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A DIRECT METHOD FOR THE CALCULATION OF ASTRONOMICAL REFRACTION

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The astronomical refraction integral is non-convergent for the zenith distance of 90°. The refraction determination method in this paper calculates refraction interatively, on the basis of an atmosphere model, through a direct application of refraction laws. The advantage of this method is that it can be also applied to the zenith distance of 90°. The astronomical refraction is calculated through this method for the case in which the Earth is considered a sphere or a revolution ellipsoid. The same refraction values are obtained for the two models for zenith distances under 70° and slightly different values for the neighbourhood of the horizon. In the neighbourhood of the horizon, for latitudes greater than or equal with 45°, the refraction values calculated in the case of the ellipsoidal model are greater than those for the spherical model.

Key words: astronomical refraction, refraction computation at the horizon.

1. INTRODUCTION

The refraction integral is deduced by means of refraction laws and of some geometric considerations. Generally, its calculation is made approximately through series development for small zenith distances and through numerical integration for large zenith distances. In keeping with the astronomical refraction study throughout time, the refraction integral evaluation method depends on the zenith distance values, namely z between 0° and 74°, z between 74° and 84°, z between 84° and 90°.

In the first case the Laplace-Oriani theorem ensures the integral evaluation through a series development with two terms, without making any hypothesis on the atmosphere, except that it is subjected to Mariotte and Gay Lussac's laws and that the temperature decreases in height. Thus, the atmosphere can be considered as being made up of plane parallel or concentric spherical layers (Laplace, Radau, Andoyer, Danjon et al. [2]), or concentric ellipsoidal [6].

In the second case, the refraction integral series is prolonged with three more terms, and a knowledge of the atmosphere structure is required, at least up to the altitude of 80 km, because the densities of the upper layers of this height become very small and the refraction influence negligibly. In this case various hypotheses are made on density variation with height (Newton, Danjon, Blajko etc.), or on temperature variation (Ivory et al.).

For the determination of refraction at zenith distances larger than 84°, the refraction series development can no longer be used, because the series is no longer convergent in the neighbourhood of 90°. However, the refraction integral evaluation is made using also a hypothesis on the atmosphere structure. Numerous attempts have been made to this effect. We mention in the first place the hypothesis on density variation with height, on which the theories given by Bouguer, Laplace, Bessel [2] were based. The theories given by Newton, Bauernfeind, Gylden, Schmidt [2] used the relation between temperature and height. Ivory, Kowalski, Oppolzer, Radau founded their theories on refraction on the relationship between density and temperature, while Lubbok and Pizzeti based their theories on the relationship between temperature and pressure.

There were also other ways in which the refraction issue has been approached. Harzer [4] is the first astronomer who deduced the formulas for refraction calculation using exclusively the meteorological observations available, made both on the Earth's surface, as well as at various heights, the number of these observations has increased very much, their accuracy improving as well.

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Another way of refraction evaluation, very much used at present, is the numerical integration method. As far as the vecinity of the horizon is concerned, the knowledge of air density depending on height is of capital importance. F. Link approached this method in 1934, 1937 and then in 1957 [5], calculating the tables of refraction values for average latitudes. We would also like to mention that in a study dedicated to refraction in the neighbourhood of horizon A.T. Young [10] underlines the importance of approaching refraction calculation in this case by means of numerical integration without resorting to series developments.

I.S. Guseva [3], on the other hand, proposes the calculation of the refraction integral in the horizon neighbourhood by numerical integration from a convenient height. She calculates the integral analitically, from the ground up to that height, introducing the variation of the refraction index depending on the distance from the atmosphere model center to the air layer of that refraction index.

The direct method, developed in this paper, determines the astronomical refraction by dividing the Earth's atmosphere in very thin concentric layers. Starting from the data in the observation point, we calculate refraction iteratively, in the opposite direction of light propagation, by means of refraction laws and of an atmosphere model. Consequently, this method avoids the refraction integral, which is non-convergent at the horizon of the place.

