THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

*PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,*
*Volume 6, Number 2/2005, pp. 000-000*

# ON THE GEOMETRIZATION OF NON-HOLONOMIC MECHANICAL SYSTEMS

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In this paper the non-holonomic mechanical systems are studied from a geometric point of view using the Lagrange Geometry. One introduces the concept of non-holonomic Lagrange space and one use the differential geometry of Lagrange spaces to study the non-holonomic mechanical systems, by means of its canonical semispray.

*Key words:* non-holonomic mechanical system, Lagrange space, non-linear connection, semispray.

## 1. INTRODUCTION

The geometry of Lagrange spaces studied and developed by R. Miron [1], [2], [3], [5], have important applications in Theoretical Mechanics, Variational Calculus, Mathematical Biology and another.

I have been studied, [6], the Lagrangian geometrical theory of Riemann mechanical systems $\Sigma = (M, F_i)$, $M$ being the configuration space and $F_i$ the external forces. In this paper I study the non-holonomic scleronomic mechanical systems using this theory.

The Analytical Mechanics have like study object the movement of systems of material points under the action of external forces and keep the constrains (the link conditions). We shall consider the material systems, which are composed of a finite number of material points, under the action of the external forces and the geometrical and the motion links.

If the material system has non-integrable motion links, it is a **non-holonomic** system. A time dependent link of a material system is given by an equation, which depend explicitly by time t. Otherwise, we say that is a time independent link. A **rheonomic** material system is a system that have time dependent links. Otherwise, if the links of system are not dependent explicitly on time, it is a **sclerhonomic** system.

The section 2 of this article is an overview of the geometry of the tangent bundle $TM$, (phase space), with the basic tools that have an important role in my paper: the Liouville vector field $C$, the almost tangent structure $J$, the concept of semispray and the concept of non-linear connection which is central in my study.

In section 3 one gives a geometrization of non-holonomic mechanical systems using the geometry of tangent bundle $TM$ and one obtains the Lagrange equations of the non-holonomic mechanical system.

The canonical semispray $S^*$, the non-linear connection $N^*$ generated by the mechanical system and the $N^*$-linear connection for the non-holonomic mechanical system are studied in section 4 of this article.

## 2. THE TANGENT BUNDLE (TM, π, M)

Let $M$ be a smooth $C^\infty$ manifold of finite dimension $n$ and $(TM, \pi, M)$ be its tangent bundle. We denote by $(x^i)$, $i = 1,2,...,n$ the local coordinates on $M$ and by $(x^i, y^i)$ the local coordinates on $TM$.

Reccomended by Radu MIRON, member of the Romanian Academy

The tangent bundle

$$E=(TM,\pi,M) \tag{2.1}$$

has the total space $E=TM$ which is a 2n-dimensional, real manifold. In a domain of a local chart $U\subset E$, the points $(x,y)\in E$ have the local coordinates $(x^i,y^i)$.

The canonical projection $\pi:E\to M$, is defined by:

$$\pi(x,y)=x,\ \forall u=(x,y)\in E. \tag{2.2}$$

A change of local coordinates on $E$ has the following form:

$$\tilde{x}^i=\tilde{x}^i(x^1,x^2,..,x^n)\ ,\qquad \tilde{y}^i=\frac{\partial\tilde{x}^i}{\partial x^j}y^j,\qquad \det\left\|\frac{\partial\tilde{x}^i}{\partial x^j}\right\|\neq 0. \tag{2.3}$$

The natural basis of tangent space $T_uE$ at the point $u=(x,y)\in U\subset E$ is given by:

$$\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^i}\right)_{|u}. \tag{2.4}$$

The coordinates transformation (2.3), determines the transformations of the natural basis as follows:

$$\frac{\partial}{\partial x^i}=\frac{\partial\tilde{x}^j}{\partial x^i}\frac{\partial}{\partial\tilde{x}^j}+\frac{\partial\tilde{y}^j}{\partial x^i}\frac{\partial}{\partial\tilde{y}^j};\qquad \frac{\partial}{\partial y^i}=\frac{\partial\tilde{y}^j}{\partial y^i}\frac{\partial}{\partial\tilde{y}^j}, \tag{2.5}$$

where

$$\frac{\partial\tilde{y}^j}{\partial y^i}=\frac{\partial\tilde{x}^j}{\partial x^i}\ ;\ \frac{\partial\tilde{y}^j}{\partial x^i}=\frac{\partial^2\tilde{x}^j}{\partial x^i\partial x^h}y^h.$$

We know that $TM$ admits a natural tangent structure $J:\chi(E)\to\chi(E)$, defined by:

$$J\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial}{\partial y^i}\ ;\ J\left(\frac{\partial}{\partial y^i}\right)=0\quad \text{for}\quad i=1,2,...,n \tag{2.6}$$

By a direct calculation, we find that $J\circ J=0$ and the Nijenhuis tensor $N_J$ vanishes.

