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LIFE INSURANCE PLANS AND STOCHASTIC ORDERS

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In risk theory several stochastic order relations among distributions are commonly used: increasing convex, stochastic and in mortality or hazard rate. We show their meaning in life insurance and point out a new type of dominance stronger than the stochastic dominance but weaker than the dominance in mortality.

1. LIFE INSURANCE SCHEMES FAIR IN EXPECTATION.

We deal with two partners: the *insured* **A** and the *insurer* **B**.

There are many kinds of *life insurance schemes* (LIS). We deal with the simplest ones which run as follows:

The two partners agree at time $t = 0$ to sign a *contract*:

- at moment t_0 (or from t_0 to t_1) **A** pays to **B** a cash amount of C monetary units (MU) (or a cash flow c) in order that
- at time t_2 **B** pay to **A** S MU ***provided that* A *is still alive*** at moment t_2 (or **B** pay to **A** a cash flow s starting at t_2 until the death of **A**).

They agree on a given *instantaneous interest rate* (IIR) denoted by δ.

So we deal with four types of LIS:

(i) LIS of type $\Pi_{1,1}(C,S;\ t_0,t_2;\ \delta,T)$: at t_0 **A** pays to **B** C MU and at time t_2 **B** pays to **A** S MU provided that **A** is still alive. The index "$_{1,1}$" means that it involves 1 payment from **A** to **B** and 1 payment from **B** to **A**. We call this an LIS *of type* 1-1. T is the lifetime of **A**; the payment is made only if $T > t_2$.

(ii) LIS of type $\Pi_{1,c}(C,s;\ t_0,t_2;\ \delta,T)$: at t_0 **A** pays to **B** C MU and from t_2 until the random time T when **A** dies, **B** pays to **A** *a cash flow of intensity s* MU. A *cash flow is a function* $s:[0,\infty) \to [0,\infty)$ *with the meaning that the cash amount paid on the time interval* $[a,b]$ *is* $\int_a^b s(x)\mathrm{d}x$ MU. The index "$_{1,c}$" means that this type involves 1 payment from **A** to **B** and a cash flow payment from **B** to **A**. We call this an LIS *of type* 1-c.

(iii) LIS of type $\Pi_{c,1}(c,S;\ t_0,t_1,t_2;\ \delta,T)$: in the interval $[t_0,\ t_1]$ **A** pays to **B** *a cash flow of intensity c* MU and at time t_2 **B** pays to **A** S MU provided **A** is still alive at t_2. The index "$_{c,1}$" means that this type involves one cash flow from **A** to **B** and a single pay from **B** to **A**. We call this an LIS *of type* c-1.

(iv) LIS of type $\Pi_{c,c}(c,s;\ t_0,t_1,t_2;\ \delta,T)$: in the interval $[t_0,\ t_1]$ **A** pays to **B** *a cash flow of intensity c* MU and from t_2 untill T, **B** pays to **A** a cash flow of intensity s MU. The index "$_{c,c}$" means that this type involves one cash flow from **A** to **B** and a single pay from **B** to **A**. We call this an LIS *of type* c-c. Sometimes it is called a *pension plan.*

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Definition 1.1. *An* ***instantaneous interest rate*** (IIR) *is any function* $\delta:[0,\infty) \to (0,\infty)$ *which is right-continuous and has limit on the left. Its meaning is that* 1 MU *borrowed at time* $t_0 = 0$ *costs* $\sigma(t_1) = \exp\left(\int_0^{t_1} \delta(x)\mathrm{d}x\right)$ MU *at time* $t_1 = t$. *The function* σ *is the* ***fructification factor***. *Throughout this paper we shall accept that* $\sigma(\infty) = \infty$. *The function* $\underline{\Phi}:[0,\infty) \to [0,1]$ *defined by* $\underline{\Phi}(t) = e^{-\int_0^t \delta(u)\mathrm{d}u} = \frac{1}{\sigma(t)}$ *is called the* ***actualization factor***. *For any function which is right continuous and with finite limits to the left we shall use the abbreviation* ***CADLAG***. *All the cash-flows will be supposed to be CADLAG, too.*

We shall use the analogy between the IIR denoted δ and the failure rate of a lifetime τ considered in [8]. The idea is since $\underline{\Phi}$ is non increasing, right-differentiable, $\underline{\Phi}(0) = 1$ and $\underline{\Phi}(\infty)= 0$, it can be considered to be the survival function of some absolutely continuous lifetime denoted by τ. If the density of τ is denoted by φ then $\delta = \varphi / \underline{\Phi}$ is the failure rate of τ. Let us denote the failure rate of an absolutely continuous lifetime T by r_T, its distribution function $P(T \le t)$ by $F_T(t)$ and its survival function $P(T > t)$ by $\underline{F}_T(t)$. Thus $\delta = r_\tau$. Notice that the actuaries call $r_T(t)$ the *mortality force of* T at moment t (see [2], p. 49) and denote it by μ_t . Instead of $\underline{F}_T(t)$ they write ${}_tp_0$; however this notation is cumbersome for mathematicians.

In general the connection between the survival function of some lifetime T and r_T is given by

$$P(T \ge t) := \underline{F}_T(t) = \exp\left(-\int_0^t r_T(x)\mathrm{d}x \right)\mathrm{d}x . \tag{1.1}$$

Remark. For the sake of shortening proofs, all the lifetimes T in the sequel will be supposed to be absolutely continuous and with finite moments of any order. All the lifetimes actuaries use are like that. However, many results are true without such drastical assumptions.

