, independently, by Bekenstein and Hawking [17, 18]. The starting point is the mass variation formula for a black hole, at constant angular momentum J and constant electric charge Q, namely $\delta M=[g(A)/(8\pi G)]\delta A$. Here, A is the horizon surface, $g(A)=GM/R_H^2(A)$ is the gravitational acceleration of the horizon and $R_H=\sqrt{A/4\pi}$ is the horizon radius. All the quantities in the formula of δM are gauge invariant and, accordingly, do hold equally for a point-like black hole. On the other hand, an acceleration applied to a box filled with radiation is equivalent to a temperature increase. Bekenstein conjectured that the same relationship between acceleration and absolute temperature does hold for a black hole, which is not a simple topological singularity, but rather a physical systems to which the concepts of entropy and absolute temperature are applicable. By writing

$$\delta E=c^2\delta M=\frac{c^2g(A)}{8\pi G}\delta A=\left[\frac{c^2l_P^2g(A)}{2\pi kG}\right]\delta\left[k\frac{1}{4}\frac{A}{l_P^2}\right]\equiv T_{BH}\,\delta S_{BH} \tag{7}$$

we may identify, for the black hole, an absolute temperature [17]

$$T_{BH}=\frac{c^2l_P^2g(A)}{2\pi k\,G} \tag{8a}$$

and an entropy [17]

$$S_{BH}=k\frac{1}{4}\frac{A}{l_P^2}. \tag{8b}$$

Let us now calculate the horizon surface by resorting to the natural gauge ($\nu+\sigma=0$). We can write [16]

$$A=\lim_{R_1\to 0}4\pi R_1^2\left(\sqrt{1+\Phi^2}+\Phi\right)^2,\ \Phi\equiv\frac{GM}{c^2R_1}. \tag{9}$$

If, in addition, we use the fine structure formula $\hbar c=137Q^2$, we come to the expression

$$S_{BH}=\lim_{R_1\to 0}\frac{k\pi G}{\hbar c}\left(M+\sqrt{M^2+137(R_1/l_P)^2Q^2/G}\right)^2,\ l_P\equiv\sqrt{\frac{\hbar G}{c^3}}\,. \tag{10}$$

To obtain just the Bekenstein – Hawking entropy formula, we have only to assume that in the proximity of the horizon surface the discrete structure of space becomes relevant and that, accordingly, the condition $R_1\to 0$ should be replaced by the more realistic one $R_1\to l_P/\sqrt{137}=Q\sqrt{G}/c^2$. In this way we reach the famous formula (for a charged black hole) [19]

$$S_{BH}=\frac{k\pi G}{\hbar c}\left(M+\sqrt{M^2+Q^2/G}\right)^2. \tag{11}$$

If $Q=0$, then $R_1\to 0$ and $A=16\pi\left(GM/c^2\right)^2$ as already known.

5. SCHWARZSCHILD EXTERIOR METRIC IN THE NEW GAUGE, DETERMINED BY THE CONJECTURE ABOUT POINT-LIKE BLACK HOLE, AND ITS MAIN FEATURES

The natural gauge ($\nu+\sigma=0$) is not a trivial coordinate transformation of the Schwarzschild metric justified by the Equivalence Principle. It is that gauge, long time looked for by the astronomers and physicists, which, by a natural incorporation of the Inertia Principle into the GRT, should be able to solve some difficult problems of Cosmology, Astrophysics and Black Hole Physics. The problems solved by the gauge ($\nu+\sigma=0$) and not by the gauge $\sigma=0$ are coming from those proprieties of a black hole which are not gauge invariant.

The metric and the field equations in the one body case, for a point-like body containing a point-like black hole, may be obtained out of formulas (5)

$$\begin{gathered}\mathrm{d}s^2=\frac{c^2\mathrm{d}t^2}{\left(\sqrt{1+\Phi^2}+\Phi\right)^2}-\left(\sqrt{1+\Phi^2}+\Phi\right)^2\left\{\frac{\mathrm{d}r^2}{1+\Phi^2}+\frac{(\boldsymbol{r}\times\mathrm{d}\boldsymbol{r})^2}{r^2}\right\},\\ \Delta\Phi=-4\pi\frac{G}{c^2}\rho(\boldsymbol{r}),\quad \rho(\boldsymbol{r})=M_0\delta(\boldsymbol{r}).\end{gathered} \tag{12}$$

