

## FERMI PSEUDOPOTENTIALS AND HALO BORROMEAN NUCLEI

Cornel HATEGAN\*, Remus-Amilcar IONESCU

Institute of Atomic Physics, Bucharest, Romania

Corresponding author: Remus-Amilcar IONESCU, e-mail: remus@ifin.nipne.ro

The three-body systems are studied according to the Boundary Conditions Method; the relevant interactions are related to Fermi pseudopotentials. We obtain a relation between the spatial extension of the three-body bound state, its energy, and the scattering lengths in the corresponding two-body subsystems. The results are compared with the experimental data for the borromean nucleus  $^{11}\text{Li}$ .

### 1. INTRODUCTION

The study of the properties of the nuclei near the neutron drip line is a highly interesting topic in low energy nuclear physics of the last years. The existence of a neutron halo, *i.e.* a large spatial extension of the neutron density distribution, is an experimentally well established fact for some drip line nuclei [1-3]. The borromean nuclei are loosely bound systems although the systems with one neutron removed are unbound [3]; their structure does imply a three-body description, the conventional shell-model assumptions being insufficient. The main problem is how to treat the residual neutron-neutron interaction, a perturbative treatment being not able to produce a qualitative change of the spectrum; in other words, one cannot obtain a discrete spectrum in a perturbative treatment if the unperturbed system has only a continuum spectrum.

The three-body systems can be described in the Fadeev picture [4-6] revealing the existence of a large number of spatially extended bound states in the vicinity of the threshold of breakup into three particles (Efimov effect) [7-11] provided the two-body forces are of a resonant nature.

As was pointed out by Fedorov *et al.* [12,13], the Schrödinger picture is much simpler than the Fadeev one but is unable to describe all the three two-body subsystems on an equal basis (the description of one almost-bound two-body state, expressed in terms of another set of Jacobi coordinates, requires a large number of angular momentum components). Nevertheless, we are interested in problems for which one particle has a much larger mass than the other two (we have in mind two neutrons and a core larger than  $^9\text{Li}$ ). In this case we have to treat two interacting particles (neutrons) within the core potential. This special case of the three-body problem was studied in the Schrödinger picture by Migdal [14] without obtaining spatially extended bound states.

On the other hand the Efimov effect was demonstrated in a model of two heavy particles and a light one when the light-heavy interaction leads to zero-energy two-body bound state [15-17] (the Born-Oppenheimer approximation was used).

As the resonant character of the light-light interaction is essential for the Efimov effect in a system of two light and one heavy particle [8] and the Schrödinger picture is very transparent, the goal of the present work is to describe the bound states of two interacting particles which are close to zero energy in the presence of a core. For such loosely bound states one can take into account the presence of the core by its scattering properties at zero energy, *e.g.* the particle-core scattering length for *s*-wave. In the same way the particle-particle interaction will be described by the particle-particle scattering length. The neutron-neutron scattering length is  $a=-18.5\text{ fm}$  [18] (compare with the neutron-proton singlet scattering length  $-23.7\text{ fm}$  which corresponds to a  $\sim 0.07\text{ MeV}$  virtual state [19]).

---

\* Member of the Romanian Academy

In a previous work [20] we used the hyperspherical coordinates (see *e.g.* [21]) to include in the angular equation both the particle-core interaction and the particle-particle one in an approximate way appropriate for the zero energy region.

In the present work we will use the Boundary Conditions Method and a simple parametrization for the wave function of the three body system and we obtain relations between the parameters of the borromean bound state in the three-body system (energy and spatial extension) and the two-body scattering lengths.

## 2. ON CONTACT INTERACTIONS AND THEIR DESCRIPTION IN TERMS OF BOUNDARY CONDITIONS

Let us consider a low energy level  $E = -|E_0|$  in a potential well  $U$ , obeying condition  $E_0/U \rightarrow 0$ . It is well known, *e.g.* [22], that the level's position in a potential well of magnitude  $U$  and radius  $R$ , does depend on the product  $UR^2$ . In this respect the level's energy is not changed in the limit  $U \rightarrow \infty$  and  $R \rightarrow 0$ , maintaining the condition  $UR^2 = \text{constant}$ . This way one can study the level's properties by means of contact singular potentials ( $U \rightarrow \infty, R \rightarrow 0$ ). Such singular contact potential is described by a point-like interaction,  $B\delta(\vec{r})$ , (Zero Range Potential Model). In the following we will relate the study of a Contact Potential to a Boundary Condition Method; this method was developed mainly by Soviet Scientists, *e.g.* [22-25].