When is this LIS *fair*?

The word "fair" has many interpretations. We mean "fair" with respect to the agreed IIR δ. In this interpretation we know what "fair" means: suppose we deal with an LIS of type 1-1. As 1 MU at moment t has the value $\underline{\Phi}(t)$ at moment 0, then if **A** is still alive at time t_2 "fair" would mean that $C\underline{\Phi}(t_0) = S\underline{\Phi}(t_2)$.

If we add the incertitude about the lifetime of **A**, then the problem changes a bit. What is that to claim the fairness when dealing with random variables? A lot has been written about that. One way to answer the question is to use the principle of the expected utility of Von Neumann – Morgenstern (see for instance [7]). But if we accept that the insurer is *risk neutral* - meaning that his estimation of a risk X is its expectation (that may happen if the insurer has many insured people) – then we could think as follows. Any LIS involves two random variables: X and Y . X is the total cash amount paid by **A** to **B** actualized at moment 0 and Y is the total cash amount paid by **A** to **B** actualized at the same moment $t = 0$. Then we have

Definition 1.2. An **LIS** *is fair in expectation* (FIE) *iff* $\mathrm{E}X = \mathrm{E}Y$.

Now, we shall see what this definition implies in the case of our four types of LIS :

(i) LIS of type $\Pi_{1,1}(C,S;\, t_0,t_2;\, \delta,T)$: $X = C\underline{\Phi}(t_0)1_{(T>t_0)}$ and $Y = S\underline{\Phi}(t_2)1_{(T>t_2)}$;

(ii) LIS of type $\Pi_{1,c}(C,s;\, t_0,t_2;\, \delta,T)$: $X = C\underline{\Phi}(t_0)1_{(T>t_0)}$ and $Y = \int_{\min(T,t_2)}^{T} s(t)\,\underline{\Phi}(t)\mathrm{d}t$. ;

(iii) LIS of type $\Pi_{c,1}(c,S;\, t_0,t_1,t_2;\, \delta,T)$: $X = \int_{\min(T,t_0)}^{\min(T,t_1)} c(t)\,\underline{\Phi}(t)\mathrm{d}t$ and $Y = S\underline{\Phi}(t_2)\; 1_{(T>t_2)}$;

(iv) LIS of type $\Pi_{c,c}(c,s;\, t_0,t_1,t_2;\, \delta,T)$: $X = \int_{\min(T,t_0)}^{\min(T,t_1)} c(t)\,\underline{\Phi}(t)\mathrm{d}t$ and $Y = \int_{\min(T,t_2)}^{T} s(t)\,\underline{\Phi}(t)\mathrm{d}t$.

Thus we have the following result:

Proposition 1.1. Let us denote by $\underline{F}$ the survival function $\underline{F}_T$ .Then

(i) *an* LIS $\Pi_{1,1}(C,S;\ t_0,t_2;\ \delta,T)$ *is* FIE *iff* $C\underline{\Phi}(t_0)\underline{F}(t_0) = S\underline{\Phi}(t_2)\,\underline{F}(t_2)$;

(ii) *an* LIS $\Pi_{1,c}(C,s;\ t_0,t_2;\ \delta,T)$ *is* FIE *iff* $C\underline{\Phi}(t_0)\underline{F}(t_0) = \int_{t_2}^{\infty} s(t)\,\underline{\Phi}(t)\underline{F}(t)\mathrm{d}t$;

(iii) *an* LIS $\Pi_{c,1}(c,S;\ t_0,t_1,t_2;\ \delta,T)$ *is* FIE *iff* $\int_{t_0}^{t_1} c(t)\,\underline{\Phi}(t)\underline{F}(t)\mathrm{d}t = S\underline{\Phi}(t_2)\underline{F}(t_2)$;

(iv) *an* LIS $\Pi_{c,c}(c,S;\ t_0,t_1,t_2;\ \delta,T)$ *is* FIE *iff* $\int_{t_0}^{t_1} c(t)\,\underline{\Phi}(t)\underline{F}(t)\mathrm{d}t = \int_{t_2}^{\infty} s(t)\,\underline{\Phi}(t)\underline{F}(t)\mathrm{d}t$.

Proof. The first claim is obvious since $\mathrm{E}X = C\underline{\Phi}(t_0)P(T > t_0) = C\underline{\Phi}(t_0)\underline{F}(t_0)$ and $\mathrm{E}Y = C\underline{\Phi}(t_2)\underline{F}(t_2)$. For the other ones, we use the following formula which is easily proved using Fubini's theorem: if f is continuous and right-differentiable, then $\mathrm{E}f(T) - f(0) = \int_0^{\infty} f'(x)\,\underline{F}(x)\,\mathrm{d}x$, where f' stands for the right-derivative of f. If we take $f(x) = \int_{\min(x,t_2)}^{x} s(t)\,\underline{\Phi}(t)\,\mathrm{d}t$ then $f(x) = 0$ for any $x \le t_2$, $f(x) = \int_{t_2}^{x} s(t)\,\underline{\Phi}(t)\,\mathrm{d}t$ for any $x > t_2$, hence $f'(x) = s(x)\underline{\Phi}(x)$ for $x > t_2$, $f'(x) = 0$ for $x < t_2$; thus, $\mathrm{E}Y = \mathrm{E}f(T) - f(0) = \int_0^{\infty} f'(x)\,\underline{F}(x)\,\mathrm{d}x = \int_{t_2}^{\infty} s(t)\,\underline{\Phi}(t)\underline{F}(t)$ $\mathrm{d}t$. To compute $\mathrm{E}X$ in cases (iii) and (iv), we use the function $f(x) = \int_{\min(x,t_0)}^{\min(x,t_1)} c(t)\,\underline{\Phi}(t)\,\mathrm{d}t$; this time $f(x) = 0$ for $x \le t_0$, $f(x) = \int_{t_0}^{t_1} c(t)\,\underline{\Phi}(t)\,\mathrm{d}t = \text{const}$ for $x > t_1$, $f(x) = \int_{t_0}^{x} c(t)\,\underline{\Phi}(t)\,\mathrm{d}t$ for $t_0 < x < t_1$ hence $f'(x) = c(x)\underline{\Phi}(x)$ for $t_0 < x < t_1$ and $f'(x) = 0$ on $[0,t_0) \cup (t_1,\infty)$. Thus $\mathrm{E}X = \int_{\min(T,t_0)}^{\min(T,t_1)} c(t)\,\underline{\Phi}(t)\,\mathrm{d}t = \int_{t_0}^{t_1} c(t)\,\underline{\Phi}(t)\underline{F}(t)\,\mathrm{d}t$.

