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ON THE RATIO *X/Y* FOR THE ELLIPTICALLY SYMMETRIC BESSEL DISTRIBUTION

Saralees NADARAJAH', Samuel KOTZ''

'Department of Statistics, University of Nebraska, Lincoln, NE 68583
''Department of Engineering Management and Systems Engineering
The George Washington University, Washington, D.C. 20052
Corresponding author: Saralees NADARAJAH, e-mail: snadaraj@unlserve.unl.edu

The distribution of X/Y is derived when (X,Y) has the elliptically symmetric Kotz type distribution.

Key words: Bessel function, elliptically symmetric Bessel distribution, hypergeometric functions.

1. INTRODUCTION

For a bivariate random vector *(X, Y),* the distribution of the ratio *X/Y* is of interest in problems in biological and physical sciences, econometrics, and ranking and selection. Examples include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics. The distribution of *X/Y* has been studied by several authors especially when *X* and *Y* are independent random variables and come from the same family. For instance, see Marsaglia [1] and Korhonen and Narula [2] for normal family, Press [3] for Student's *t* family, Basu and Lochner [4] for Weibull family, Shcolnick [5] for stable family, Hawkins and Han [6] for non-central chi-squared family, and Provost [7] for gamma family. However, there is relatively little work of this kind when *X* and *Y* are correlated random variables. Some of the known work include Hinkley [8] for bivariate normal family, Kappenman [9] for bivariate *t* family, and Lee *et al* [10] for bivariate gamma family.

In this paper, we study the distribution of *X/Y* when *(X, Y)* has the elliptically symmetric Bessel distribution given by the joint probability density function (pdf)

$$f(x,y) = M\left\{(x-\alpha)^2+(y-\beta)^2-2\rho(x-\alpha)(y-\beta)\right\}^{a/2} K_a\left(\frac{\sqrt{(x-\alpha)^2+(y-\beta)^2-2\rho(x-\alpha)(y-\beta)}}{b\sqrt{1-\rho^2}}\right) \qquad (1)$$

for $-\infty < x < \infty,\ -\infty < y < \infty,\ -\infty < \alpha < \infty,\ -\infty < \beta < \infty,\ a > -1,\ b > 0,$ and $-1 < \rho < -1,$ where $K_a(.)$ is the modified Bessel function of the third kind defined by

$$K_a(x) = \frac{\pi\{I_{-a}(x) - I_a(x)\}}{2\sin(a\pi)}$$

for $a \neq n$ with $K_n(x) = \lim_{\nu\to n} K_\nu(x),$

$$I_a(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(a+k+1)}\left(\frac{x}{2}\right)^{a+2k},$$

and the normalizing constant *M* satisfies

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$$\frac{1}{M} = 2^{a+1} \pi b^{a+2} \Gamma(a+1)(1-\rho^2)^{(a+1)/2}.$$

For details on the theory and applications of the Bessel functions see Korenev [11].

When $a = 0$ and $b = \sigma / \sqrt{2}$, (1) reduces to the elliptically symmetric Laplace distribution given by the joint probability density function (pdf)

$$f(x, y) = \frac{1}{\pi\sigma^2 \sqrt{1-\rho^2}} K_0 \left(\frac{\sqrt{2}\sqrt{(x-\alpha)^2 + (y-\beta)^2 - 2\rho(x-\alpha)(y-\beta)}}{\sigma\sqrt{1-\rho^2}} \right)$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \alpha < \infty$, $-\infty < \beta < \infty$, $\sigma > 0$, and $-1 < \rho < -1$, where $K_0(.)$ is the modified Bessel function of the third kind of order zero. The parameter ρ is the correlation coefficient between the x and y components. For details on properties of these distributions see Jensen [12] and Fang et al. [13].

Bessel random variables are generalizations of Laplace random variables and have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. They are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data. In many of the application areas, one would be interested in the correlation or dependence between two Bessel random variables. Some examples are:

- in communication theory, X and Y could represent the random noise corresponding to two different signals.
- in ocean engineering, X and Y could represent distributions of navigation errors.
- in finance, X and Y could represent distributions of log-returns of two different commodities.
- in image and speech recognition, X and Y could represent "input" distributions.

In each of the above examples, the elliptically symmetric Bessel distribution given by (1) could be used to model the dependence between X and Y.