2. METHOD PRESENTATION

We deal first with the case of spherical Earth. Refraction takes place in the vertical plane of the celestial body. Because the phenomenon takes place in the same way in any vertical plane, for simplicity's sake we limit ourselves to the vertical plane which coincides with the meridian plane of the place. Let *Oxy* be a system of coordinates in this plane, *O* being the center of the Earth, *Ox* the intersection of the equatorial plane with the meridian plane of the place, and *Oy* the rotation axis of the Earth. In the observation point P_0 we know the zenith distance observed z_0 , as well as the place latitude φ_0. In this system, the point P_0 will have the following coordinates

$$\begin{aligned} x_0 &= R_0 \cos \varphi_0, \\ y_0 &= R_0 \sin \varphi_0, \end{aligned} \tag{1}$$

where R_0 is the Earth's radius.

Depending on the zenith distance, a division of the light ray (taken in the opposite direction of propagation) can be established, the lengths of the division intervals increasing towards the superior limit of the atmosphere. Thus, the trajectory of the light ray through the atmosphere is approximated with a polygonal line. When the norm of the division considered tends to zero, the real trajectory of the light ray is obtained.

The method determines the refraction angle and the incidence angle in every node of the polygonal line, by means of the two refraction laws. Let P_0, P_1, P_2, …, P_m be the nodes of the polygonal line, P_0 – the observation place and P_m – the entrance point in the atmosphere of the ray light.

The coordinates of point P_1 in the system *Oxy* are easily determined, if the length P_0P_1 (chosen depending on the zenith distance) is known and they have the following expressions

$$\begin{aligned} x_1 &= x_0 + P_0P_1 \cos(\varphi_0 - z_0), \\ y_1 &= y_0 + P_0P_1 \sin(\varphi_0 - z_0). \end{aligned} \tag{2}$$

The atmospheric layer in which refraction is studied is divided in *m* concentric layers which pass through the nodes P_1, P_2, …, P_m. Thus, the circle which passes through P_1 is of ray $R_0 + h_1$, where

$$h_1 = \sqrt{x_1^2 + y_1^2} - R_0. \tag{3}$$

From (2) the latitude of point P_1 is deduced immediately, namely

$$\varphi_1 = \arctan \frac{y_1}{x_1}. \tag{4}$$

The refraction angle of P_1 results from the triangle P_0P_1O (Fig. 1),

$$r_1 = z_0 + \varphi_1 - \varphi_0. \tag{5}$$

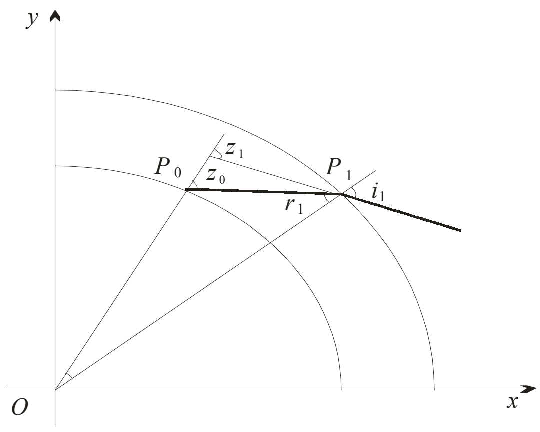

Fig. 1

By means of an atmosphere model (for example TSA-60 [1] or US Standard Atmosphere 1968 [9]) the air density at the height h_1 can be determined and depending on it the corresponding refraction index is established.

From the second refraction law the incidence angle in P_1 can be deduced,

$$i_1 = \arcsin\left(\frac{n_0}{n_1}\sin r_1\right) \tag{6}$$

Thus, with relations (5) and (6) the astronomical refraction in P_1 is

$$\Delta R_1 = i_1 - r_1 \tag{7}$$

and astronomical refraction in P_0 is

$$R = z_1 - z_0, \tag{8}$$

where $z_1 = i_1 + \varphi_0 - \varphi_1$.

Having the data at height h_1 in point P_1 of latitude φ_1, the data for P_2, with the length P_1P_2 given by the chosen division , can be deduced similarly. If the refraction in P_k is

$$\Delta R_k = i_k - r_k, \tag{9}$$

then the astronomical refraction is given by the formula

$$R = \sum_{k=1}^{m} \Delta R_k \tag{10}$$

or by the formula

$$R = z_m - z_0 \tag{11}$$

where $z_m = i_m + \varphi_0 - \varphi_m$.

