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GENERALIZED VARIATIONAL INEQUALITIES INVOLVING SOME TYPES OF RELAXED LIPSCHITZ AND RELAXED MONOTONE OPERATORS

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We introduce some new classes of generalized relaxed Lipschitz operators and relaxed monotone operators. An iterative algorithm for solving generalized variational inequalities is studied . The convergence of the iterative sequence generated by this algorithm is obtained under weaker assumptions, improving some known results in the recent literature.

1. INTRODUCTION

Variational inequalities are an important area of mathematics, with applications in industry, physics, economics and engineering. In recent years, variational inequalities have been extended in several directions [2,4,8]. One of them is the class of generalized variational inequalities, introduced by Verma [7]. Also, Verma and others considered the (GWI) problem associated with this type of variational inequality and established its equivalence to a nonlinear equation which imply projection operators [1,5,6].

Here we extend this result, introducing some new concepts of generalized relaxed Lipschitz continuity and generalized relaxed monotonicity and prove the convergence of an iterative algorithm for solving generalized variational inequalities involving (k_1, k_2, k_3)-relaxed Lipschitz and (c_1, c_2, c_3)-relaxed monotone multivalued operators and (r_1, r_2, r_3)-strongly monotone singlevalued operators.

2. PRELIMINARIES

Let K be a nonempty closed convex subset of a real Hilbert space H with inner product $<\cdot,\cdot>$ and norm $\|\cdot\|$.

We consider the following generalized variational inequality. Find $x\in H, w\in S(x), z\in T(x)$ such that $f(x)\in K$ and

$$\left\langle w-z, v-f(x) \right\rangle \geq 0 \text{ for all } v\in K, \tag{1}$$

where $f: H\to H$ is a given singlevalued operator and $S,T: H\to 2^H$ are multivalued operators. Now,as in Verma [8], we consider the corresponding (GVI) problem.Find $x\in H, w\in S(x)$ and $z\in T(x)$ such that $f(x)\in K$ and

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$$\left\langle f(x)-(w-z), v-f(x)\right\rangle \geq 0 \text{ for all } v\in K \tag{2}$$

In what follows we shall introduce some general classes of operators and prove some properties of these classes.

Definition 2.1. Let $r>0$. An operator $f: H\to H$ is called r-expansive if $\|f(u)-f(v)\|\geq r\|u-v\|$ for all $u,v\in H$. When r=1, we say that f is expansive.

Definition 2.2. Let $s>0$. An operator $f: H\to H$ is called s-Lipschitz continuous if $\|f(u)-f(v)\|\leq s\|u-v\|$ for all $u,v\in H$. Denote by $L(s)$ the class of all s-Lipschitz continuous operators $f: H\to H$.

Definition 2.3. Let $m>0$. An operator $S: H\to 2^H$ is called m-Lipschitz continuous if for all $u,v\in H$ and $w_1\in S(u)$, $w_2\in S(v)$, we have $\|w_1-w_2\|\leq m\|u-v\|$. Denote by $L(m)$ the class of all m-Lipschitz continuous operators $S: H\to 2^H$.

Definition 2.4. Let k_1, k_2 and k_3 be nonnegative real numbers. An operator $S: H\to 2^H$ is called (k_1, k_2, k_3)-relaxed Lipschitz if for all $u,v\in H$ and $w_1\in S(u)$, $w_2\in S(v)$, we have $\left\langle w_1-w_2, u-v\right\rangle \leq -k_1\|u-v\|^2+k_2\|w_1-w_2\|^2+k_3\|u-v\|\,\|w_1-w_2\|$. Denote by $RL(k_1, k_2, k_3)$ the class of all (k_1, k_2, k_3)-relaxed Lipschitz operators $S: H\to 2^H$.

Remark 2.5. We note that $RL(k_1, 0, 0)$ corresponds to the class of k_1-relaxed Lipschitz operators (see[2]).