The aim of this paper is to calculate the distribution of the ratio X / Y when (X, Y) has the joint pdf (1). The calculations of this paper involve the hypergeometric functions defined by

$$G(\alpha; \beta, \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k (\gamma)_k} \frac{x^k}{k!}$$

and

$$H(\alpha, \beta; \gamma, \delta, \eta; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k (\delta)_k (\eta)_k} \frac{x^k}{k!},$$

where $c_k = c(c+1) \cdots (c+k-1)$ denotes the ascending factorial. We also need the following lemmas.

Lemma 1 (Equation (2.16.2.3), Prudnikov et al [14], volume 2) *For* $a > 0$, $\beta > 0$ *and* $\alpha > \nu$,

$$\int_0^a x^{a-1} (a-x)^{\beta-1} K_\nu (cx) dx = A(\nu) + A(-\nu),$$

where

$$A(\nu) = 2^{\nu-1} a^{\alpha+\beta-\nu-1} c^{-\nu} \Gamma(\nu) B(\beta, \alpha-\nu) \times H\left(\frac{\alpha-\nu}{2}, \frac{1+\alpha-\nu}{2}; 1-\nu, \frac{\alpha+\beta-\nu}{2}, \frac{1+\alpha+\beta-\nu}{2}; \frac{a^2 c^2}{4} \right).$$

Lema 2 (Equation 2.16.2.4), Prudnikov et al [14], volume 2). *For* $a>0$ *and* $a>\nu$,

$$\int_a^\infty x^{a-1}(a-x)^{\beta-1}K_\nu(cx)\,\mathrm{d}x = A(\nu)+A(-\nu)-C(\nu)$$

where

$$A(\nu)=2^{\nu-1}a^{\alpha+\beta-\nu-1}c^{-\nu}\Gamma(\nu)B(\beta,1-\alpha-\beta+\nu)H\left(\frac{\alpha-\nu}{2},\frac{1+\alpha-\nu}{2};1-\nu,\frac{\alpha+\beta-\nu}{2},\frac{1+\alpha+\beta-\nu}{2};\frac{a^2c^2}{4}\right)$$

$$B(\nu)=2^{\nu-1}a^{\alpha+\beta-\nu-1}c^{-\nu}\Gamma\left(\frac{\alpha+\beta+\nu-1}{2}\right)\Gamma\left(\frac{\alpha+\beta-\nu-1}{2}\right)H\left(\frac{1-\beta}{2},1-\frac{\beta}{2};\frac{1}{2},\frac{3-\alpha-\beta-\nu}{2},\frac{3+\nu-\alpha-\beta}{2};\frac{a^2c^2}{4}\right)$$

and

$$C(\nu)=2^{\alpha+\beta-4}ac^{2-\alpha-\beta}(\beta-1)\Gamma\left(\frac{\alpha+\beta+\nu}{2}-1\right)\Gamma\left(\frac{\alpha+\beta-\nu}{2}-1\right)H\left(1-\frac{\beta}{2},\frac{3-\beta}{2};\frac{3}{2},\frac{\nu-\alpha-\beta}{2},2-\frac{\nu+\alpha+\beta}{2};\frac{a^2c^2}{4}\right)$$

Lemma 3 (Equation (2.16.2.2.), Prudnikov et al [14], volume 2). *For* $c>0$ *and* $\alpha>\nu$,

$$\int_a^\infty x^{a-1}(a-x)^{\beta-1}K_\nu(cx)\,\mathrm{d}x = 2^{\alpha-2}c^{-\alpha}\Gamma\left(\frac{\alpha+\nu}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right).$$

Further properties of the above special functions can be found in Prudnikov et al [14] and Gradshteyn and Ryzhik [15]

2. PDF

Theorem 1 derives an explicit expression for the pdf of $Z=X/Y$ in terms of the hypergeometric functions.