On the manifold $E$ there exists a vertical distribution $V$, generated by $n$ local vectors fields $\left(\frac{\partial}{\partial y^1},\frac{\partial}{\partial y^2},...,\frac{\partial}{\partial y^n}\right)$,

$$V:u\in E\to V_u\subset T_uE, \tag{2.7}$$

where the $n$-dimensional linear space $V_u$ generated by the fields $\left(\frac{\partial}{\partial y^1},\frac{\partial}{\partial y^2},...,\frac{\partial}{\partial y^n}\right)_{|u}$ is a linear subspace of tangent bundle $T_uE$.

A non-linear connection in $E$ is a distribution:

$$N:u\in E\to N_u\subset T_uE, \tag{2.8}$$

which is supplementary to the vertical distribution $V$,

$$T_uE=N_u\oplus V_u,\quad \forall u=(x,y)\in E. \tag{2.9}$$

The local basis adapted to the decomposition (2.9) is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}. \quad (2.10)$$

$N_i^j(x,y)$ are real functions, locally defined on $E$ and subject to the following transformation rule under (2.3):

$$\tilde{N}_m^j \frac{\partial \tilde{x}^m}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^m} N_i^m - \frac{\partial \tilde{y}^j}{\partial x^i}. \quad (2.11)$$

The coordinates transformation (2.3) determines the transformation of the local basis adapted to the decomposition (2.9) as follows:

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j}; \quad \frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j}. \quad (2.12)$$

## 3. NON-HOLONOMIC MECHANICAL SYSTEMS

A sclerhonomic non-holonomic mechanical system is a quadruple

$$\Sigma = (M, g, F, Q_\sigma)$$

where $M$ is a real $n$-dimensional manifold (configuration space), $(M, g)$ is a Riemannian space, $F = (F_i)$ is a vector field on $M$ (the external forces) and $Q_\sigma \equiv (a_{\sigma i}(x)), \sigma = p+1,...,n$ are the supplementary forces determined by the non-holonomic constraints given by the relations

$$Q_\sigma(x)dx \equiv a_{\sigma i}(x)dx^i = 0. \quad (3.1)$$

For a non-holonomic system, $\Sigma$, we consider the vector field of the forces given by

$$F + \lambda^\sigma Q_\sigma, \quad (3.2)$$

where $\lambda^\sigma : R \to R$, $\sigma = p+1,...,n$ are the **Lagrange multipliers** and $\lambda^\sigma Q_\sigma$ are the components of the so-called non-holonomic constraint force**.**

Let consider the Riemannian manifold $(M, g)$ and $\nabla$ its Levi-Civita connection. A trajectory of non-holonomic mechanical system $\Sigma$ is a differentiable curve $c$ in $M$, $c : t \in I \to (x^i(t)) \in V \subset M$ ($I \subset R, V$ is a domain of a local chart of $M$), who verify the following Lagrange equations:

$$\nabla_{\dot{c}}\dot{c} = F \circ c + \lambda^\sigma Q_\sigma \circ c \quad (3.3)$$

In local coordinates in $M$, (3.3) may be written

$$\frac{d^2x^k}{dt^2} + \Gamma_{ij}^k(x)\frac{dx^i}{dt}\frac{dx^j}{dt} = F^k(x) + \lambda^\sigma a_\sigma^k(x) \quad (3.4)$$

where $\{\Gamma_{ij}^k\}$ are the Christoffel symbols of the connection $\nabla$ of the Riemannian metric tensor $g$ and $F^k(x) = g^{ki}(x)F_i(x)$, $a_\sigma^i(x) = g^{ij}a_{\sigma j}(x)$.