Proposition 2.6. For nonnegative real constants k_1, k_2, k_3 we have

i) $RL(k_1,0,0)\subset RL(k_1,k_2,0)\subset RL(k_1,k_2,k_3)\subset RL(0,k_2,k_3)$;

ii) $RL(0,k_2,0)\subset RL(0,k_2,k_3)$;

iii) If $0\leq k_2m^2+k_3m<k_1$, then $RL(k_1,0,0)\subset RL(k_1-k_2m^2-k_3m,0,0)$;

iv) If $0\leq k_2m^2+k_3m<k_1$, then $RL(k_1,k_2,k_3)\cap L(m)\subset RL(k_1-k_2m^2-k_3m,0,0)$.

Remark 2.7. The oposite inclusion in iii) does not hold, so it makes sense to obtain results about the class $RL(k_1,k_2,k_3)$ which is more general than $RL(k_1, 0, 0)$. In order to prove this, we give

Example 2.8. Let $S: R\to 2^R$, S(x)= {-x}. Let $m=\frac{11}{10}>0, k_1=3>0, k_2=\frac{1}{4}>0, k_3=2>0$. It is easy to show that $S\in L\left(\frac{11}{10}\right)$ and $S\in RL\left(3,\frac{1}{4},2\right)$, so that $S\in RL\left(3-\frac{1}{4}\cdot\frac{11}{10}-2\cdot\left(\frac{11}{10}\right)^2\right)=RL\left(\frac{61}{200}\right)$, but $S\notin RL(3,0,0)$.

Definition 2.9. Let r_1,r_2,r_3 be nonnegative real numbers. An operator $f: H\to H$ is called (r_1, r_2, r_3)-strongly monotone if $\left\langle f(u)-f(v), u-v\right\rangle \geq r_1\|u-v\|^2-r_2\|f(u)-f(v)\|^2-r_3\|u-v\|\,\|f(u)-f(v)\|$ for all $u,v\in H$. Denote by $SM(r_1, r_2, r_3)$ the class of all (r_1, r_2, r_3)-strongly monotone operators $f: H\to H$.

Remark 2.10. We note that $SM(r_1, 0,0)$ corresponds to the class of r_1-strongly monotone operators (see [7]).

Proposition 2.11. For nonnegative real constants s, r_1, r_2, r_3 , we have

i) $SM(r_1,0,0) \subset SM(r_1,r_2,r_3)$;

ii) if $0 \le r_2 s^2 + r_3 s < r_1$, then $SM(r_1,0,0) \subset SM(r_1 - r_2 s^2 - r_3 s,0,0)$;

iii) if $0 \le r_2 s^2 + r_3 s < r_1$, then $SM(r_1,r_2,r_3) \cap L(s) \subset SM(r_1 - r_2 s^2 - r_3 s,0,0)$.

Remark 2.12. The oposite inclusion in ii) does not hold, so it makes sense to obtain results about the class $SM(r_1, r_2, r_3)$, which is more general than $SM(r_1, 0,0)$. In order to prove this, we give

Example 2.13. Let $f : R \to R$, f(x)=x+1. Let $S = \frac{11}{10} > 0, r_1 = 3 > 0, r_2 = \frac{1}{4} > 0, r_3 = 2 > 0$. It is easy to show that $f \in L\left(\frac{11}{10}\right)$ and $f \in SM\left(3, \frac{1}{4}, 2\right)$, so that

$$f \in SM\left(3 - \frac{1}{4} \cdot \frac{11}{10} - 2 \cdot \left(\frac{11}{10}\right)^2\right) = SM\left(\frac{61}{200}\right),$$

but $f \notin SM(3,0,0)$.

Definition 2.14. Let c_1, c_2, c_3 be nonnegative real numbers. An operator $T : H \to 2^H$ is called (c_1, c_2, c_3)-relaxed monotone if for all $u, v \in H$ and $z_1 \in T(u), z_2 \in T(v)$, we have $\langle z_1 - z_2, u - v \rangle \ge -c_1 \|u - v\|^2 - c_2 \|z_1 - z_2\|^2 - c_3 \|u - v\| \, \|z_1 - z_2\|$. Denote by $RM(c_1, c_2, c_3)$ the class of all (c_1, c_2, c_3)-relaxed monotone operators $T : H \to 2^H$.