Now we show the probabilistic meaning of these relations.

Recall that if T is the lifetime of **A** and $t > 0$, then the new random variable $T^*: \{T > t\} \to [0,\infty)$ defined on the new probability space $\{T > t\}$ with the new probability $P_{\{T>t\}}$ by the formula $T^* := T - t$ (the usual notation of actuaries is $(T - t \mid T > t)$!) is called *the residual lifetime at age t* or even *time-up-to death* at age t (see [3]). It is denoted by $T(t)$, but we prefer to denote it by $T_{(t)}$ to avoid possible confusions. Its survival function $P(T_{(t)} > x) = \dfrac{P(T - t > x)}{P(T > t)} = \dfrac{\underline{F}(t + x)}{\underline{F}(t)}$ is usually denoted by $\underline{F}_{(t)}(x)$ and its hazard rate by $r_{(t)}(x) := r(t+x)$.

Now remark that if $\underline{\Phi}$ and $\underline{F}$ are survival functions, then $\underline{\Phi}\cdot\underline{F}$ is a survival function too. If $\underline{\Phi}(t) = P(\tau>t)$ and $\underline{F}(t) = P(T > t)$ then $\underline{\Phi}\cdot\underline{F}(t) = P(\min(T,\tau) > t)$, provided that T and τ are independent. Thus, if we suppose that our fictive τ described by δ is independent of the real lifetime T we could restate the above proposition as

Corollary 1.2. Let us denote by $\underline{F}$ the survival function $\underline{F}_T$ .Then

(i) *an* LIS $\Pi_{1,1}(C,S;\ t_0,t_2;\ \delta,T)$ *is* FIE *iff* $C\underline{F}_{\min(T,\tau)}(t_0) = S\underline{F}_{\min(T,\tau)}(t_2)$;

(ii) *an* LIS $\Pi_{1,c}(C,s;\ t_0,t_2;\ \delta,T)$ *is* FIE *iff* $C\underline{F}_{\min(T,\tau)}(t_0) = \mathrm{E}S(\min(T,\tau))$ *where* $S(t) = \int_{\min(t,t_2)}^{t} s(x)\,\mathrm{d}x$*;*

(iii) *an* LIS $\Pi_{c,1}(c,S;\ t_0,t_1,t_2;\ \delta,T)$ *is* FIE *iff* $\mathrm{E}C(\min(T,\tau)) = S\underline{F}_{\min(T,\tau)}(t_2)$ *where* $C(t) = \int_{\min(t,t_0)}^{\min(t,t_1)} c(x)\,\mathrm{d}x$*;*

(iv) *an* LIS $\Pi_{c,c}(c,S;\ t_0,t_1,t_2;\ \delta,T)$ *is* FIE *iff* $\mathrm{E}C(\min(T,\tau)) = \mathrm{E}S(\min(T,\tau))$ *with C,S as above*.

2. STOCHASTIC ORDERS AND FAIR LIS

Definition 2.1. Let T_1, T_2 be two absolutely continuous lifetimes. Let $\underline{F}_1$, $\underline{F}_2$ be their survival functions and r_1, r_2 their hazard rates. We say (see, for instance [6] or [7])

- T_1 *is stochastically dominated by* T_2 (and write $T_1 \prec_{\mathbf{st}} T_2$) iff $\underline{F}_1 \le \underline{F}_2$;
- T_1 *is dominated by* T_2 *in mortality* (and write $T_1 \prec_{\mathbf{m}} T_2$) iff $(T_1)_{(t)} \prec_{\mathbf{st}} (T_2)_{(t)}\ \forall\ t \ge 0$;
- T_1 *is increasing convex dominated by* T_2 (and write $T_1 \prec_{\mathbf{icx}} T_2$) iff $\mathrm{E}(T_1 - t)_+ \le \mathrm{E}(T_2 - t)_+\ \forall\ t \ge 0$.

It is well known (see for instance [5], [6] or [7]) that $T_1 \prec_{\mathbf{st}} T_2 \Leftrightarrow \mathrm{E}w(T_1) \le \mathrm{E}w(T_2)$ for any increasing nonnegative w; that $T_1 \prec_{\mathbf{m}} T_2 \Leftrightarrow r_1 \ge r_2$ and that $T_1 \prec_{\mathbf{icx}} T_2 \Leftrightarrow \mathrm{E}w(T_1) \le \mathrm{E}w(T_2)$ for any increasing convex nonnegative w. However, we will not need that in the sequel.