Theorem 1 Suppose X and Y are jointly distributed according to (1) and let

$$A=1-\rho\sin(2\theta), \tag{2}$$

$$B=(\rho\beta-\alpha)\cos\theta+(\rho\alpha-\beta)\sin\theta, \tag{3}$$

$$C=\alpha^2+\beta^2-2\rho\alpha\beta, \tag{4}$$

and

$$D=(1-\rho^2)(\alpha\sin\theta-\beta\cos\theta)^2/A. \tag{5}$$

Furthermore, define

$$g_1(\theta)=\frac{M}{2A}\left\{-\Delta+\frac{B}{\sqrt{A}}\sum_{k=0}^{\infty}\binom{-1/2}{k}\left(\frac{B^2}{A}-C\right)^k\Omega(k)\right\} \tag{6}$$

and

$$g_2(\theta)=\frac{MB}{A^{3/2}}\sum_{k=0}^{\infty}\binom{-1/2}{k}\left(\frac{B^2}{A}-C\right)^k\Lambda(k), \tag{7}$$

where

$$\Delta = 2^{a-1} b^a \Gamma(a) C^4 (1-\rho^2)^{a/2} G\left(1;1-a,2;\frac{C^4}{4b^2(1-\rho^2)}\right) + b\Gamma(-a)\Gamma(1+a) C^{2a+2}\sqrt{1-\rho^2}\, I_{a+1}\left(\frac{C^2}{b\sqrt{1-\rho^2}}\right) \quad (8)$$

$$\begin{aligned}\Omega(k) = &\frac{2^a b^a \Gamma(a) C^{2(1-2k)}(1-\rho^2)^{a/2}}{2k-1} G\left(\frac{1}{2}-k;1-a,\frac{3}{2}-k;\frac{C^4}{4b^2(1-\rho^2)}\right) + \\ &+ \frac{\Gamma(-a) C^{2(2a-2k+1)}}{2^a b^a (1-\rho^2)^{a/2}(2k-2a-1)} G\left(a+\frac{1}{2}-k;1+a,a+\frac{3}{2}-k;\frac{C^4}{4b^2(1-\rho^2)}\right),\end{aligned} \quad (9)$$

and

$$\begin{aligned}\Lambda(k) = &\frac{2^a b^a \Gamma(a) C^{2(1-2k)}(1-\rho^2)^{a/2}}{1-2k} G\left(\frac{1}{2}-k;1-a,\frac{3}{2}-k;\frac{C^4}{4b^2(1-\rho^2)}\right) + \\ &+ \frac{\Gamma(-a) C^{2(2a-2k+1)}}{2^a b^a (1-\rho^2)^{a/2}(2a-2k+1)} G\left(a+\frac{1}{2}-k;1+a,a+\frac{3}{2}-k;\frac{C^4}{4b^2(1-\rho^2)}\right) - \\ &- \frac{2^a b^a \Gamma(a) D^{2(1-2k)}(1-\rho^2)^{a/2}}{1-2k} G\left(\frac{1}{2}-k;1-a,\frac{3}{2}-k;\frac{D^4}{4b^2(1-\rho^2)}\right) - \\ &- \frac{\Gamma(-a) D^{2(2a-2k+1)}}{2^a b^a (1-\rho^2)^{a/2}(2a-2k+1)} G\left(a+\frac{1}{2}-k;1+a,a+\frac{3}{2}-k;\frac{D^4}{4b^2(1-\rho^2)}\right).\end{aligned} \quad (10)$$

If $B \geq 0$ then the pdf of $Z = X / Y$ can be expressed as

$$f(z) = \frac{g_1(\arctan(z)) + g_1(\pi + \arctan(z)) + g_2(\pi + \arctan(z))}{1+z^2}. \quad (11)$$

On the other hand if $B < 0$ then

$$f(z) = \frac{g_1(\arctan(z)) + g_1(\pi + \arctan(z)) + g_2(\arctan(z))}{1+z^2}. \quad (12)$$

Proof: Set $(X,Y) = (T\sin\theta, T\cos\theta)$. Under this transformation, the Jacobian is *T* and so one can express the joint pdf of (T,θ) as

$$g(t,\theta) = Mt(At^2 + 2Bt + C)^{a/2} K_a\left(\frac{\sqrt{At^2+2Bt+C}}{b\sqrt{1-\rho^2}}\right), \quad (13)$$

where *A, B* and *C* are given by (2), (3) and (4), respectively. Set $z(t) = At^2 + 2Bt + C$ and note that

$$\frac{dz^{-1}(t)}{dt} = \pm 2\sqrt{B^2 - A(C-z)}.$$

Note further that *z(t)* is an increasing function of *t* with *z(0)* = *C* if $B \geq 0$. On the other hand, if *B* < 0 then *z(t)* decreases between $0 \leq t \leq z^{-1}(D)$ before increasing for all $t \geq z^{-1}(D)$, where *D* is given by (5). Thus, the marginal pdf of θ can be expressed as $g(\theta) = g_1(\theta)$ if $B \geq 0$ and as $g(\theta) = g_1(\theta) + g_2(\theta)$ if *B* < 0, where