The above formula does hold equally for a spherical source of field, in the outward region $r>R$. Onto the surface $r=R$ of the spherical source, the potential function Φ and its derivative $\mathrm{d}\Phi(r)/\mathrm{d}r$ must go continuously into their counterparts, which are solutions of the Einstein's field equations inside the source ($r<R$). If formula (12) would be valid in the inward region, provided that only amendment to be brought is the replacing of the point-like mass distribution $M_0\delta(\vec{r})$ by arbitrary mass distribution $\rho(\vec{r})$, then we can speak about a" functional solution" of the one body problem, namely $\nu=\nu(\Phi),\lambda=\lambda(\Phi),\sigma=\sigma(\Phi)$. Such a solution cannot be exactly obtained in the framework of GRT. For practical applications, however, it is possible to obtain a first order functional approximation

$$e^{\nu}\approx\left(1-\Phi+\frac{1}{2}\Phi^2\right)^2,\ e^{\lambda}\approx e^{\sigma}\approx\left(1+\Phi+\frac{1}{2}\Phi^2\right)^2. \tag{13}$$

The only relativistic theory of gravitation, delivering exactly a functional one body metric, is the Rosen's theory devised in 1974 [20][2]. The first order approximation of the Rosen's one body metric does coincide with the approximation (13), coming from Einstein's GRT. So, one can speak about an almost functional one body metric, delivered by the GRT, within the conjecture about the existence of point-like black hole.

[2] See also Ion Dobrescu and Nicholas Ionescu-Pallas, *Variational and Conservative Principles in Rosen's Theory of Gravitation,* Balkan J. of Geom. And Its Appl.,Vol 4, nr. 1 (1999), pp. 31-43

6. GAUGE AS INTEGRABILITY CONDITIONS. CONTINUITY CONDITIONS ONTO THE SPHERICAL SURFACE OF SOURCE

It is opportune now to point out some aspects connected to the concepts of integrability and gauge. Only the components E_{00} and E_{22} of the Einstein's field tensor contain second order derivatives, namely:

$$\begin{aligned} & E_{00}\exp(\lambda-\nu)=\Delta\sigma+\textit{first order derivatives},\\ & -E_{11}=\textit{first order derivatives},\\ & \frac{1}{r^2}E_{22}\exp(\lambda-\sigma)=-\frac{1}{2}\Delta(\sigma+\nu)+\textit{first order derivatives}. \end{aligned} \tag{14}$$

Accordingly, there are two obvious cases of integrability: 1) $\sigma=0$ (Schwarzschild) and 2) $\nu+\sigma=0$ (Natural gauge). The gauges, defined as integrability cases, do hold, in the same form, for either inside or outside source solutions. Nevertheless, accounting for the geometrical radius of the source, which is not gauge invariant, and, at the same time, for the second derivatives of the metrical functions, which have saltus onto the source surface, we conclude that the transformation $r_s=\sqrt{r^2+\mu^2}+\mu$ performs the going over from the gauge $\sigma=0$ to the gauge $\nu+\sigma=0$ only outside the source. The interior solutions in the gauge $\nu+\sigma=0$ must be worked out separately, even though the homologous solutions for the gauge $\sigma=0$ are known. Of course, there are many other integrability conditions which are less natural, because resort equally to first order derivatives.

7. BIMETRIC EXPLANATION OF THE POINT-LIKE BLACK HOLE AND OF ZEROTH LAW IN TERMS OF REPULSIVE GRAVITATION ACTING AT VERY SMALL INTERBODY DISTANCE

Irrespective of the fact that some theorists agree or not, Einstein's GRT acquires an alternative interpretation in the framework of the doctrine of bimetrism. Bimetrism is a concept, used for the first time in 1922 by the mathematician Alfred Whitehead, in conjunction with an actio-in-distance theory of gravitation, in view of ensuring a more natural incorporation of the inertia principle into the relativistic theory of gravitation [21]. By inserting a flat metric, besides the curved one, Whitehead intended to avoid some inherent difficulties, yielded by the geometry depending on the position and motion of masses. In 1940, Nathan Rosen has the inspired idea to apply the concept of bimetrism just to the GRT [22]. In 1963, he extended his study acquiring new noteworthy results [23]. Among the relevant achievements, brought by us in the field of bimetrism of GRT, we may mention 1) The gravitational energy tensor (Ioan Gottlieb and Nicholas Ionescu-Pallas, 1986) [24]; 2) Gravitational Lorentz force (Nicholas Ionescu-Pallas, 1993) [2].