Our goal is to make the connection between these concepts and the LIS which are fair in expectation.

To start with, suppose that $\Pi_{1,1}(C,1;\ t_0,t_2;\ \delta,T)$ is FIE. This means that $C\underline{\Phi}(t_0)\underline{F}(t_0) = \underline{\Phi}(t_2)\underline{F}(t_2)$. As S is only a proportionality factor we may as well assume that $S = 1$ MU; write then the fairness condition as

$$C\ =\ C_{1,1}(t_0,t_2;\delta,T)\ \frac{\underline{\Phi}(t_2)\underline{F}(t_2)}{\underline{\Phi}(t_0)\underline{F}(t_0)}. \tag{2.1}$$

Suppose now that the insurer **B** deals with two insured persons $\mathbf{A_1}$ and $\mathbf{A_2}$. Suppose that he knows their survival functions $\underline{F}_1$ and $\underline{F}_2$ (that really happens if he knows their ages, residence or sex). When is it fair to charge $\mathbf{A_1}$ less than $\mathbf{A_2}$ for the same reimbursement S?

Proposition 2.1. Let T_1, T_2 be two absolutely continuous lifetimes. Then

(i) $C_{1,1}(s,t;\ \delta,T_1) \le C_{1,1}(s,t;\ \delta,T_2)$ *for every* $t,\ t > s$ *iff* $(T_1)_{(s)} \prec_{\mathbf{st}} (T_2)_{(s)}$;

(ii) $C_{1,1}(0,t;\ \delta,T_1) \le C_{1,1}(0,t;\ \delta,T_2)$ *for every* $t \ge 0$ *iff* $T_1 \prec_{\mathbf{st}} T_2$;

(iii) $C_{1,1}(s,t;\ \delta,T_1) \le C_{1,1}(s,t;\ \delta,T_2)$ *for every* $s,t,\ 0 \le s < t$ *iff* $T_1 \prec_{\mathbf{m}} T_2$.

Proof. For instance, $C_{1,1}(s,t;\ \delta,T_1) \le C_{1,1}(s,t;\ \delta,T_2) \Leftrightarrow \dfrac{\underline{\Phi}(t)\underline{F}_1(t)}{\underline{\Phi}(s)\underline{F}_1(s)} \le \dfrac{\underline{\Phi}(t)\underline{F}_2(t)}{\underline{\Phi}(s)\underline{F}_2(s)} \Leftrightarrow \dfrac{\underline{F}_1(t)}{\underline{F}_1(s)} \le \dfrac{\underline{F}_2(t)}{\underline{F}_2(s)}$ and the last inequality can be written as $P((T_1)_{(s)} > t\text{-}s) \le P((T_2)_{(s)} > t\text{-}s)\ \forall\ t \ge s$, proving claim (i). In particular, if s=0 then we check the second claim and if the inequality holds for every $s < t$ this is the very definition of the domination by mortality.

Let us consider now an LIS of type $\Pi_{1,c}(C,s;\ t_0,t_2;\ \delta,T)$. We know that this type of LIS is FIE iff $C\underline{\Phi}(t_0)\underline{F}(t_0) = \int_{t_2}^{\infty} s(t)\,\underline{\Phi}(t)\underline{F}(t)\,\mathrm{d}t$; write that as

$$C\ =\ C_1,c(s;\ t_0,t_2;\delta,T)\ =\ \frac{1}{\underline{\Phi}(t_0)\underline{F}(t_0)}\int_{t_2}^{\infty} s(t)\underline{\Phi}(t)\underline{F}(t)\,\mathrm{d}t \tag{2.2}$$

The analog of Proposition 2.1. is

Proposition 2.2. Let T_1, T_2 be two absolutely continuous lifetimes. Then

(i) $C_{1,c}(s; x, t_2; \delta, T_1) \le C_{1,c}(s; x, t_2; \delta, T_2)$ *for any cash flow s and any* t_2 , $t_2 > x$, *iff* $(T_1)_{(x)} \prec_{\mathbf{st}} (T_2)_{(x)}$;

(ii) $C_{1,c}(s; 0, t; \delta, T_1) \le C_{1,c}(s; 0, t; \delta, T_2)$ *for any cash flow s and any* $t \ge 0$, *iff* $T_1 \prec_{\mathbf{st}} T_2$;

(iii) $C_{1,c}(s; t_0, t_2; \delta, T_1) \le C_{1,c}(s; t_0, t_2; \delta, T_2)$ *for any cash flow s and any* t_0, t_2 , $t_2 > t_0$, *iff* $T_1 \prec_{\mathbf{m}} T_2$;

(iv) $C_{1,c}(1; x, t_2; \delta, T_1) \le C_{1,c}(1; x, t_2; \delta, T_2)$ *for every* t_2 , $t_2 > t_0$, *iff* $(\min(T_1,\tau))_{(x)} \prec_{\mathbf{icx}} \min((T_2,\tau)_{(x)}$;

(v) $C_{1,c}(1; 0, t; \delta, T_1) \le C_{1,c}(1; 0, t; \delta, T_2)$ *for every* $t \ge 0$, *iff* $(\min(T_1,\tau)) \prec_{\mathbf{icx}} \min(T_2,\tau)$

(vi) $C_{1,c}(1; x, t_2; \delta, T_1) \le C_{1,c}(1; x, t_2; \delta, T_2)$ *for any* IIR δ *and any* t_2 , $t_2 > x$, *iff* $(T_1)_{(x)} \prec_{\mathbf{st}} (T_2)_{(x)}$;

(vii) $C_{1,c}(1; 0, t; \delta, T_1) \le C_{1,c}(1; 0, t; \delta, T_2)$ *for any* IIR δ *and any* $t \ge 0$, *iff* $T_1 \prec_{\mathbf{st}} T_2$

(viii) $C_{1,c}(1; t_0, t_2; \delta, T_1) \le C_{1,c}(1; t_0, t_2; \delta, T_2)$ *for any* IIR δ *and any* t_0, t_2 , $t_2 > t_0$, *iff* $T_1 \prec_{\mathbf{m}} T_2$.