Consider now the matching of the wave function inside potential well,  $\chi_1 = (r\Psi)_1$  to that of outside potential,  $\chi_2 = (r\Psi)_2$ ,  $\chi_2 \sim \exp(-ar)$ , at the radius  $R$ ,  $(d\ln\chi/dr)|_R = -a$ . As the level's energy does not change in the above limit,  $U \rightarrow \infty, R \rightarrow 0$ , this procedure should not result into a modification of the above boundary condition. The states in the deep potential well could be described in terms of logarithmic derivative  $d\ln\chi/dr$ , calculated at origin,  $R \rightarrow 0$ . The description in terms of the logarithmic derivative appears to be a general property of Schrödinger equation.

The basic idea of Contact or Zero Range Potential Models is to replace the solution of the Schrödinger equation inside the potential well by Boundary Conditions on the Wave Function at center of the potential well

$$d/dr(r\Psi)|_{R=0} = -a(r\Psi)|_{R=0}$$

For a bound state  $\chi = r\Psi \sim \exp(-ar)$ ,  $\alpha = \sqrt{-2mE/\hbar^2}$ , one obtains  $E = -\hbar^2\alpha^2/2m$ . For a scattering state with energy  $E = \hbar^2k^2/2m$ , the wave function is given by asymptotic boundary condition,

$$\Psi = \exp(i\vec{k}\vec{r}) + f\exp(ikr)/r$$

The boundary condition does relate the scattering amplitude,  $f$ , to the  $\alpha$  parameter,

$$f = -1/(\alpha + ik)$$

One obtains that the negative of zero-energy scattering amplitude is related to logarithmic derivative

$$a = -f(0) = 1/\alpha$$

The quantity  $a = -1/(d\ln\chi/dr)|_{R=0}$  is called Scattering Length; for a bound state it is positive,  $a > 0$ , while for a virtual state it is negative,  $a < 0$ . The boundary condition for both cases is  $d\chi/dr|_0 = -1/a\chi|_0$ .

The scattering amplitude, in Born Approximation, is the Fourier Transform of the interaction potential. For a contact potential,  $U(r) = B\delta(\vec{r})$ , the scattering amplitude becomes

$$f = -a = -mB/2\pi\hbar^2$$

resulting in a relation connecting the potential strength and scattering length,  $B \sim a$ .

The Schrödinger equation's Zero-Range Potential becomes now,  $U(\vec{r}) = \frac{2mV(r)}{\hbar^2} = 4\pi a\delta^3(\vec{r})$  and it is known as Fermi's Pseudopotential, [26]. The Fermi Pseudopotential is repulsive for a nuclear medium described by a positive scattering length. The Fermi Pseudopotential is attractive for a nuclear medium with a negative scattering length. In concrete terms a slow neutron is attracted by a virtual state, ( $a < 0$ ), and it is repelled by a bound state, ( $a > 0$ ). These physical results were used in confining ultra-cold neutrons, *e.g.* [26]. For an ultracold energy less than Fermi's Pseudopotential strength the neutrons are totally reflected by material walls, provided neutron-material's nucleus interaction is given by a positive scattering length.

This idea can be extrapolated to the case of a neutron outside a nuclear medium with a negative scattering length. A neutron is attracted by a nucleus provided their (interaction) scattering length is negative, *i.e.* the neutron and nucleus (core) do form a virtual state. However this (virtual) system's state is unstable and could manifest itself only via a scattering process. Now consider that another neutron does approach this (virtual state) system formed by the core-nucleus and initial neutron. The additional neutron and core-nucleus is another virtual-state system. Also the two-neutrons system is a virtual state one because  $n$ - $n$  scattering length is negative. Let us approach this problem of Particle "Scattering" on two "Centers" (either neutron "scattering" on system of core plus initial neutron, or the core-nucleus "scattering" on two neutrons system), in terms of the Fermi Zero Range Potential Model, *i.e.* in terms of Boundary Conditions Method.