So far, nobody tried to explain, in the framework of bimetrism, the existence of the black holes. This lateness was determined by the missing of the point-like black hole concept. The common opinion among specialists is, in present, that the black holes may exist only as some rigid balls of finite radial size. As the radial size is not a gauge invariant quantity, there is no scientific argument for preferring a certain geometrical picture of the black hole. In this situation, the single criteria of choosing must rely on the physical consequences coming from the applying of a certain gauge. But, we know already that Schwarzschild gauge leads to great difficulties, as the infinite cosmical pressure (De Donder, 1922) [5] and the existence of two mass distribution inside a star (R. Oppenhaimer and G. Volkoff, 1939) [25][3] No such difficulty is to be found, when the point-like black hole concept is adopted. Once the gauge $\nu+\sigma=0$, corresponding to the point-like black hole, was adopted, a natural explanation of the black hole in terms of the attractive and repulsive forces may be put forward. The free falling down of a test body, in the radial direction, in the gravitational field yielded by a point-like massive source, is described by the formulas

[3] Similar difficulties are present for Schwarzschild gauge ($\sigma=0$) as well as for other gauges ($\sigma=\lambda$, $\lambda=\nu$); N. Ionescu-Pallas, Symp. in Bistritza, 1993 and 1995.

$$m_0\ddot{r} = -\frac{\partial}{\partial r}U(r), U(r) = -G\frac{m_0 M_0}{r}\chi(X),$$
$$\chi(X) = \frac{X^3(X^2+1)}{\left(\sqrt{X^2+1}+1\right)^5}, X = \frac{c^2 r}{GM_0}. \tag{15}$$

For $r = r_0$, $r_0 = 4.4505914(GM_0/c^2)$, $U'(r) = 0$ and the two kinds of forces do reciprocally cancel. For $r < r_0$ the repulsive force becomes dominant, preventing the reaching of the point $r = 0$ in a finite time. The gauge invariant horizon surface may be interpreted as the scattering cross section onto the repulsive potential.

8. EXACT SCHWARZSCHILD SOLUTION INSIDE THE SOURCE IN THE NEW GAUGE AND ITS PHYSICAL SELFCONSISTENCY

The accurate metric inside the source is close to formula (12) of the exterior metric

$$e^{\nu} = \left(\sqrt{1+\Phi^2}+\Phi\right)^{-2}, \ \exp\left[-(\nu+\lambda)\right] = 1+\zeta(r), \nu+\sigma = 0,$$
$$\zeta(r) = \frac{32\pi G}{c^4 r^2}\int_0^r r^3 p e^{-\nu}\mathrm{d}r \approx (r\Phi')^2 + 8\pi\frac{G}{c^4}pr^2, \Delta\Phi = -4\pi\frac{G}{c^4}\rho_E, \tag{16}$$
$$\rho_E = \left(e^{\lambda}\mathrm{ch}\frac{\nu}{2}\right)\left[\rho c^2 + \left(\rho H + 2p\right)\right] + \frac{c^4}{16\pi G}\nu'\left[\nu'\exp\left(\nu/2\right) + \lambda'\mathrm{ch}\frac{\nu}{2}\right].$$

It may be proved that $\rho_E = 0$ for $r > R$; moreover, by working out the necessary calculations in the expression of ρ_E we reach the result

$$\rho_E \approx \left\{\rho_m c^2 - \frac{GM}{R}\rho_m + \frac{1}{2}\rho_m U(r) - p\right\} + \left\{3p - \frac{1}{2}\rho_m U(r)\right\} + O\left(\frac{1}{c^2}\right). \tag{17}$$

Here $U(r) = c^2\Phi$ is the Newton's potential function, ρ_m is the invariant rest mass density in flat space and p is the equilibrium pressure.

Taking into account that [2]

$$p \approx -\int_r^R \rho_m U'(r)\mathrm{d}r, \quad \int_0^R\left(\frac{1}{2}\rho_m U(r) - 3p\right)4\pi r^2\mathrm{d}r \approx 0, \tag{18}$$

we prove that the energy stored inside the source is the total energy expected from an ideal self-gravitating fluid in the framework of Special Relativity Theory. The location of the total energy inside the source, as well as the description of the gravitational field in terms of a single mass distribution, are specific features, adequate for successfully entering upon the problem of cosmical pressure, to be found solely in the case of the natural gauge $\nu + \sigma = 0$.