Proof. (i). We have $C_{1,c}(s; x, t_2; \delta, T_1) \le C_{1,c}(s; t_0, x; \delta, T_2) \Leftrightarrow \frac{1}{F_1(x)}\int_{t_2}^{\infty} s(t)\underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t \le \frac{1}{F_2(x)}\int_{t_2}^{\infty} s(t)\underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t$ for any bounded *CADLAG* $s:[0,\infty) \to [0,\infty)$. If $s = 1_{[a,b)}$ for some a, $b > t_2$, we get $\frac{1}{\underline{F}_1(x)}\int_a^b \underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t \le \frac{1}{\underline{F}_2(x)}\int_a^b \underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t$. As the integrands are continuous, this means that $\frac{\underline{\Phi}(t)\underline{F}_1(t)}{\underline{F}_1(x)} \le \frac{\underline{\Phi}(t)\underline{F}_2(t)}{\underline{F}_2(x)}$ $\forall\, t > t_2$ and, as t_2 can be any moment greater than x, we get the inequality $\frac{\underline{F}_1(t)}{\underline{F}_1(x)} \le \frac{\underline{F}_2(t)}{\underline{F}_2(x)}$ $\forall\, t > x$ which is the same as $(T_1)_{(x)} \prec_{\mathbf{st}} (T_2)_{(x)}$.

Claims (ii) and (iii) are easy consequences of (i). We prove (iii). Now, $s(x) = 1$ is constant, hence the inequality becomes

$$\frac{1}{\underline{\Phi}(x)\underline{F}_1(x)}\int_{t_2}^{\infty}\underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t \le \frac{1}{\underline{\Phi}(x)\underline{F}_2(x)}\int_{t_2}^{\infty}\underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t \tag{2.3}$$

for any $t_2 > x$. Let $T_j^* = \min(T_j,\tau)$, $j = 1,2$. Recall that τ is supposed to be independent of T_1 and T_2. Then $P(T_j^* > t) = \underline{\Phi}(t)\underline{F}_j(t)$, hence the above inequality becomes

$$\frac{1}{P(T_1^* > x)}\int_{t_2}^{\infty} P(T_1^* > t)\,\mathrm{d}t \le \frac{1}{P(T_2^* > x)}\int_{t_2}^{\infty} P(T_2^* > t)\,\mathrm{d}t \tag{2.4}$$

for any $t_2 > x$. Let $h = t_2 - x$. Remark that for any lifetime T we have $\mathrm{E}(T_{(x)} - h)_+ = \mathrm{E}((T-x-h)_+ \mid T > x) = \frac{E((T-x-h)_+; T > x)}{P(T > x)} = \frac{E(T-x-h; T > x+h)}{P(T > x)} = \frac{E(T-t_2; T > t_2)}{P(T > x)} = \frac{\int_{t_2}^{\infty} P(T > t)\,\mathrm{d}t}{P(T > x)}$. Thus the inequality (2.4) can be written as $\mathrm{E}((T_1^*)_{(x)} - h)_+ \le \mathrm{E}((T_2^*)_{(x)} - h)_+$ for every $h \ge 0$. This is the very definition of the “icx” domination. Now, (v) is an easy consequence when $x = 0$.

To prove (vi), write a instead of t_2 and let some $b > a$. Choose a sequence $(\delta_n)_n$ of interest rates with the property that $\underline{\Phi}_n 1_{[a,b]} \to 1_{[a,b]}$. Here $\underline{\Phi}_n(t) = \exp\left(-\int_0^t \delta_n(y)\,\mathrm{d}y\right)$. (For instance, take $\delta_n(t) = 1/n$ if $t < b$ and $\delta_n(t)$

$= n$ for $t > b$). According to (2.3) we have $\frac{1}{\underline{F}_1(x)}\int_{t_2}^{\infty}\underline{\Phi}_n(t)\underline{F}_1(t)\mathrm{d}t \le \frac{1}{\underline{F}_2(x)}\int_{t_2}^{\infty}\underline{\Phi}_n(t)\underline{F}_2(t)\mathrm{d}t$ for every n. Letting $n \to \infty$ and applying Lebesgue's domination principle we get

$$\frac{1}{\underline{F}_1(x)}\int_a^b \underline{F}_1(t)\mathrm{d}t \le \frac{1}{\underline{F}_2(x)}\int_a^b \underline{F}_2(t)\mathrm{d}t \tag{2.5}$$

for any x,a,b such that $x<a<b$.

As $\underline{F}_j$ are continuous, it folllows that $\frac{\underline{F}_1(t)}{\underline{F}_1(x)} \le \frac{\underline{F}_2(t)}{\underline{F}_2(x)}$ $\forall\ t > x \ \Leftrightarrow (T_1)_{(x)} \prec_{\mathbf{st}} (T_2)_{(x)}$. The converse implication is a consequence of (i). Now, (vii) and (viii) are consequences of (vi).