Consequently, the Lagrange equations of the non-holonomic mechanical system $\Sigma$ are (3.3) and

$$Q_\sigma(x)dx \equiv a_{\sigma i}(x)dx^i = 0, \ (\sigma = p+1,...,n).$$

We will study the properties of non-holonomic mechanical system $\Sigma$ associating the Lagrange space $L^{*n} = (M, L^*(x, y))$, with the fundamental function

$$L^*(x,y) = L(x,y) + \lambda^\sigma a_{\sigma i}(x) y^i \tag{3.5}$$

where $L(x, y)$ is the regular Lagrangian, given by kinetic energy:

$$L(x,y) = g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}.$$

In order to determine the Lagrange multipliers $\lambda^\sigma$ (which, in general depend on the material points $x^i$) we adopt the following postulate:

***The Lagrangians*** $L(x, y)$ ***and*** $\boldsymbol{L^*(x, y) = L(x, y) + \lambda^\sigma a_{\sigma i}(x) y^i}$ ***are equivalent.***

This condition means that $L(x, y)$ and $L^*(x, y)$ satisfy the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L^*}{\partial y^i} - \frac{\partial L^*}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0.$$

Using (3.5), one obtains:

$$\left[\frac{\partial \lambda^\sigma}{\partial x^i} a_{\sigma j} - \frac{\partial \lambda^\sigma}{\partial x^j} a_{\sigma i} + \lambda^\sigma \left( \frac{\partial a_{\sigma i}}{\partial x^j} - \frac{\partial a_{\sigma j}}{\partial x^i} \right)\right] y^j = 0.$$

Deriving with respect to $y^j$ we obtain:

$$\frac{\partial \lambda^\sigma}{\partial x^i} a_{\sigma j} - \frac{\partial \lambda^\sigma}{\partial x^j} a_{\sigma i} + \lambda^\sigma \left( \frac{\partial a_{\sigma i}}{\partial x^j} - \frac{\partial a_{\sigma j}}{\partial x^i} \right) = 0. \tag{3.6}$$

Let us consider the 1-form

$$\lambda^\sigma(x) Q_\sigma(x) dx = \lambda^\sigma(x) a_{\sigma i}(x) dx^i \tag{3.7}$$

This 1-form is closed if and only if the equations (3.6) hold.
Indeed, it is not difficult to see that the equations (3.6) are equivalent to the exterior equations

$$d[\lambda^\sigma(x) Q_\sigma(x) dx] = 0. \tag{3.8}$$

**Remark 3.1.** Applying the Carathedory theorem, we have directly that the Lagrangians $L$ and $L^*$ are equivalent if (3.8) holds.

Resuming:

**The Lagrange equations of the non-holonomic mechanical system $\Sigma$ are:**

$$\begin{gathered} \frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(x) \frac{dx^i}{dt}\frac{dx^j}{dt} = F^k(x) + \lambda^\sigma(x) a^k_\sigma(x) \\ \lambda^\sigma(x) a_{\sigma i}(x) dx^i = 0; \ d[\lambda^\sigma(x) a_{\sigma i}(x)] \wedge dx^i = 0. \end{gathered} \tag{3.9}$$

## 4. CANONICAL SEMISPRAY AND CANONICAL NON-LINEAR CONNECTION OF Σ

The Lagrange space $L^n = (M, L(x,y))$ has a canonical semispray

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$2G^k(x,y) = \Gamma^k_{ij}(x) y^i y^j.$$

Looking at the Lagrange equations (3.9) we remark that the system of functions

$$G^{*i}(x,y) = G^i(x,y) - \frac{1}{2}(F^i(x) + \lambda^\sigma(x) a^i_\sigma(x))$$

determines the coefficients of a semispray on the phase space $TM$.

So, we have:

**Theorem 4.1.** There exists a semispray $S^*$ on the phase space $TM$,

$$S^* = y^i \frac{\partial}{\partial x^i} - 2G^{*i}(x,y)\frac{\partial}{\partial y^i} \tag{4.1}$$

which depend only on the non-holonomic mechanical system Σ. Its coefficients are given by

$$2G^{*k}(x,y) = \Gamma^k_{ij}(x) y^i y^j - (F^k(x) + \lambda^\sigma(x) a^k_\sigma(x)) \tag{4.2}$$

and the Lagrange multipliers $\lambda^\sigma(x)$ satisfying the equations

$$d[\lambda^\sigma(x) a_{\sigma i}(x)] \wedge dx^i = 0.$$

$S^*$ is called the **canonical semispray** of the mechanical system $\Sigma = (M, g, F, Q_\sigma)$.