$$g_1(\theta) = \frac{M}{2A}\int_c^\infty z^{a/2} K_a\left(\frac{\sqrt{z}}{b\sqrt{1-\rho^2}}\right)\left\{1+\frac{B}{\sqrt{B^2-A(C-z)}}\right\}dz \tag{14}$$

and

$$g_2(\theta) = \frac{MB}{A}\int_D^C \frac{z^{a/2}}{\sqrt{B^2-A(C-z)}} K_a\left(\frac{\sqrt{z}}{b\sqrt{1-\rho^2}}\right)dz. \tag{15}$$

These two expressions actually reduce to those given by (6) and (7), respectively, as shown below. Consider (14). Note that $\left|(B^2-AC)/(Az)\right| \le 1$ for all $z \ge C$. Thus, using the series expansion

$$(1+x)^{-1/2} = \sum_{k=0}^\infty \binom{-1/2}{k} x^k,$$

one can expand (11) as

$$\begin{aligned}
g_1(\theta) &= \frac{M}{2A}\int_C^\infty z^{a/2} K_a\left(\frac{\sqrt{z}}{b\sqrt{1-\rho^2}}\right)\left\{1+\frac{B}{\sqrt{Az}}\sum_{k=0}^\infty \binom{-1/2}{k}\left(\frac{B^2-AC}{Az}\right)^k\right\}dz \\
&= \frac{M}{2A}\int_C^\infty z^{a/2} K_a\left(\frac{\sqrt{z}}{b\sqrt{1-\rho^2}}\right)dz \\
&\quad + \frac{B}{\sqrt{A}}\sum_{k=0}^\infty \binom{-1/2}{k}\left(\frac{B^2-AC}{A}\right)^k \int_C^\infty z^{a/2-k-1} K_a\left(\frac{\sqrt{z}}{b\sqrt{1-\rho^2}}\right)dz \\
&= \frac{M}{A}\left[\int_C^\infty y^{a+1} K_a\left(\frac{y}{b\sqrt{1-\rho^2}}\right)dy\right. \\
&\quad \left. + \frac{B}{\sqrt{A}}\sum_{k=0}^\infty \binom{-1/2}{k}\left(\frac{B^2-AC}{A}\right)^k \int_C^\infty y^{a-2k} K_a\left(\frac{y}{b\sqrt{1-\rho^2}}\right)dy\right],
\end{aligned} \tag{16}$$

where the last step follows by substituting $y=\sqrt{z}$. Application of Lemma 2 and properties of the hypergeometric functions shows that the two integrals in (16) reduce as

$$\int_C^\infty y^{a+1} K_a\left(\frac{y}{b\sqrt{1-\rho^2}}\right)dy = -\frac{\Delta}{2}$$

and

$$\int_C^\infty y^{a-2k} K_a\left(\frac{y}{b\sqrt{1-\rho^2}}\right)dy = \frac{\Omega(k)}{2},$$

respectively, where Δ and $\Omega(k)$ are given by (8) and (9), respectively. Substituting these into (16), one notes that (14) reduces to (6). By using Lemma 1, one can similarly show that (15) reduces to (7). The result of the theorem follows by noting that the pdf of $Z=\tan\theta$ can be expressed as

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1 + z^2} \tag{17}$$

and that $g(\theta) = g_1(\theta)$ if $B \geq 0$ and $g(\theta) = g_1(\theta) + g_2(\theta)$ if $B < 0$. ■

Figures 1 and 2 illustrate possible shapes of the pdfs (11)-(12) for a range of values of α, β, a and υ. Note the diminishing scale of the densities with increasing values of a and how their location depends on the relative magnitudes of α and β.