As an interesting byproduct, we notice

Corollary 2.3. *Let T_1 and T_2 be two life times on a probability space in which absolutely continuous distributed random variables do exist . Then*

$T_1 \prec_{\mathbf{st}} T_2 \ \Leftrightarrow \min(T_1,\tau) \prec_{\mathbf{icx}} \min(T_2,\tau)$ *for any τ independent of T_1 and T_2;*

$T_1 \prec_{\mathbf{m}} T_2 \ \Leftrightarrow (\min(T_1,\tau))_{(t)} \prec_{\mathbf{icx}} (\min(T_2,\tau))_{(t)}$ *for any τ independent of T_1 and T_2 and for any $t \ge 0$.*

Notice that the "⇒" is indeed obvious via the well known fact that $T_1 \prec_{\mathbf{st}} T_2 \Leftrightarrow$ there exist versions of T_j, say T'_j such that $T'_1 \le T'_2$ a.s. (see for instance [7])

Now we deal with an LIS of type "c-1". According to Proposition 1.1(iii), an LIS $\Pi_{c,1}(c,S;\ t_0,t_1,t_2;\ \delta,T)$ is FIE if

$$S = S_{c,1}(c\ ;\ t_0,\ t_1,\ t_2;\ \delta,\ T) = \int_{t_0}^{t_1} c(t)\,\underline{\Phi}(t)\underline{F}(t)\mathrm{d}t\,/\,(\underline{\Phi}(t_2)\underline{F}(t_2)) \tag{2.6}$$

Proposition 2.4. Let T_1, T_2 be two absolutely continuous lifetimes. Then

(i) $S_{c,1}(c\ ;\ t_0,\ t_1,\ b;\ \delta,T_1) \ge S_{c,1}(c\ ;\ t_0,\ t_1,\ b;\ \delta,T_2)\ \forall\ t_0,\ t_1 \in [0,b)$, $t_0 < t_1$ *iff* $\frac{c(t)\underline{F}_1(t)}{\underline{F}_1(b)} \ge \frac{c(t)\underline{F}_2(t)}{\underline{F}_2(b)}\ \ \forall\ t < b;$

(ii) $S_{c,1}(1\ ;\ t_0,\ t_1,\ b;\ \delta,\ T_1) \ge S_{c,1}(1\ ;\ t_0,\ t_1,\ b;\ \delta,\ T_2)\ \forall\ t_0,\ t_1 \in [0,b)$, $t_0 < t_1$ *iff* $\frac{\underline{F}_1(t)}{\underline{F}_1(b)} \ge \frac{\underline{F}_2(t)}{\underline{F}_2(b)}\ \ \forall\ t < b;$

(iii) *the following assertions are equivalent*:

(a) $S_{c,1}(c\ ;\ t_0,\ t_1,\ b;\ \delta,\ T_1) \ge S_{c,1}(c\ ;\ t_0,\ t_1,\ b;\ \delta,\ T_2)\ \forall\ t_0,\ t_1,\ t_2$ *such that* $t_0 < t_1 < t_2$ *and for any c;*

(b) $S_{c,1}(1\ ;\ t_0,\ t_1,\ b;\ \delta,\ T_1) \ge S_{c,1}(1\ ;\ t_0,\ t_1,\ b;\ \delta,\ T_2)\ \forall\ t_0,\ t_1,\ t_2$ *such that* $t_0 < t_1 < t_2$;

(c) $T_1 \prec_{\mathbf{m}} T_2$.

(iv) *if* $S_{c,1}(1\ ;\ 0,\ t_1,\ t_2;\ \delta,\ T_1) \ge S_{c,1}(1\ ;\ 0,\ t_1,\ t_2;\ \delta,\ T_2)\ \ \forall\ \ t_1,\ t_2$ *such that* $t_1 < t_2$ *then* $T_1 \prec_{\mathbf{st}} T_2$;

(v) $S_{c,1}(1\ ;\ t_0,\ t_2,\ t_2;\ \delta,\ T_1) \ge S_{c,1}(1\ ;\ t_0,\ t_2,\ t_2;\ \delta,\ T_2)\ \ \forall\ t_0,\ t_2$ *such that* $t_0 < t_2$, *for any* IIR δ *iff* $T_1 \prec_{\mathbf{m}} T_2$.

Proof.: (i) We have $S_{c,1}(c\ ;\ t_0,\ t_1,\ b;\ \delta,\ T_1) \ge S_{c,1}(c\ ;\ t_0,\ t_1,\ b;\ \delta,\ T_2) \Leftrightarrow \int_{t_0}^{t_1} c(t)\,\underline{\Phi}(t)\underline{F}_1(t)\ \mathrm{d}t\ /\underline{F}_1(b) \ge \int_{t_0}^{t_1} c(t)\,\underline{\Phi}(t)\underline{F}_2(t)\mathrm{d}t\,/\,\underline{F}_2(b)$ for every $t_0 < t_1 < b$. Let $t < b$, $t_0 = t$, $t_1 = t + h$, $h > 0$ small enough . If we divide the last inequality by h , let $h \to 0$ and use the fact that the integrands are right-continuous we get $\frac{c(t)\underline{F}_1(t)}{\underline{F}_1(b)} \geq \frac{c(t)\underline{F}_2(t)}{\underline{F}_2(b)}$. The converse is obvious.