The Boundary Conditions Method was generalized, [27], to many centers scattering problems; one introduces now, boundary conditions corresponding to every non-overlapping of centers. The Boundary Condition Method describing the "scattering" of a neutron on a Core-nucleus (1) and another neutron (2), are

$$d/d\rho_1(\rho_1\Psi)|_{\rho_1=0} = -(1/a_1)(\rho_1\Psi)|_{\rho_1=0}$$

$$d/d\rho_2(\rho_2\Psi)|_{\rho_2=0} = -(1/a_2)(\rho_2\Psi)|_{\rho_2=0}$$

where  $\rho_i = r - r_i$ , ( $i=1,2$ ). In absence of second "scatterer" (well) the wave function outside the first scatterer well is  $\Psi_1 = C_1 \exp(-\alpha\rho_1)/\rho_1$ , where  $\alpha$  it would have significance of  $1/a_1$ . In a similar way, outside the second well and in absence of the first one, the wave function would be  $\Psi_2 = C_2 \exp(-\alpha\rho_2)/\rho_2$ , now with  $\alpha \sim 1/a_2$  significance. If the both centers are present, then

$$\Psi = C_1 \exp(-\alpha\rho_1)/\rho_1 + C_2 \exp(-\alpha\rho_2)/\rho_2$$

where  $\alpha$  has to be determined by procedures of the Boundary Conditions Method. By using this wave function  $\Psi$  in above Boundary Conditions Equations, one obtains a set of two equations for the constants  $C_1$  and  $C_2$ ,

$$(\alpha - 1/a_1) \cdot C_1 + \exp(-\alpha R)/R \cdot C_2 = 0$$

$$\exp(-\alpha R)/R \cdot C_1 + (\alpha - 1/a_2) \cdot C_2 = 0$$

with  $R = |\vec{r}_1 - \vec{r}_2|$  as a distance parameter for the two scattering centers. The solution of this set of two equations implies the determinant condition

$$(\alpha - 1/a_1)(\alpha - 1/a_2) - \exp(-2\alpha R)/R^2 = 0$$

The energies of the particle in the fields of two centers are  $E_i = -\hbar^2\alpha_i^2(R)/2m$ , where  $\alpha_1$  and  $\alpha_2$  are roots of the above equation and  $m$  is particle (reduced) mass.

Now consider a near zero energy state, *i.e.*  $\alpha \rightarrow 0$ . In this case one obtains  $\exp(-2\alpha R)/R^2 \approx 1/R^2 - 2\alpha/R + 2\alpha^2$ , ( $|\alpha R| \ll 1$ ), and the equation becomes

$$\alpha^2 + (1/a_1 + 1/a_2 - 2/R)\alpha + (1/R^2 - 1/a_1 a_2) = 0$$

resulting for

$$2\alpha = -(1/a_1 + 1/a_2 - 2/R) \pm \sqrt{(1/a_1 + 1/a_2)^2 + 4/a_1 a_2 - 4(1/a_1 + 1/a_2)/R}$$

Consider now a model case with  $|a_1| = |a_2| = a$ . If  $a_1 = a_2 = a > 0$  then  $\alpha$  is complex for  $a > R$ . If  $a_1 = -a_2 = a > 0$  (or  $-a_1 = a_2 = a > 0$ ) the solution is a complex number too. The two repulsive potentials and the one repulsive - one attractive potential result into no bound state solutions, (otherwise  $\alpha$  should be real). In second case one obtains a virtual state of the additional particle, provided  $a > R$ ,

$$\alpha = 1/R \pm i/a$$

Consider now the case of two virtual states  $a_1 = a_2 = -a$ , ( $a > 0$ ); one obtains two real solutions

$$\alpha = 1/a + 1/R \pm \sqrt{2/a(1/a + 1/R)}$$

The first solution (+) should be rejected because it is not compatible with initial constraint,  $\alpha R \ll 1$ . The second solution (-) has to be accepted because always  $1 + 2a/R \gg 0$  (the scattering length is much larger than internuclear distances). It does correspond to a real negative energy  $E_i = -\hbar^2 \alpha^2 / 2m$ ; for the two scatterers the total energy is  $E_i = -\hbar^2 \alpha^2 / m$ .