In the case of the ellipsoidal Earth we use an analogous procedure for the light ray which is situated in the meridian plane of the place. Depending on the geodetic latitude φ_0, the point P_0 has the following coordinates

$$x_0 = \frac{a_0 \cos\varphi_0}{\sqrt{1-e^2 \sin^2 \varphi_0}} \quad ; \quad y_0 = \frac{a_0\left(1-e^2\right)\sin\varphi_0}{\sqrt{1-e^2 \sin^2 \varphi_0}}, \tag{12}$$

where a_0 and e are the great semi-axis and the eccentricity of the meridian ellipse which passes through P_0. Point P_1 situated on the polygonal line which approximates the light ray will have the following coordinates

$$x_1 = x_0 + P_0P_1 \cos\left(\varphi_0 - z_0\right) \quad ; \quad y_1 = y_0 + P_0P_1 \sin\left(\varphi_0 - z_0\right), \tag{13}$$

where the length P_0P_1 is chosen depending on the zenith distance z_0. The atmosphere being divided in m concentric ellipsoidal layers, we have to determine the height of each layer. Knowing the coordinates of point P_1, we get

$$h_1 = \sqrt{x_1^2 + \frac{y_1^2}{1-e^2}} - a_0. \tag{14}$$

From the atmosphere model chosen we determine the air density δ_1 depending on h_1 and adequately n_1. The latitude of point P_1 is found out through

$$\varphi_1 = \arctan\left(\frac{1}{1-e^2}\frac{y_1}{x_1}\right) \tag{15}$$

The refraction angle in P_1 is obtained from the quadrangle $P_0P_1A_1A_0$ (Fig. 2),

$$r_1 = z_0 + \varphi_1 - \varphi_0 \tag{16}$$

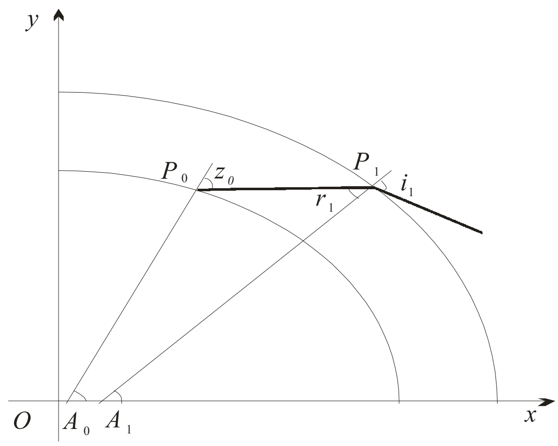

Fig. 2

The procedure continues identically as with the spherical case. If $e = 0$ and $a_0 = R_0$, the formulas of the ellipsoidal case change into the formulas of the spherical case.

3. RESULTS

The astronomical refraction determination method presented above has been applied in the following three situations:

1. The spherical model with concentric spherical layers, in which the Earth is considered to be a sphere with a radius of $R_0 = 6371.21\,\text{km}$, and the atmospheric layer significant for refraction is divided in m concentric spherical layers.

2. The spherical model with concentric spherical layers, with the sphere radius equal with the radius of curvature of the terrestrial ellipsoid in the point considered, $r_c = a_0\sqrt{1 - e^2}/(1 - e^2 \sin^2 \varphi_0)$, where a_0 is the great semi-axis of the ellipsoid, equal with 6378.14 km, e the ellipsoid eccentricity, equal with 0.081819772, φ_0 the geodetic latitude of the point considered. The atmospherical layer is divided in m concentric spherical layers.

3. The ellipsoidal model with concentric ellipsoidal layers, in which the terrestrial ellipsoid has the great semi-axis $a_0 = 6378.14$ km, the ellipsoid eccentricity $e = 0.081819772$, and the atmospherical layer is divided in m homothetic ellipsoidal layers.

In all three models, the maximum height of the atmosphere has been considered to be of 80 km, this being enough for the calculation of astronomical refraction with an error of $0''.005$. The atmosphere models TSA-60 [1] and US Standard Atmosphere 1976 [9] give approximately equal final results. The results presented in this paper are calculated on the basis of the model TSA-60.