Now, (ii) is an easy consequence. In (iii), the implication (a) ⇒ (b) is obvious; for (b) ⇒ (c) use (ii) put in the form $a < b \Rightarrow \frac{\underline{F}_1(b)}{\underline{F}_1(a)} \leq \frac{\underline{F}_2(b)}{\underline{F}_2(a)} \Leftrightarrow (T_1)_{(a)} \prec_{\mathbf{st}} (T_2)_{(a)}\ \forall\, a \geq 0 \Leftrightarrow T_1 \prec_{\mathbf{m}} T_2$. As for (c) ⇒ (a), it is easy.

For (iv), just let $t_1 \to 0$ and divide by t_1 the inequality $\int_0^{t_1} \underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t / \underline{F}_1(t_2) \geq \int_0^{t_1} \underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t / \underline{F}_2(t_2)$: it follows that $1/\underline{F}_1(t_2) \geq 1/\underline{F}_2(t_2)$. In (vi). the novelty is that now $t_1 = t_2$. Let $a = t_0$ and $b = t_2$. So, the hypothesis is that

$$\int_a^b \underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t / \underline{F}_1(b) \geq \int_a^b \underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t / \underline{F}_2(b) \tag{2.7}$$

for every $a < b$ and for every IIR δ. Let $a < b$ be fixed and let $c \in (a,b)$. Let $(\delta_n)_n$ be a sequence of IIR such that $\underline{\Phi}_n \to 1_{[0,c]}$ (for instance, $\delta_n(t) = 1/n$ if $t < c$ and $\delta_n(t) = n$ for $t > c$). Replacing in (2.7) $\underline{\Phi}$ by $\underline{\Phi}_n$ and letting $n \to \infty$, one gets the inequality $\int_a^c \underline{F}_1(t)\mathrm{d}t / \underline{F}_1(b) \geq \int_a^c \underline{F}_2(t)\mathrm{d}t / \underline{F}_2(b)$, true for any $0 < a < c < b$. Replacing c by $a+h$, dividing by h and letting $h \to 0$ it follows that the inequality $\underline{F}_1(a) / \underline{F}_1(b) \geq \underline{F}_2(a) / \underline{F}_2(b)$ holds for any $a < b$; but this is precisely the definition of "$T_1 \prec_{\mathbf{m}} T_2$".

Finally, we deal with an LIS of type "c-c" – with pension plans.

By Proposition 1.1(iv), an LIS $\Pi_{c,c}(c,s;\, t_0,t_1,t_2;\, \delta,T)$ is FIE iff $\int_{t_0}^{t_1} c(t)\,\underline{\Phi}(t)\underline{F}(t)\mathrm{d}t = \int_{t_2}^{\infty} s(t)\,\underline{\Phi}(t)\underline{F}(t)\mathrm{d}t$. This time we shall suppose that c and s are constant cash flows: $c(t) = c$, $s(t) = s$. Then the fairness condition becomes

$$c\int_{t_0}^{t_1} \underline{\Phi}(t)\underline{F}(t)\mathrm{d}t = r\int_{t_2}^{\infty} \underline{\Phi}(t)\underline{F}(t)\mathrm{d}t \tag{2.8}$$

Suppose that $r = 1$ and denote the corresponding c from the above equality by $c(t_0, t_1, t_2;\, \delta,T)$. Thus

$$c(t_0, t_1, t_2;\, \delta,T) = \int_{t_2}^{\infty} \underline{\Phi}(t)\underline{F}(t)\,\mathrm{d}t \Big/ \int_{t_0}^{t_1} \underline{\Phi}(t)\underline{F}(t)\,\mathrm{d}t \tag{2.9}$$

Proposition 2.5. Let T_1, T_2 be two absolutely continuous lifetimes and let τ be a lifetime independent of them with δ as hazard rate. Then

(i) $c(t_0, t_1, t_2;\, \delta,T_2) \leq c(t_0, t_1, t_2;\, \delta,T_2)$ *for any* $0 \leq t_0 < t_1 < t_2$ *iff* $(\min(T_1,\tau))_{(t)} \prec_{\mathbf{icx}} (\min(T_2,\tau))_{(t)}\ \forall\, t \geq 0$;

(ii) *The following assertions are equivalent:*
  (a). $c(t_0, t_1, t_2;\, \delta,T_2) \leq c(t_0, t_1, t_2;\, \delta,T_2)$ *for any* $t_0 < t_1 < t_2$ *and for any* IIR δ;
  (b). $c(a, b, b;\, \delta,T_2) \leq c(a, b, b;\, \delta,T_2)$ *for any* $0 \leq a < b$ *and for any* IIR δ;
  (c). $T_1 \prec_{\mathbf{m}} T_2$

(iii) *If* $c(0, a, b;\, \delta,T_2) \leq c(0, a, b;\, \delta,T_2)$ *for any* $0 \leq a < b$ *and for any* IIR δ, *then* $T_1 \prec_{\mathbf{st}} T_2$.