Now we have to recover the initial meaning of the  $\alpha$  parameter (as  $1/\text{scattering length}$ ). The positive  $\alpha$  does correspond to a bound state, in agreement with above evaluation of energy. Let us have a look on the two other cases,  $a = R$ ,  $a < R$ , in spite they contradict the initial constraint on a  $R$ . For  $a = R$  one obtains for  $\alpha$  a positive solution ( $\alpha = 4/a = 4/R$ ) and also a zero one, ( $\alpha = 0$ ), corresponding to a bound state and to a zero-energy "scattering" state. For  $a < R$  one obtains for  $\alpha$  both positive and negative solutions corresponding to a bound state and to a virtual one. However bound state solutions for  $a = R$  and  $a < R$  are not compatible with initial constraint  $\alpha R \ll 1$  and must be disregarded.

Let us remark that in this model case the initial transcendental equation may be written

$$\alpha R + \frac{R}{a} = \pm \exp(-\alpha R)$$

and the above equations can be considered as equations for  $x = \alpha R$ . These equations have only one real solution,  $x$ , which is positive when the intercenter distance is small,  $R < a$ , and negative otherwise. In the limit  $R \rightarrow 0$  the solution  $x$  satisfies the equation  $x = \exp(-x)$  which has a numerical solution  $x = 0.567$ . Therefore, the parameter  $\alpha$  behaves as  $0.567/R$  for small separation distance,  $R$  (see also [16]). If the solution is small,  $\alpha R \ll 1$ , one obtains  $\exp(-\alpha R) \cong 1 - \alpha R$  and the solution can be approximated by

$$\alpha R = \frac{1}{2} \left( 1 - \frac{R}{a} \right)$$

which results into  $x = \alpha R = 0.5$  in the limit  $R \rightarrow 0$ ; this approximation is only slightly different from the numerical solution of the initial equation  $x = 0.567$ . This behaviour for small  $R$  means a dipolar attractive interaction between the two centers and this fact can cause, for sufficiently large mass of the centers, a Thomas effect (the existence of states with arbitrary large binding energy). Nevertheless, the inclusion of the term related to the finite interaction range of the interaction,  $r_{eff}$ , in the boundary conditions modifies this behaviour for small  $R$  and a Thomas effect will be not present anymore (this inclusion might be expressed in the form

$$d/d\rho(\rho\Psi)|_{\rho=0} = -(1/a)(\rho\Psi)|_{\rho=0} - 1/2r_{eff} d^2/d\rho^2(\rho\Psi)|_{\rho=0}$$

In the previous approach the two centers were considered to be the two neutrons, (equal scattering lengths core-neutrons); the internuclear distance  $R$  was that between the two neutrons. Perhaps it is questionable if  $R$  could be treated as a parameter, because of neutron small masses. Another partition of the three-body system is core-neutron plus the additional neutron; the two scattering lengths are that of neutron-core and neutron-neutron. Now the internuclear distance  $R$  is that of neutron-core.

The evaluation of the energy,  $(1/\alpha)$ , of the borromean nucleus needs internuclear distance  $R$  and the two scattering lengths,  $a_1$  and  $a_2$ . One can adopt also the inverse procedure, to calculate the internuclear distances  $R_{nn}$  (neutron-neutron) and  $R_{nC}$  (neutron-Core) if we know the energy and the two scattering lengths. The neutron-neutron scattering length is  $a_{nn} = -18.5 fm$ , [18], and  $n-^9Li$  scattering length (for  $^{11}Li$  borromean nucleus) is  $a_{nC} < -20 fm$ , [28]. The  $2n$  bound neutrons separation energy in  $^{11}Li$  borromean nucleus is  $E_0 = -300 keV$ ; the corresponding  $\alpha$  parameter is  $\alpha = 0.5810 \exp(12) cm^{-1}$ . For two neutrons scatterers one obtains, from the determinant equation,  $R_{nn} = 6.7 fm$ . For neutron-Core nucleus as scatterers one obtains  $R_{nC} = 7.9 fm$ . One can easily verify, in both cases, that  $\alpha R \approx 0.4 - 0.5$ ,  $(\alpha R)^2 / 2! \approx 0.1$  and  $(\alpha R)^3 / 3! \approx 0.01$  and the neglecting of last or higher order terms is justified. In this schematic model the two neutrons are basis corners and Core nucleus is top-corner of an isosceles triangle. A physical conclusion is the two neutrons are located far-away from each other; they do not take form of a dineutron. Indeed, the deuteron 'radius' for both bound and virtual states is  $2 - 2.5 fm$ , e.g. [22], while the  $n-n$  distance in halo is  $\sim 6.7 fm$ . It is interesting to compare the above results for halo's radius to those reported in literature, [31, 32]. The reported distance of  $^9Li$  Core to the mass center of the two neutrons is  $\sim 8 fm$ ; according to present calculations this value is  $\sqrt{R_{nC}^2 - R_{nn}^2} / 4 = 7.15 fm$ .