Remark 2.15. We note that $RM(c_1,0,0)$ coresponds to the class of c_1-relaxed monotone operators (see[7]).

Proposition 2.16. Let m,c_1,c_2,c_3 be nonnegative real constants and $T : H \to 2^H$ a multivalued operator. Then

i) $T \in RM(0,0,1)$;

ii) $RM(c_1,0,0) \subset RM(c_1,c_2,c_3)$;

iii) $RM(c_1,c_2,c_3) \cap L(m) \subset RM(c_1 + c_2 m^2 + c_3 m,0,0)$.

Remark 2.17. The oposite inclusion in ii) does not hold, so it makes sense to obtain results about the class $RM(c_1,c_2,c_3)$, which is more general than $RM(c_1,0,0)$. In order to prove this, we give

Example 2.18. Let $T : R \to 2^R, T(x) := \{-x\}$ .Let $m = 2 > 0, c_1 = \frac{1}{2} > 0, c_2 = 1 > 0, c_3 = 2 > 0$ .It is easy to show that $T \in L(2)$ and $T \in RM\left(\frac{1}{2}, 1, 2\right)$, so $T \in RM\left(\frac{1}{2} + 1 \cdot 2^2 + 2 \cdot 2\right) = RM\left(\frac{17}{2}\right)$, but $T \notin RM\left(\frac{1}{2}, 0, 0\right)$.

3. MAIN RESULTS

We consider an iterative method for solving the (GVI) problem defined in Section 2, which has been introduced by Verma [8]. In this section, we prove the convergence of this method under the hypotheses of m-Lipschitz continuity, (k_1, k_2, k_3)-relaxed Lipschitz continuity, (r_1, r_2, r_3)-strong monotonicity and (c_1, c_2, c_3)-relaxed monotonicity.

Lemma 3.1 ([3]). *Let $f: H \to H$ be a singlevalued operator, S, $T: H \to 2^H$ multivalued operators, $x \in H, w \in S(x), z \in T(x)$ such that $f(x) \in K$. Then x,w and z are a solution set for* (*GVI*) *problem* (2) *iff they satisfy the equation*(*in t*)

$$f(x) = P_k[(1-t)f(x) + t(w-z)] \tag{3}$$

Based on Lemma 3.1, Verma [8] considered the iterative

Algorithm 3.2. For $n = 0, 1, 2, \ldots\ldots$ $f(x_{n+1}) = P_k[(1-t)f(x_n) + t(w_n - z_n)]$

Theorem 3.3. *Let K be a nonempty closed convex subset of a real Hilbert space H, $f: H \to H$ an s-Lipschitz continuous and* (r_1, r_2, r_3)- *strongly monotone operator, where s,r_1, r_2, r_3 are nonnegative real constants. Let $S: H \to 2^H$ be m- Lipschitz and $T: H \to 2^H$ a d- Lipschitz continuous and* (c_1, c_2, c_3)-*relaxed monotone operators, where m, d, k_1, k_2, k_3, c_1, c_2 and c_3 are nonnegative real constants.Assume also that* $r>0$, $k>0$, $1-2r+s^2>0$ *and*

$$1 + (k-c) > p(p-r) + \left\{[1 + 2(k-c) + (m+d)^2 - p^2][1-(r-p)^2]\right\}^{1/2}, \tag{4}$$

$$\begin{gathered}\left| t - \frac{1+k-c-p(p-r)}{1+2(k-c)+(m+\mathrm{d})^2-p^2} \right| < \\ < \left\{[1+(k-c)-p(p-r)]^2 - [1+2(k-c)+(m+\mathrm{d})^2-p^2][1-(r-p)^2\right\}^{1/2} \cdot \\ \cdot [1+2(k-c)+(m+\mathrm{d})^2-p^2]^{-1},\end{gathered} \tag{5}$$

where $r = r_1 - r_2 s^2 - r_3 s$, $k = k_1 - k_2 m^2 - k_3 m, c = c_1 - c_2 d^2 - c_3 d$, $p = (1 - 2r + s^2)^{1/2}$.