9. SUCCESSFULLY SOLVING THE SEELIGER'S PRESSURE PARADOX, BY TAKING AGAIN THE HYDRODYNAMIC MODEL IN THE CONDITION OF THE NEW GAUGE

Assuming $\rho = const.$ and $\rho_E/c^2 \approx \rho$ one obtains for $r < R$

$$\Phi \approx \frac{GM}{c^2 R}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right), M \approx \frac{4}{3}\pi\rho R^3, \mu = \frac{GM}{c^2}, \ p = \rho c^2\left(\frac{\sqrt{1+\Phi^2}+\Phi}{\sqrt{1+\mu^2/R^2}+\mu/R} - 1\right). \tag{19}$$

For $(GM/c^2R) << 1$, $p \to (2/3)\pi G\rho^2 R^2(1-r^2/R^2)$; for $R \to \infty$, $p \to \rho c^2/2$. It is for the first time that a reliable solution of the gravitational paradox is delivered in the framework of GRT, using the classical concept of local equilibrium between gravitational and fluidodynamic forces. The finiteness of the cosmical pressure is a direct consequence of the non-linear relationship between pressures p and potential Φ, which, in its turn, is a consequence of the existence of point-like black hole.

10. GAUGE AS FIRST ORDER PARTIAL DERIVATIVE EXPRESSION, NECESSARY FOR COMPLETING THE DEFINITION OF GRAVITATIONAL FIELDS

The only inconvenient aspect of the point-like black hole theory, so far outlined, is the non covariant form of the assumed gauge ($\nu + \sigma = 0$). This may be easily remedied by resorting to Rosen's bimetrism [22, 23]. According to this doctrine, we define simultaneously two metrics – a curved one, including the gravitational field, and a flat one, in the absences of gravitation,

$$\mathrm{d}s_C^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta > 0, \mathrm{d}s_F^2 = a_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta > 0\,, \tag{20}$$

using the same set of coordinates (for instance spherical ones or Cartesian ones). The quantity $J = \sqrt{-g}/\sqrt{-a}$ is a Minkowskian invariant and the same property should be assigned to any scalar $\Lambda(J)$. As the gauge condition may be a differential equation with first order partial derivatives, we are entitled to write

$$H^{\alpha\beta}_{/\beta} = 0, \quad H^{\alpha\beta} = J\,g^{\alpha\beta} + \Lambda(J)\,a^{\alpha\beta}\,. \tag{21}$$

Further on, we determine the function $\Lambda(J)$ asking the recovering of the known integrability condition, in the case of spherical coordinates. Thus, one obtains:

1) De Donder-Lanczos-Rosen-Fock gauge [5, 26, 22, 27]

$$\nu + \lambda = 0, \Lambda = 1\,; \tag{22a}$$

2) Weyl gauge

$$\lambda - \sigma = 0, J = (2-\Lambda)\left(1+\sqrt{\Lambda-1}\right)^4, \Lambda = 1 + \frac{1}{4}\left(\frac{\mu}{r}\right)^2; \tag{22b}$$

3) Natural gauge

$$\nu + \sigma = 0, \sqrt{J} = \Lambda^{\frac{1}{2}} + \left(\Lambda - \frac{1}{\Lambda}\right)^{\frac{1}{2}}, \Lambda = \left(1 + \frac{\mu^2}{r^2}\right)^{\frac{1}{2}}. \tag{22c}$$

The relationship between J and Λ is obtained by eliminating r between $J(r)$ and $\Lambda(r)$. $J(r)$ is obtained, in its turn, though the intermediary of $f(r)$ as $J(r) = (f/r)^2 f'(r)$, with $r_s = f(r)$, id est:

1) $$\nu + \lambda = 0, \ J(r) = \left(1 + \frac{\mu}{r}\right)^2 \tag{23a}$$

2) $$\nu + \sigma = 0, J(r) = \left[\left(1 + \frac{\mu^2}{r^2}\right)^{\frac{1}{4}} + \frac{\mu}{r}\left(1 + \frac{\mu^2}{r^2}\right)^{-\frac{1}{4}}\right]^2 \tag{23b}$$

3) $$\lambda-\sigma=0,\ J(r)=\left(1+\frac{1}{2}\frac{\mu}{r}\right)^{4}\left(1-\frac{1}{4}\frac{\mu^{2}}{r^{2}}\right) \tag{23c}$$

The derivative in (21) is covariant in flat space-time. The formulation of the Weyl gauge in the form (21) is for the first time given in this work.