### 3. CONCLUSIONS

The results obtained allow us to conclude that one can use the Fermi contact pseudopotentials (Boundary Conditions Method) to describe the weakly bound states in a three-body system. This method relates the properties of the two-body interactions to the properties of the three-body bound states: spatial extension and energy. To illustrate the method we have compared our results with the data available for the borromean nucleus  $^{11}Li$ .

The method could be used for other borromean nuclei whose loosely bound neutrons are expected to be in the  $s$ -state, like  $^{14}Be$ . Moreover, the extension for studying of a neutron-core resonance in  $p$ -wave close to zero energy is straightforward (this might be necessary for borromean nuclei in which the neutrons are expected to be in a  $p$ -state, e.g.  $^6He$ ).

### REFERENCES

1. A.S. Jensen, K. Riisager, D.V. Fedorov - Rev. Mod. Phys. **76**, p. 215, 2004
2. K. Riisager - Rev. Mod. Phys. **66**, p. 1105, 1994
3. M.V. Zhukov, B.V. Danilin, D.V. Fedorov, J.M. Bang, I.J. Thompson, J.S. Vaagen - Phys. Rep. **231**, p. 151, 1993
4. L.D. Faddeev - Sov. Phys. JETP **12**, p. 151, 1961
5. A.G. Sitenko - *Lectures in Scattering Theory*, Pergamon Press, Oxford, 1971
6. A.G. Sitenko - *Theory of Nuclear Reactions*, World Scientific, Singapore, 1990
7. V. Efimov - Phys. Lett. **33B**, p. 563, 1970
8. V. Efimov - Nucl. Phys. **A210**, p. 157, 1973
9. V. Efimov - Phys.Rev. **C47**, p. 1876, 1993
10. D.V. Fedorov, A.S. Jensen - Phys. Rev. Lett. **71**, p. 4103, 1993
11. D.V. Fedorov, A.S. Jensen, K. Riisager - Phys. Rev. Lett. **73**, p. 2817, 1994
12. D.V. Fedorov, A.S. Jensen, K. Riisager - Phys. Rev. **C49**, p. 201, 1994
13. D.V. Fedorov, A.S. Jensen, K. Riisager - Phys. Rev. **C50**, p. 2372, 1994
14. A.B. Migdal - Sov. J. Nucl. Phys. **16**, p. 238, 1973
15. A.C. Fonseca, E.F. Redish, P.E. Shanley - Nucl. Phys. **A320**, p. 273, 1979
16. B.G. Giraud, Y. Hahn - Nucl. Phys. **A588**, p. 653, 1995