Therefore: **The geometrical theory of non-holonomic mechanical system is the Lagrange geometry of the triple**

$$(TM, S^*, d(\lambda^\sigma a_{\sigma i}) \wedge dx^i = 0).$$

One proves, by (4.1), (4.2):

**Theorem 4.2.** The integral curves of the canonical semispray $S^*$ are given by the equations (3.9) of mechanical system Σ.

So, the non-linear connection $N^*$ determined by the canonical semispray $S^*$ (called canonical non-linear connection of Σ) has the coefficients

$$N^{*i}_j = \frac{\partial G^{*i}}{\partial y^j} = \Gamma^i_{jk}(x) y^k. \tag{4.3}$$

We obtain:

**Theorem 4.3.** The canonical non-linear connection $N^*$ of the non-holonomic mechanical system $\Sigma = (M, g, F_i, Q_\sigma)$ does not depend on the forces $F_i$ and $Q_\sigma$.

If we have the canonical non-linear connection $N^*$ with the coefficients (4.3) on the Lagrange space $L^{*n}$, we can define a $N^*$-linear connection $D$, [1]:

A linear connection $D$ on $TM$ is called an $N^*$-linear connection if:

1. $D$ preserves by parallelism the horizontal distribution $N^*$;
2. $J$ is absolute parallel with respect to $D$, that is $D_X J = 0,\ \forall X \in \chi(E)$.

In the local basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ adapted to the decomposition (2.9), an $N^*$-linear connection can be uniquely written in the form:

$$D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = L^i_{jk}(x,y)\frac{\delta}{\delta x^i};\ D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = L^i_{jk}(x,y)\frac{\partial}{\partial y^i} \tag{4.4}$$

$$D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} = C^{\ i}_{jk}(x,y)\frac{\delta}{\delta x^i};\ D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = C^{\ i}_{jk}(x,y)\frac{\partial}{\partial y^i}\ . \tag{4.4'}$$

The system of functions { $L^k_{ij}(x,y)$, $C^k_{ij}(x,y)$ } are called the coefficients of the $N^*$-linear connection $D$. It is important to remark that $C^k_{ij}(x,y)$ are the coordinates of a d-tensor field of type (1,2).

In our case $G^{*i}(x,y)$ are the coefficients of the semispray $S^*$ (4.1); it is easily to see that $\left(\frac{\partial^2 G^{*i}}{\partial y^i \partial y^j}, 0\right)$ are the coefficients of an $N^*$-linear connection on $TM$, $N^*$ having the coefficients $N^{*i}_{\ j} = \frac{\partial G^{*i}}{\partial y^j}$.

**Theorem 4.4.** [1] 1. There exists a unique $N^*$-linear connection $D$ on $T\tilde{M}$ verifying the axioms

$$g_{ij|k} = 0;\ g_{ij}\big|_k = 0;$$
$$T^k_{ij} = 0;\ S^k_{ij} = 0, \tag{4.5}$$

where "$|$" and "$\big|$" are the *h*-covariant derivation and *v*-covariant derivative and $T^k_{ij} = L^k_{ij} - L^k_{ji}$; $S^k_{ij} = C^k_{ij} - C^k_{ji}$.

2.This connection has the coefficients

$$L^i_{kj} = \frac{1}{2}\, g^{ih}\left(\frac{\delta g_{hk}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^k} - \frac{\delta g_{kj}}{\delta x^h}\right);\ C^i_{kj} = \frac{1}{2}\, g^{ih}\left(\frac{\partial g_{hk}}{\partial y^j} + \frac{\partial g_{hj}}{\partial y^k} - \frac{\partial g_{kj}}{\partial y^h}\right). \tag{4.6}$$

3.The previous connection depend only on the fundamental function $L^*(x,y)$ of the Lagrange space.

The connection $D$ from the previous theorem will be called **canonical metrical connection** on the space $L^{*n}$.

Now we consider, more general case when the equations of movement for the non-holonomic, sclerhonomic system are given by

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt} = \frac{1}{2} g^{kj}(x) F_j(x,y) + \lambda^\sigma(x) a^k_\sigma(x)\ ;\ y^i = \frac{dx^i}{dt}. \tag{4.7}$$

where $F_i(x,y)$ is a *d*-tensorial field on $TM$ and the Lagrange multipliers $\lambda^\sigma$ verify (3.6).