11. GAUGE INDEPENDENT AND GAUGE DEPENDENT PHYSICAL EFFECTS

The various physical effects, whose explanation is given by resorting to one body metric, may be shared in two categories: 1) Gauge invariant effects and 2) Gauge dependent effects. The famous tests of GRT [a) Perihelion advance, b) gravitational bending of a light ray and c) gravitational red shift of light] belong to the first category. Using the notations we already adopted: $r_s = f(r)$, and assuming $f(r)/r = 1+\alpha(\mu/r)+\beta(\mu/r)^2+.....$, we proved that, in the firs relativistic approximation, the values for the three tests are not α-dependent, although the motion equation exhibits such dependence [29]. Independence of the tests on the η function was also studied

$$\begin{gathered}\boldsymbol{a}=-\frac{GM_0}{r^3}\left[1-2C_1\frac{\mu}{r}+2C_2\frac{\boldsymbol{v}^2}{c^2}-2C_3\frac{(\boldsymbol{vr})^2}{c^2r^2}\right]\boldsymbol{r}+2C_4\frac{GM_0}{c^2r^3}(\boldsymbol{vr})\boldsymbol{v},\\ C_1=\alpha+1,\ C_2=1-\frac{1}{2}\alpha,\ C_3=\frac{3}{2}(1-\alpha),\ C_4=\alpha+1,\ \mu=\frac{GM_0}{c^2}.\end{gathered} \tag{24}$$

The fourth test (I. Shapiro, 1964) [13] has an intermediary position between the two mentioned categories of effects, as far as it turns out to be slightly α-dependent [29]. Concerning the gauge dependent effects, we have to point out, above all, the necessity to use a frame of inertia. Among the effects pertaining to this category, we mention: a) pressure finiteness in an infinite fluid medium, b) equilibrium of a self-gravitating fluid sphere in terms of a single mass density, c) non-contradictory falling down in gravitational field [28]. All these effects are very sensitive to small departures of the frame from the rigorous state of inertia. This critical aspect prevents us from the using of approximate expressions of $f(r)$ derived from many-body studies [31]. For evaluating such effects we need not only few terms in the expression of $f(r)/r$, but the sum of the infinite series. Owing to the very tedious calculations in gravitational many-body systems, this seems to be impossible.

12. PARAMETRIZED GAUGE. ATTEMPT TO DETERMINE THE GAUGE FROM THE COSMICAL CONSTRAINT OF PRESSURE FINITENESS

However, assuming the constraint $\lim\limits_{R\to\infty}\left[f(r)/(r\Phi)\right]=$ finite quantity, we may enter upon a study whose conclusion is more reliability for the natural gauge. The starting point, complying with the conjecture about the behavior at infinity, is $f(r)/r=(1+2\beta\mu^2/r^2)^{1/2}+\alpha(\mu/r)\approx\ 1+\alpha(\mu/r)+\beta(\mu/r)^2+\ldots$. Accordingly, inside the spherical source, we have

$$\frac{f(r)}{r}\to\exp\left(\frac{\sigma}{2}\right)=\sqrt{1+2\beta\Phi^2}+\alpha\Phi;\ e^{\nu}\to 1-2\frac{\mu}{r}\exp\left(-\frac{\sigma}{2}\right)\to\frac{\sqrt{1+2\beta\Phi^2}-(2-\alpha)\Phi}{\sqrt{1+2\beta\Phi^2}+\alpha\Phi},\text{whence}$$

$$e^{\nu}=\frac{1+\left[2\beta-(2-\alpha)^2\right]\Phi^2}{\left[\sqrt{1+2\beta\Phi^2}+\alpha\Phi\right]\left[\sqrt{1+2\beta\Phi^2}+(2-\alpha)\Phi\right]}. \tag{25}$$