Then the sequences $\{x_n\}$, $\{w_n\}$, $\{z_n\}$*and* $\{f(x_n)\}$, *generated by Algorithm* 3.2 *with* $x_0 \in H, w_0 \in S(x_0), z_0 \in T(x_0)$ *and* $f(x_0) \in K$, *converge to x, w, z and f*(*x*), *respectively, solution of equation* (3).

Theorem 3.3 has the following corollaries.

Corollary 3.4 ([7]). *Let K be a nonempty closed convex subset of a real Hilbert space H, $f: H \to H$ an s-Lipschitz continuous and* (r_1, r_2)-*strongly monotone operator, where s,r_1, r_2 are nonnegative real constants. Let $S: H \to 2^H$ be an m-Lipschitz continuous and* (k_1, k_2)-*relaxed Lipschitz operator ,let $T: H \to 2^H$ a d- Lipschitz continuous and* (c_1, c_2)- *relaxed monotone operator, where m, d, k_1, k_2, c_1 and c_2 are nonnegative real constants.*

Assume also that $r>0$, $k>0$, $1 - 2r + s^2>0$ *and conditions* (4) *and* (5) *hold, where* $r = r_1 - r_2 s^2$, $k = k_1 - k_2 m^2$, $c = c_1 + c_2 d^2$, $p = (1 - 2r + s^2)^{1/2}$.

Then the sequences $\{x_n\}$, $\{w_n\}$, $\{z_n\}$ *and that* $\{f(x_n)\}$, *generated by Algorithm* 3.2 *with* $x_0 \in H, w_0 \in S(x_0), z_0 \in T(x_0)$ *and* $f(x_0) \in K$, *converge to x, w, z and f*(*x*), *respectively, solution of equation* (3).

Corollary 3.5 ([7]). *Let K be a nonempty closed convex subset of a real Hilbert space H and let $f: H \to H$ be a strongly monotone and Lipschitz continuous operator with corresponding constants* $r>0$ *and* $s>0$. *Let $S: H \to 2^H$ be a relaxed Lipschitz and Lipschitz continuous operator with corresponding* *constants* $k \geq 0$ *and* $m \geq 0$. *Let* $T : H \to 2^H$ *be a relaxed monotone and Lipschitz continuous operator with corresponding constants* $c>0$ *and* $d>0$. *Also, we asume* $1 - 2r + s^2 > 0$ *and conditions* (4) *and* (5) *hold.*

Then the sequences $\{x_n\}$, $\{w_n\}$, $\{z_n\}$*and* $\{f(x_n)\}$, *generated by Algorithm* 3.2 *with* $x_0 \in H, w_0 \in S(x_0), z_0 \in T(x_0)$ *and* $f(x_0) \in K$, *converge to x, w, z and f*(*x*), *respectively, solution of equation* (3).

Corollary 3.7 ([2]). *Let f be the identity and* $S : K \to H$ *a relaxed Lipschitz continuous operator. Then the sequence* $\{x_n\}$, *generated by*

$$x_{n+1} = P_k[(1-t)x_n + tS(x_n)] \text{ , } n = 0, 1, 2, \ldots \tag{6}$$

where x_0 *is in K* $0 < t < \dfrac{2(1+k)}{1+2k+m^2}$, *converges to the unique fixed point of S.*

Corollary 3.8 ([9]). *Let f be the identity and* $S : K \to H$ *a Lipschitz continuous operator with Lipschitz constant* $m \geq 1$. *Then the sequence* $\{x_n\}$ *generated by* (6), *where* $x_0 \in K$ *and* $0 < t < \dfrac{2}{1+m}$, *converges to the unique fixed point of S.*

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