Proof. By (2.8) the inequality $c(t_0, t_1, t_2;\, \delta,T_2) \leq c(t_0, t_1, t_2;\, \delta,T_2)$ can also be written as

$$\left(\int_{t_2}^{\infty}\underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t\right)\Big/\left(\int_{t_0}^{t_1}\underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t\right)\le\left(\int_{t_2}^{\infty}\underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t\right)\Big/\left(\int_{t_0}^{t_1}\underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t\right) \tag{2.10}$$

or as $\left(\int_{t_2}^{\infty}\underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t\right)\left(\int_{t_0}^{t_1}\underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t\right)\le\left(\int_{t_2}^{\infty}\underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t\right)\left(\int_{t_0}^{t_1}\underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t\right)$. Let $t_1 = t_0 + h$. Divide the inequality by h, write t instead of t_0 and let $h \to 0$; one gets the inequality

$$\underline{F}_2(t)\underline{\Phi}(t)\int_{t_2}^{\infty}\underline{\Phi}(t)\underline{F}_1(t)\,\mathrm{d}t \le \underline{F}_1(t)\underline{\Phi}(t)\int_{t_2}^{\infty}\underline{\Phi}(t)\underline{F}_2(t)\,\mathrm{d}t \tag{2.11}$$

for any $t \le t_2$. Writing the inequality as $\mathrm{E}[(\min(T_1,\tau))_{(t)} - (t_2 - t))_+] \le \mathrm{E}[(\min(T_1,\tau))_{(t)} - (t_2 - t))_+]$ $\forall\ t_2 > t_0$, we see that we obtain the very definition of the fact that $(\min(T_1,\tau))_{(t)} \prec_{\mathbf{icx}} (\min(T_2,\tau))_{(t)}$. To get the converse implication, just integrate (2.11) from t_0 to t_1.

(ii) The only non-trivial implication is (b) $\Rightarrow$ (c). Let $c > b$ and choose a sequence $(\delta_n)_n$ of IIRs, such that δ_n converge to $1_{[0,c]}$ as $n \to \infty$. Put in (2.9) $t_0 = a$, $t_1 = t_2 = b$, δ_n instead of δ and let $n \to \infty$. One gets

$$\int_b^c \underline{F}_1(t)\mathrm{d}t \Big/ \int_a^b \underline{F}_1(t)\mathrm{d}t \le \int_b^c \underline{F}_2(t)\mathrm{d}t \Big/ \int_a^b \underline{F}_2(t)\,\mathrm{d}t \tag{2.12}$$

for any $a < b < c$. Write now $\underline{F}_2 = \Lambda\underline{F}_1$, put $c = b + h$, divide (2.11) by h and let $h \to 0$. It follows that

$$\underline{F}_1(b)/\int_a^b \underline{F}_1(t)\,\mathrm{d}t \le \underline{F}_2(b)/\int_a^b \underline{F}_2(t)\,\mathrm{d}t \Leftrightarrow \int_a^b (\Lambda(t)-\Lambda(b))\underline{F}_1(t)\,\mathrm{d}t \le 0 \tag{2.13}$$

for any $a < b$. Repeat the trick with $a = b - h$; one gets

$$\int_b^c \underline{F}_1(t)\,\mathrm{d}t/\underline{F}_1(b) \le \int_b^c \underline{F}_2(t)\,\mathrm{d}t/\underline{F}_2(b) \Leftrightarrow \int_b^c (\Lambda(b)-\Lambda(t))\underline{F}_1(t)\,\mathrm{d}t \le 0 \tag{2.14}$$

for any $b < c$. As inequality (2.14) holds for ***any*** $b < c$ we may as well put a instead of b and b instead of c, just to get the same integration limits in (2.13) and (2.14). Addding the two inequalities, we get $\int_b^c (\Lambda(a)-\Lambda(b))\underline{F}_1(t)\,\mathrm{d}t \le 0$. As $a < b$ the meaning is that Λ is non-decreasing. But this is another way to describe the fact $T_1 \prec_{\mathbf{m}} T_2$.

(iii) In (2.9) do the same trick with $\delta_n \to 1_{[0,c]}$ as before. Get $\int_b^c \underline{F}_1(t)\mathrm{d}t/\int_0^a \underline{F}_1(t)\,\mathrm{d}t \le \int_b^c \underline{F}_2(t)\,\mathrm{d}t/\int_0^a \underline{F}_2(t)\,\mathrm{d}t$ and let $a \to 0$, $c \to b$ to obtain $\underline{F}_1(b) \le \underline{F}_2(b)$; as b is arbitrary, $T_1 \prec_{\mathrm{st}} T_2$.

Remark. The only cases without equivalence are (iv) from Proposition 2.4 and (iii) from Proposition (2.5). Actually the following equivalences hold:

(a) $S_{c,1}(1 ; 0, a, b; \delta, T_1) \ge S_{c,1}(1 ; 0, a, b; \delta, T_2)$ *for any* $0 \le a < b$

(b) $c(0, a, b; \delta, T_2) \le c(0, a, b; \delta, T_2)$ *for any* $0 \le a < b$

(c) $\underline{F}_1(b)/(\int_0^a \underline{F}_1(t)\mathrm{d}t) \le \underline{F}_2(b)/(\int_0^a \underline{F}_2(t)\mathrm{d}t)$ *for any* $0 < a < b$

They imply "$T_1 \prec_{\mathbf{st}} T_2$" and are implied by "$T_1 \prec_{\mathbf{m}} T_2$". Thus this new stochastic order lies somewhere between stochastic domination and domination in mortality and is different from both of them. For example, if $\underline{F}_1(t) = e^{-t}$ $(t \geq 0)$ and $\underline{F}_2(t) = \Lambda(t)\underline{F}_1(t)$ with $\Lambda(t) = (1+t)1_{[0,3)}(t) + (7-t)1_{[3,4)}(t) + 3\cdot 1_{[4,\infty)}(t)$, then the reader may check that (c) is fulfiled (hence (a) and (b) are fulfilled too !) but it is not true that $T_1 \prec_{\mathbf{m}} T_2$. To give an example when $T_1 \prec_{\mathrm{st}} T_2$ but (c) is not true is even easier.

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