The pressure inside a spherical source of radius R, at equilibrium, for constant mass density ρ, is given by the formulas

$$p = \rho c^2 \left[\frac{\exp\left[-(1/2)\,\nu(r)\right]}{\exp\left[-(1/2)\,\nu(R)\right]} - 1 \right] = \rho c^2 \left(\sqrt{A}\sqrt{B}\sqrt{C} - 1 \right),\ A = \frac{1 + \left[2\beta - (2-\alpha)^2 \right] (\mu / R)^2}{1 + \left[2\beta - (2-\alpha)^2 \right] \Phi^2},$$
$$B = \frac{\sqrt{1 + 2\beta\Phi^2} + \alpha\Phi}{\sqrt{1 + 2\beta\,(\mu/R)^2} + \alpha(\mu/R)} \equiv B(\beta,\alpha),\ C = C(\beta,\alpha) = B(\beta, 2-\alpha). \tag{26}$$

We distinguish two cases:

Case 1)

$$2\beta - (2-\alpha)^2 > 0,\ \alpha + \sqrt{2\beta} > 0,\ 2 - \alpha + \sqrt{2\beta} > 0,$$
$$\sqrt{A} \to \lim_{R\to\infty} \frac{(\mu/r)}{\Phi} \to \frac{2}{3},\ B \to \lim_{R\to\infty} \frac{\Phi}{(\mu/r)} \to \frac{3}{2},\ C \to B \to \frac{3}{2},$$
$$p \to \rho c^2 \left(\frac{2}{3}\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}} - 1 \right) = 0.$$

As this case is unphysical, we examine the following

Case 2)

$$2\beta - (2-\alpha)^2 = 0,\ \alpha + \sqrt{2\beta} > 0,\ 2 - \alpha + \sqrt{2\beta} > 0.$$

Thus, one obtains

$$\beta = \frac{1}{2}(2-\alpha)^2 = 0,\ \alpha \neq 2,$$
$$p \to \rho c^2 \left(\sqrt{3/2}\sqrt{3/2} - 1 \right) = \frac{1}{2}\rho c^2. \tag{27}$$

There are two alternatives emerging from this case: either $\alpha = 1$, $\beta = 1/2$ (i.e. natural gauge), or $\alpha = -1$, $\beta = 9/2$ (scalar gravitation [34]). The special case $\alpha = 2$, $\beta = 0$ leads also to a finite value of pressure $p = \rho c^2 \left(\sqrt{3/2} - 1 \right)$, but the position space is not isotropic with respect to a light ray propagation. This case corresponds to the elimination of the inaccessible space of the Schwarzschild metric, without associating this transformation with the recovering of the inertia frame. Owing to the Weyl's condition concerning the existence of gravitational waves [35], id est $\alpha = 1$, the only possible determination of the constant is that corresponding to natural gauge $(\alpha = 1, \beta = 1/2)$.

13. CONCLUDING REMARKS

It is useful to summarize the main consequences, coming as a result of adopting the natural gauge ($\nu + \sigma = 0$).

1. The entire three-dimensional space of position, excepting for the origin ($r = 0$), is accessible to physical experience.
2. Seeliger's gravitational paradox is avoided, allowing to devise a model of matter, filling up homogeneously and isotropically the infinite three-dimensional space, and lying permanently in a stable mechanical equilibrium, as a result of cancellation between attractive forces of the Newtonian type and repulsive forces of the Eulerian type. The equilibrium pressure is $\sim 10^{-16}\,\text{atm}$.
3. The unphysical picture of the interior region of a star, in terms of two mass densities, pointed out for the first time in 1939 by J. R. Oppenheimer and G. M. Volkoff, is avoided.

4. There is full compliance between natural gauge and black hole physics (which is based essentially on gauge invariant properties).
5. It is possible to "derive" even the famous Bekenstein-Hawking mass formula, provided that the space region, close to the horizon surface, should be discretized.
6. The bi-dimensional infinitesimal invariant, associated with a gravitational rotator, does not depend on the mass of source-body.
7. Te position subspace is almost conform onto the three-dimensional Euclidian space.
8. The one body metric is almost functional, id est the metric functions (ν, λ, σ) may be written as $\nu \approx \nu(\Phi)$, $\lambda \approx \lambda(\Phi)$ and $\sigma \approx \sigma(\Phi)$, Φ being the gravitational potential.
9. Natural gauge goes asymptotically into De Donder-Lanczos-Rosen-Fock gauge for $r \to \infty$.
10. Natural gauge may be written as the vanishing divergence of a certain Minkowskian tensor, in close analogy to De Donder-Lanczos-Rosen-Fock gauge.

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