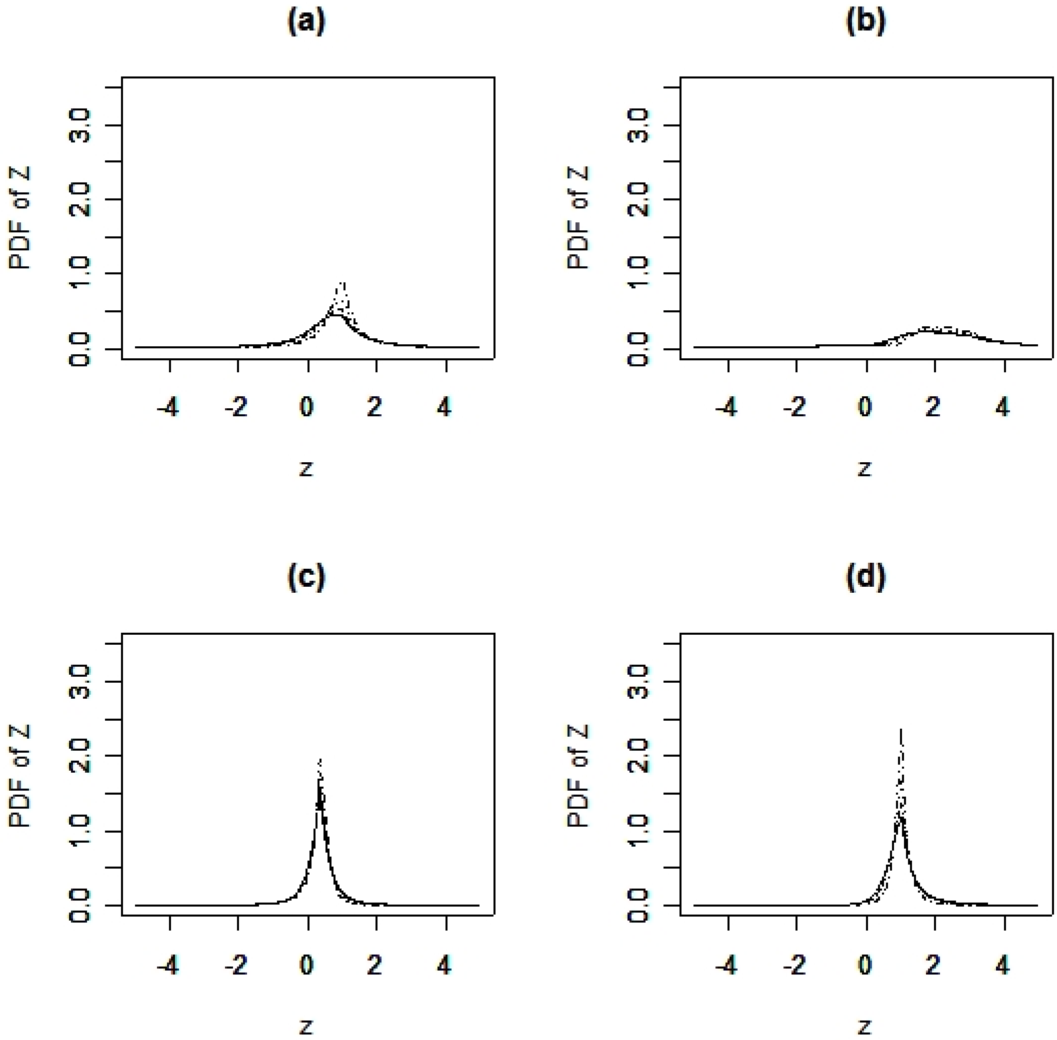

Fig.1 Plots of the pdfs (11)-(12) for $a = 0, b = 1$ and (a): $\alpha = 1$ and $\beta = 1$; (b): $\alpha = 1$ and $\beta = 3$; (c): $\alpha = 3$ and $\beta = 1$; (d): $\alpha = 3$ and $\beta = 3$. The four curves in each plot are: the solid curve for ρ =0.2; the curve of dots for ρ =0.4; the curve of dots for ρ =0.6; and, the curve of dots and dots for ρ =0.8.