We put

$$2\tilde{G}^k(x,y) = \Gamma^k_{ij}(x) y^i y^j - \frac{1}{2} g^{kj}(x) F_j(x,y) - \lambda^\sigma(x) a^k_\sigma(x) \tag{4.8}$$

and the equation (4.7) give us the integral curves of the semispray

$$\tilde{S} = y^i \frac{\partial}{\partial x^i} - 2\tilde{G}^i(x,y) \frac{\partial}{\partial y^i}. \tag{4.9}$$

This semispray $\tilde{S}$ determine the non-linear connection $\tilde{N}$ with the coefficients

$$\tilde{N}^i_j = \frac{\partial \tilde{G}^i}{\partial y^j} = \frac{\partial G^i}{\partial y^j} - \frac{1}{4}\frac{\partial F^i}{\partial y^j}, \tag{4.10}$$

where $F^i(x,y) = g^{ij}(x) F_j(x,y)$ and $2G^k(x,y) = \Gamma^k_{ij}(x) y^i y^j$.

The study of the non-holonomic, sclerhonomic mechanical system will be made in Lagrange space $L^{*n} = (M, L^*(x,y))$, with the fundamental function $L^*(x,y)$ from (3.5) and the non-linear connection $\tilde{N}$ with the coefficients given by (4.10).

Using the non-linear connection $\tilde{N}$ we can consider the adapted basis $\left(\frac{\delta}{\delta \tilde{x}^i}, \frac{\partial}{\partial y^i}\right)$ to the decomposition

$$T_u(TM) = \tilde{N}_u \oplus V_u, \quad \forall u = (x,y) \in TM, \tag{4.11}$$

with

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \tilde{N}^j_i(x,y) \frac{\partial}{\partial y^j}. \tag{4.12}$$

We may construct the $\tilde{N}$-linear connection $D$, which preserves by parallelism the horizontal distribution $\tilde{N}$ and the tangent structure $J$ is absolute parallel with respect $D$.

If $D\Gamma(\tilde{N}) = (\tilde{L}^i_{jk}(x,y), \tilde{C}^i_{jk}(x,y))$ are the coefficients of $D$ in the adapted basis $\left(\frac{\delta}{\delta \tilde{x}^i}, \frac{\partial}{\partial y^i}\right)$, denoting by $g_{ij|k}$ and $g_{ij}\big|_k$ the $h$- and $v$-covariant derivatives, then we have:

$$g_{ij|k} = \frac{\delta g_{ij}}{\delta x^h} - \tilde{L}^m_{ih} g_{mj} - \tilde{L}^m_{jh} g_{im}; \quad g_{ij}\big|_k = \frac{\partial g_{ij}}{\partial y^h} - \tilde{C}^m_{ih} g_{mj} - \tilde{C}^m_{jh} g_{im}. \tag{4.13}$$

It is not difficult to prove:

**Theorem 4.5.** In a non-holonomic Lagrange space $L^{*n} = (M, L^*(x,y))$ the following properties hold:

a)There exists a unique $\tilde{N}$-linear connection $D$ on $TM$ satisfying the axioms:

$$g_{ij|k} = 0; \quad g_{ij}\big|_k = 0; \tag{4.14}$$

$$T^k_{ij} = 0; S^k_{ij} = 0. \tag{4.14′}$$

b)The coefficients $(\tilde{L}^i_{jk}(x,y), \tilde{C}^i_{jk}(x,y))$ of this connection are:

$$\tilde{L}^i_{kj} = \frac{1}{2}\, g^{ih}\left(\frac{\delta g_{hk}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^k} - \frac{\delta g_{kj}}{\delta x^h}\right) = L^i_{kj} - \frac{1}{4} C^i_{kr}\, \frac{\partial F^r}{\partial y^j}\,;$$

$$\tilde{C}^i_{kj} = \frac{1}{2}\, g^{ih}\left(\frac{\partial g_{hk}}{\partial y^j} + \frac{\partial g_{hj}}{\partial y^k} - \frac{\partial g_{kj}}{\partial y^h}\right) = C^i_{kj}\,. \tag{4.15}$$

where $(L^i_{jk}(x,y), C^i_{jk}(x,y))$ are the coefficients of the metrical connection *N*, from (4.6).

In conclusion: **The Lagrange space $L^{*n}$ endowed with the non-linear connection $\tilde{N}$ gives us a geometrical model for the non-holonomic mechanical system $\Sigma = (M, g, F_i(x,y), Q_\sigma)$.**

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*Received March 4 , 2005*