The atmosphere refraction index on the ground has been obtained by means of Owens's formulas [7], in the standard conditions of temperature 15° C, pressure 760 mmHg, dry air, light wave length 0.593 mμ. The refraction index of the atmospheric layer k has been calculated for simplicity's sake with Laplace's law

$$n_k = \sqrt{1 + 2c\delta_k}, \tag{17}$$

δ_k being the density of the layer considered in the chosen model. The constant $c = 0.226142335$ is given by $n_0 = 1.000276986$ and $\delta_0 = 0.001225\ g \cdot cm^{-3}$ [1].

The method presented above is based on the dividing of the light ray trajectory through the atmosphere. The calculations show that by making the norm the division smaller, the results remain pratically unchanged. The division choice depends on the zenith distance observed.The length of a segment of the polygonal line, which approximates the ray of refracted light ranges between 10 m and 1,000 m for zenith distances observed under 85°, and between 12 m and 6,000 m for zenith distances observed above 85°.

Tables 1, 2 and 3 contain the values of the astronomical refractions at latitudes of 30°, 45° and 60° depending on the zenith distance observed z. We have marked with R_{sf} the astronomical refraction calculated through the direct method for the concentric spherical model, and with R_{rc} the astronomical refraction calculated for the concentric spherical model 2, and with R_{el} the astronomical refraction for the concentric elipsoidal model 3. In columns R_P we give the values of the astronomical refractions obtained from the Tables of Pulkovo Observatory 1985 [8]. Table 2 contains also the values of the astronomical refractions determined by Guseva [3], R_G, calculated also by means of the radius of curvature r_c.

It can be noticed that the values R_{sf}, R_{rc} and R_{el} have the same values for zenith distances under 70°, but that after that zenith distance, the values R_{rc} are between R_{sf} and R_{el}. The values R_P and R_G are slightly greater as against the similar model R_{rc}.

The refraction differences for the three models for which the direct method has been applied become important at zenith distances greater as 80°. On the other hand, the values R_{el} are smaller than the corresponding values R_{el} of the same ellipsoidal model, but calculated by means of the refraction integral depending on the radius of curvature [6].

Table 1 - Values of the astronomical refraction for φ_0 = 30°

z	R_{sf}	R_{rc}	R_{el}	R_P
60.0	98.47	98.47	98.47	98.52
70.0	155.38	155.38	155.38	155.47
80.0	312.33	312.32	312.31	312.49
85.0	577.89	577.85	577.75	578.19
86.0	687.26	687.20	687.04	687.62
87.0	840.02	839.92	839.64	840.45
88.0	1063.17	1062.99	1062.48	1063.68
89.0	1407.06	1406.72	1405.71	1407.60
89.5	1652.09	1651.62	1650.15	1652.80
90.0	1967.73	1967.04	1964.87	1973.78

Table 2 - Values of the astronomical refraction for $\varphi_0 = 45°$

z	R_{sf}	R_{rc}	R_{el}	R_P	R_G
60.0	98.47	98.47	98.47	98.52	98.51
70.0	155.38	155.39	155.39	155.47	155.46
80.0	312.33	312.34	312.35	312.53	312.48
85.0	577.89	577.96	577.97	578.39	578.31
86.0	687.26	687.37	687.38	687.94	
87.0	840.02	840.20	840.20	840.97	
88.0	1063.17	1063.49	1063.47	1064.61	1064.4
89.0	1407.06	1407.66	1407.59	1409.42	1409.2
89.5	1652.09	1652.95	1652.80	1655.43	
90.0	1967.73	1968.98	1968.70	1977.97	1977.6

Table 3 - Values of the astronomical refraction for $\varphi_0 = 60°$

z	R_{sf}	R_{rc}	R_{el}	R_P
60.0	98.47	98.47	98.47	98.52
70.0	155.38	155.39	155.39	155.47
80.0	312.33	312.36	312.39	312.56
85.0	577.89	578.07	578.18	578.57
86.0	687.26	687.54	687.72	688.22
87.0	840.02	840.48	840.77	841.45
88.0	1063.17	1063.99	1064.48	1065.49
89.0	1407.06	1408.61	1409.49	1411.12
89.5	1652.09	1654.30	1655.51	1657.87
90.0	1967.73	1970.92	1972.61	1981.57

In Fig. 3 the difference $R_{el} - R_{sf}$ is represented for the latitudes of 30°, 45° and 60°, depending on the zenith distance z, which ranges between 80° and 90°.

