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ON THE USE OF A THEOREM BY V. VÂLCOVICI IN PLANAR MOTION DYNAMICS

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A theorem by V. Vâlcovici that generalizes Koenig's well known angular momentum theorem finds its use in planar motion dynamics. The reference point in this type of motion is the instant center. The advantage in choosing the instant center as the reference point in angular momentum theorems is that unknown constraint forces do not appear in the equations of motions.

Key words: relative and absolute angular momentums, instant center, centrodes

1. INTRODUCTION

Consider generally a deformable body S of mass m, occupying at moment t the three dimensional domain $D(t)$. We frequently use Koenig's angular momentum theorem

$$\vec{K}_O = m\overrightarrow{OC} \times \vec{V}_C + \vec{K}'_C , \tag{1}$$

where $\vec{K}_O$ stands for the angular momentum of the body about fixed point O, $\vec{V}_C$ for the velocity of centroid C and $\vec{K}'_C$ for its relative angular momentum about its center of mass

$$\vec{K}'_C = \iiint\limits_{D(t)} \overrightarrow{CM} \times \vec{v}_M^{(C)} dm , \tag{2}$$

$\vec{v}_M^{(C)}$ being the relative velocity of point M with respect to C. Koenig's theorem was generalized by V. Vâlcovici [1] by considering instead of C an arbitrary point A, not necesarily on the body, origin of a frame of reference with axes of constant directions in space:

$$\vec{K}_O = m\overrightarrow{OA} \times \vec{V}_C + m\overrightarrow{AC} \times \vec{V}_A + \vec{K}'_A \tag{3}$$

where

$$\vec{K}'_A = \iiint\limits_{D(t)} \overrightarrow{AM} \times \vec{v}_M^{(A)} dm \tag{4}$$

is the relative angular momentum with respect to point *A*; $\vec{v}_M^{(A)}$ has a similar significance as $\vec{v}_M^{(C)}$. The absolute angular momentum in A is

$$\vec{K}_A = \iiint\limits_{D(t)} \overrightarrow{AM} \times \vec{V}_M \, dm \tag{5}$$

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Since

$$\vec{K}_O = \vec{K}_A + \overrightarrow{OA} \times \vec{H} \quad , \tag{6}$$

where $\vec{H} = m\vec{V}_C$ stands for the momentum of the body it follows from (3) and (4) that

$$\vec{K}_A = m\overrightarrow{AC} \times \vec{V}_A + \vec{K}'_A \tag{7}$$

For $A \equiv C$ it results

$$\vec{K}_C = \vec{K}'_C \tag{8}$$

V. Vâlcovici's relative angular momentum theorem with respect to point A is

$$m\overrightarrow{AC} \times \dot{\vec{V}}_A + \dot{\vec{K}}'_A = \vec{M}_A . \tag{9}$$

Here we identify $\vec{M}_A$ as the moment about point A of all external forces acting on the body. Note that the absolute angular momentum theorem about the same point is written as

$$\vec{V}_A \times \vec{H} + \dot{\vec{K}}_A = \vec{M}_A . \tag{9'}$$

For a rigid body of angular velocity $\vec{\omega}$ and inertia tensor about point A, $\overline{\overline{J}}_A$, we have

$$\begin{gathered} \vec{V}_A = \vec{v}_A \quad , \quad \vec{v}_M = \vec{v}_A + \vec{\omega} \times \overline{AM} \quad , \quad \vec{v}_M^{(A)} = \vec{v}_M - \vec{v}_A = \vec{\omega} \times \overline{AM} \quad , \\ \vec{K}'_A = \iiint_D \left[\overline{AM} \times \left(\vec{\omega} \times \overline{AM} \right) \right] \, \mathrm{d}m = \overline{\overline{J}}_A \vec{\omega} \end{gathered} \tag{10}$$

2. RELATIVE ANGULAR MOMENTUM THEOREMS FOR PLANAR MOTIONS

A remarkable point in planar motions is the instant center. It is convenient to use the equation of relative angular momentum (9) about the instant center I since in many planar motion problems reaction forces concur in this point. This circumstance is not general, but includes the important case of rolling without slipping over a fixed obstacle. However, care must be taken to distinguish I as either a point on the space centrode or a point on the body centrode (I_*)(fig. 1) since

$$\dot{\overrightarrow{IM}} = \vec{\omega} \times \overrightarrow{IM} - \vec{u}_I \quad , \quad \dot{\overrightarrow{I_* M}} = \vec{\omega} \times \overrightarrow{I_* M} \tag{11}$$

$\vec{u}_I$ being the velocity of I on the space centrode.