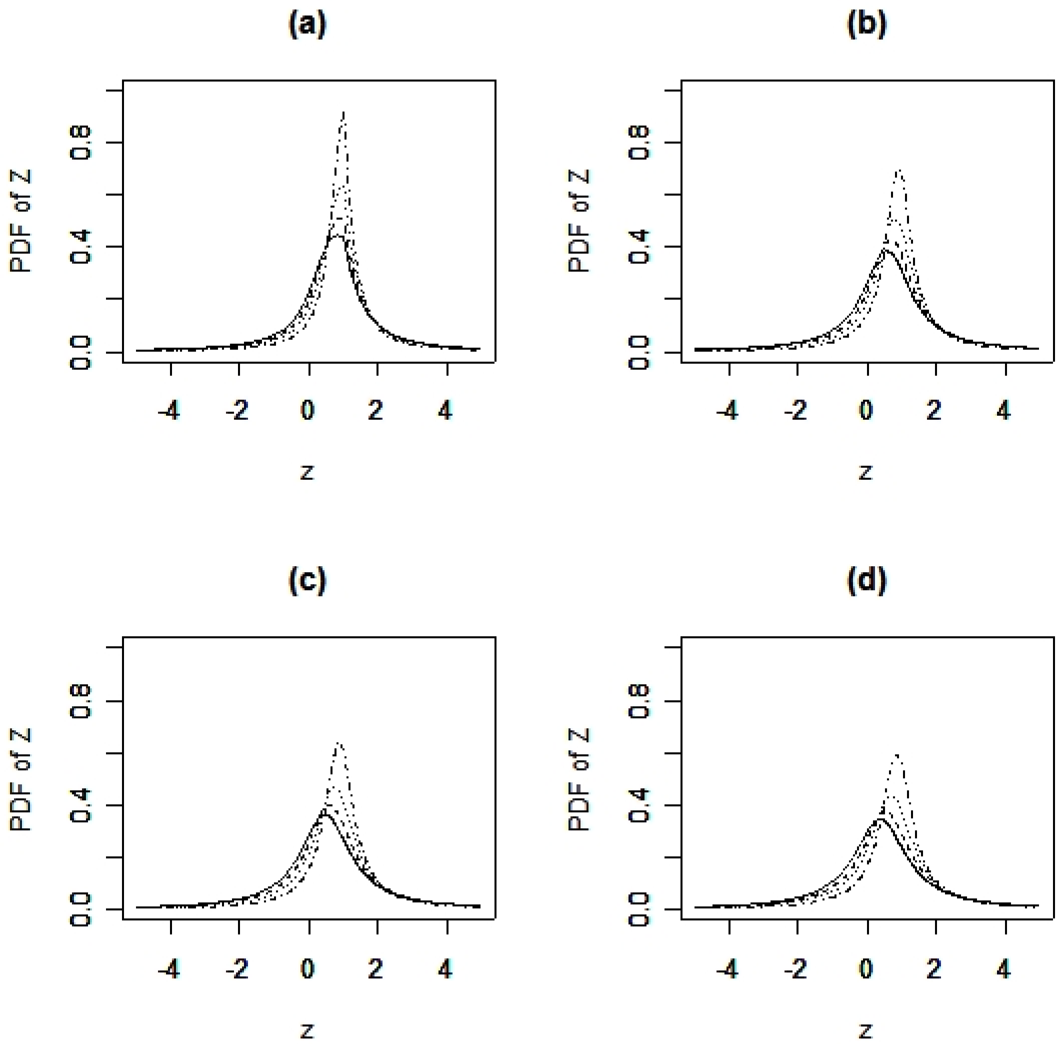

Fig.2 Plots of the pdfs (11)-(12) for $\alpha = 1, \beta = 1, b = 1$ and (a): $a = 0$; (b): $a = 0.5$; (c): $a = 1$; and (d): $a = 2$. The four curves in each plot are: the solid curve for ρ =0.2; the curve of lines for ρ =0.4; the curve of dots for ρ =0.6; and, the curve of lines and dots for ρ =0.8.

Corollary 1 considers a particular form for the pdf of Z for the case $\alpha = \beta = 0$. Note that the resulting pdf is elementary and depends only on ρ.

Corrolary 1: If X and Y are jointly distributed according to (1) and if $\alpha = \beta = 0$ *then the pdf of Z reduces to*

$$f(z) = \frac{2g(\arctan(z))}{1+z^2}, \tag{18}$$

where

$$g(\theta) = \frac{\sqrt{1-\rho^2}}{2\pi\{1-\rho\sin(2\theta)\}}$$

for $\theta \in [0, 2\pi)$.

Proof: Note that in this case (13) takes the form

$$g(t,\theta) = MA^{a/2} \mid t \mid^{a+1} K_a \left(\frac{\sqrt{A} \mid t \mid}{b\sqrt{1-\rho^2}} \right),$$

where A is given by (2). Integrating this over $0 \leq t < \infty$ yields the form for *g(.)* given in the corollary. The result of the corollary follows by noting that the pdf of $Z = \tan\theta$ can be expressed *as*

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1+z^2} \qquad (19)$$

and that $g(\arctan(\theta)) + g(\pi + \arctan(\theta))$ for the form for *g(.).*

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