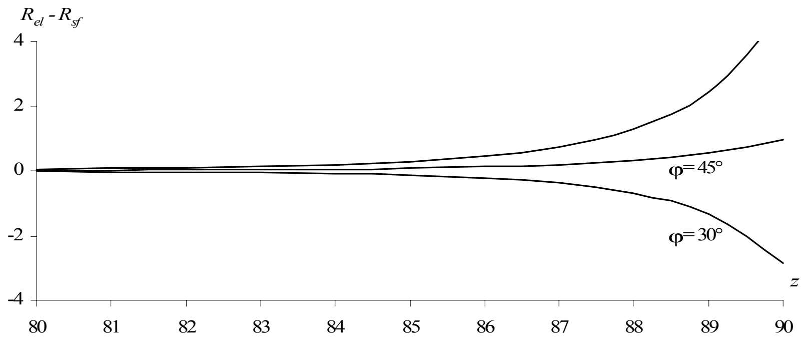

Fig. 3 – The difference of the refraction values for the ellipsoidal model and for the spherical one in the neighbourhood of the horizon

In Fig. 4 the diagram of the difference $R_P - R_{rc}$ is represented, in arcsec, for the latitude of 45°, depending on the zenith distance z, which ranges between 80° and 90°.

In Fig. 5 the diagram of the difference $R_G - R_{rc}$ is represented, in arcsec, for the latitude of 45°, depending on the zenith distance z, which ranges between 80° and 90°, R_G being calculated using also the radius of curvature.

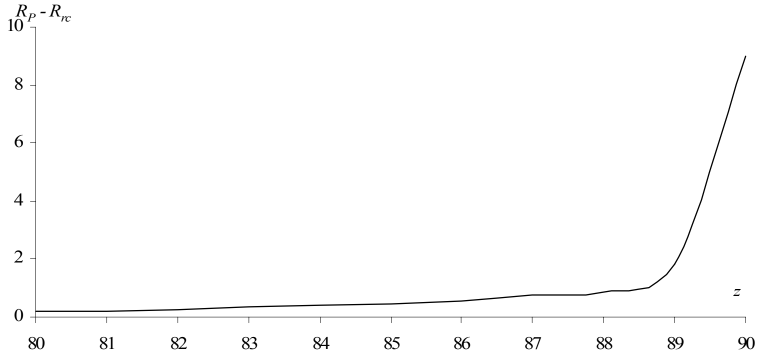

Fig. 4 – The difference of the refraction values given by the Pulkovo Tables 1985 and those given by the direct method in the neighbourhood of the horizon.

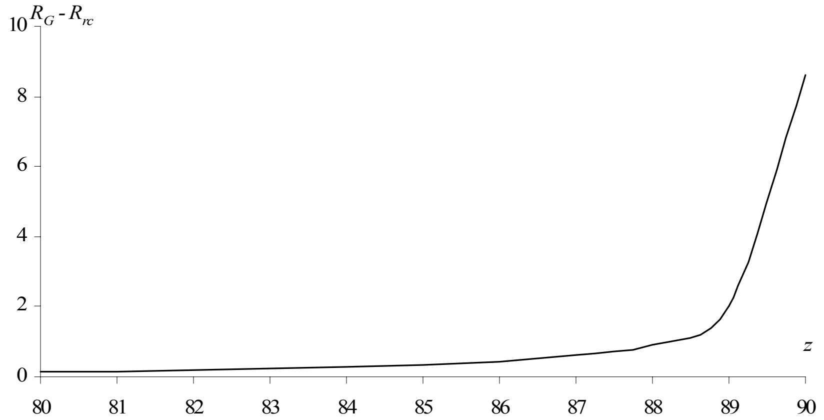

Fig. 5 – The difference of the refraction values determined by Guseva and those given by the direct method in the neighbourhood of the horizon.

It can be noticed that the values of the astronomical refraction in the neighbourhood of the horizon, calculated through the direct method, independently of the model chosen, are smaller than the values of refraction deduced by means of the refraction integral and do not become infinite for $z = 90^\circ$.

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