17. A. Estrin, M. Koštrun, Y. Hahn - Phys. Rev. **C57**, p. 50, 1998
18. C.R. Howell, Q. Chen, T.S. Carman, A. Hussein, W.R. Gibbs, B.F. Gibson, G. Mertens, C.F. Moore, C. Morris, A. Obst, E. Pasyuk, C.D. Roper, F. Salinas, I. Slaus, S. Sterbenz, W. Tornow, R.L. Walter, C.R. Whiteley, M. Whitton - Phys. Lett. **B444**, p. 252, 1998
19. L. Landau, E. Lifshitz – *Mécanique Quantique; Théorie Nonrelativiste*, Mir, Moscou, 1966
20. R.A. Ionescu, C. Hategan - Europhys. Lett. **45**, p. 296, 1999
21. C.D. Lin - Phys. Rep. **257**, p. 1, 1995
22. A.I. Baz, I.B. Zeldovich, A.M. Perelomov - *Rasseianie, Reaktsii i Raspady v Nerelativistskoi Kvantovoi Mekhanike*, Nauka, Moskva, 1971
23. G.F. Drukarev - *Stolknovenia Elektronov s Atomami i Molekulami*, Nauka, Moskva, 1978
24. Y.N. Demkov, V.N. Ostrovskii - *Metod Potentsialov Nulego Radiusa v Atomnoi Fizike*, Izd. Leningradskogo Universiteta, 1975
25. G.F. Drukarev - Adv. Quantum Chemistry **11**, p. 251, 1978
26. J. Byrne - *Neutrons, Nuclei and Matter*, IOP Publishing, Bristol, 1995
27. O. Firsov, B. Smirnov - J.E.T.P. **47**, p. 232, 1964
28. M. Thoennessen, S. Yokoyama, A. Azhari, T. Baumann, J.A. Brown, A. Galonsky, P.G. Hansen, J.H. Kelley, R.A. Kryger, E. Ramakrishnan, P. Thirolf Phys. Rev. **C59**, p. 111, 1999
29. M. Thoennessen, S. Yokoyama, P.G. Hansen - Phys. Rev. **C63**, 014308, 2000
30. M. Labiche, N.A. Orr, F.M. Marques, J.C. Angelique, L. Axelsson, B. Benoit, U.C. Bergmann, M.J.G. Borge, W.N. Catford, S.P.G. Chappell, N.M. Clarke, G. Costa, N. Curtis, A. D'Arrigo, E.D. Brennand, O. Dorvaux, G. Fazio, M. Freer, B.R. Fulton, G. Giardina, S. Grevy, D. Guillemaud-Mueller, F. Hanappe, B. Heusch, K.L. Jones, B. Jonson, C. Le Brun, S. Leenhardt, M. Lewitowicz, M.J. Lopez, K. Markenroth, A.C. Mueller, T. Nilsson, A. Ninane, G. Nyman, F. de Oliveira, I. Piqueras, K. Riisager, M.G. Saint Laurent, F. Sarazin, S.M. Singer, O. Sorlin, L. Stuttge - Phys. Rev. Lett. **86**, p. 600, 2001
31. T. Frederico, M.S. Hussein - Mod. Phys. Lett. **A8**, p. 295, 1993
32. F.M. Marqués, M. Labiche, J.C. Angelique, L. Axelsson, B. Benoit, U.C. Bergmann, M.J.G. Borge, W.N. Catford, S.P.G. Chappell, N.M. Clarke, G. Costa, N. Curtis, A. D'Arrigo, F.D. Santos, E.D. Brennand, O. Dorvaux, M. Freer, B.R. Fulton, G. Giardina, C. Gregori, S. Grevy, D. Guillemaud-Mueller,
33. F. Hanappe, B. Heusch, B. Jonson, C. Le Brun, S. Leenhardt, M. Lewitowicz,
34. M.J. Lopez, K. Markenroth, M. Motta, A.C. Mueller, T. Nilsson, A. Ninane, G. Nyman, I. Piqueras, K. Riisager, M.G. Saint Laurent, F. Sarazin, S.M. Singer, O. Sorlin, L. Stuttge - Phys. Lett. **B476**, p. 219, 2000; F.M. Marqués, M. Labiche, N.A. Orr, J.C. Angelique, L. Axelsson, B. Benoit, U.C. Bergmann, M.J.G. Borge, W.N. Catford, S.P.G. Chappell, N.M. Clarke, G. Costa, N. Curtis, A. D'Arrigo, E.D. Brennand, F.D. Santos, O. Dorvaux, G. Fazio, M. Freer, B.R. Fulton, G. Giardina, S. Grevy, D. Guillemaud-Mueller, F. Hanappe, B. Heusch, Jonson, C. Le Brun, S. Leenhardt, M. Lewitowicz, M.J. Lopez, K. Markenroth, A.C. Mueller, T. Nilsson, A. Ninane, G. Nyman, I. Piqueras, K. Riisager, M.G. Saint Laurent, F. Sarazin, S.M. Singer, O. Sorlin, L. Stuttge - Phys. Rev. **C64**, 061301, 2001

Received May 6, 2005