2.1. Relative angular momentum theorem about I, point on the space centrode (I nonmaterial point)

Relative angular momentum about point I is

$$\begin{aligned} \vec{K}'_I &= \iint_D \left(\overrightarrow{IM} \times \vec{v}_M^{(I)} \right) dm = \iint_D \left[\overrightarrow{IM} \times \left(\vec{\omega} \times \overrightarrow{IM} - \vec{u}_I \right) \right] dm = \\ &= J_I \vec{\omega} - m\overrightarrow{IC} \times \vec{u}_I = \left(J_C + m\overrightarrow{IC}^2 \right) \vec{\omega} - m\overrightarrow{IC} \times \vec{u}_I \quad , \end{aligned} \tag{12}$$

where J_C and $J_I = \iint_D \overrightarrow{IM}^2 dm$ represent the inertia moments of the rigid body with respect to axes perpendicular to the plane of motion in C and I. Equation (9) yields in this case

$$m\overrightarrow{IC}\times\dot{\vec{u}}_I+\left(J_C+m\overrightarrow{IC}^2\right)\dot{\vec{\omega}}+m\vec{\omega}\frac{d}{dt}\overrightarrow{IC}^2-m\,\dot{\overrightarrow{IC}}\times\vec{u}_I-m\overrightarrow{IC}\times\dot{\vec{u}}_I=\vec{M}_I \tag{13}$$

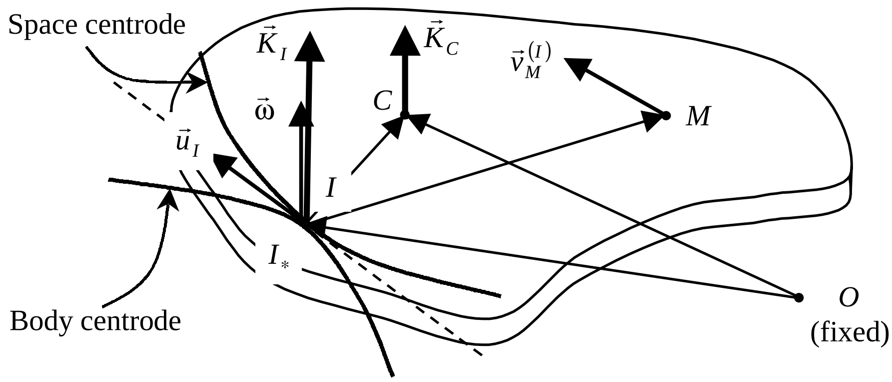

Fig.1 Flat plate in planar motion.

It follows from the substitution of the first relation (11) into (13) that

$$\dot{\overrightarrow{IC}}\times\vec{u}_I=\dot{\overrightarrow{IC}}\times\left(\vec{\omega}\times\overrightarrow{IC}-\dot{\overrightarrow{IC}}\right)=\vec{\omega}\left(\overrightarrow{IC}\cdot\dot{\overrightarrow{IC}}\right)=\frac{\vec{\omega}}{2}\frac{d}{dt}\overrightarrow{IC}^2 \tag{14}$$

Finally we have

$$\left(J_C+m\overrightarrow{IC}^2\right)\dot{\vec{\omega}}+\frac{m\vec{\omega}}{2}\frac{d}{dt}\overrightarrow{IC}^2=\vec{M}_I\,. \tag{15}$$

2.2. Relative angular momentum theorem about I, point on the body centrode ($I\equiv I_*$ material point)

For a point A on a rigid body one can write

$$\vec{K}'_A=\iint_D\left(\overrightarrow{AM}\times\vec{v}_A^{(M)}\right)dm=\iint_D\left[\overrightarrow{AM}\times\left(\vec{\omega}\times\overrightarrow{AM}\right)\right]dm=J_A\vec{\omega}, \tag{16}$$

$$\dot{\vec{K}}'_A=J_A\dot{\vec{\omega}}+\vec{\omega}\times\left(J_A\vec{\omega}\right)=J_A\dot{\vec{\omega}}, \tag{17}$$

where J_A is the inertia moment of the rigid with respect to an axis perpendicular to the plane of motion in A. Make use of equation (9) replacing A by I_*:

$$m\overrightarrow{IC}\times\vec{a}_{I_*}+\left(J_C+m\overrightarrow{IC}^2\right)\dot{\vec{\omega}}=\vec{M}_I\,. \tag{18}$$

For the acceleration of point I_* one can use the formula [4]

$$\vec{a}_{I_*}=\vec{\omega}^2\frac{\dfrac{\vec{n}'}{\rho'}-\dfrac{\vec{n}}{\rho}}{\left(\dfrac{\vec{n}'}{\rho'}-\dfrac{\vec{n}}{\rho}\right)^2}\,. \tag{19}$$

Here $\vec{n}, \vec{n}', \rho, \rho'$ stand for the unit vectors of the normals of the centrodes, respectively their curvature radiuses in I.

3. EXAMPLE

We illustrate the use of the two relative angular momentum theorems given above with the example of an excentric roller of mass m, radius r and central moment of inertia $J_C = mk^2$ rolling without slipping on the inside of a cylindrical surface of radius $R > r$ (fig. 2) [3].

$$OC = e, \quad \omega = \dot{\varphi},$$

$$\psi = \frac{r}{R-r}\varphi \quad , \quad \theta = \frac{R}{R-r}\varphi \ ,$$

$$\vec{n} \equiv \vec{n}', \quad \rho = R, \quad \rho' = r,$$

$$\vec{a}_{I_*} = \frac{rR}{R-r}\dot{\varphi}^2 \vec{n}' \ ,$$

$$\overrightarrow{IC} \times \vec{a}_{I_*} = \frac{erR}{R-r}\dot{\varphi}^2 \sin\theta \frac{\vec{\omega}}{\omega} \ ,$$

$$\overrightarrow{IC}^2 = e^2 + r^2 - 2er\cos\theta \ ,$$

$$\vec{M}_I = -mg(r\sin\psi + e\sin\varphi)\frac{\vec{\omega}}{\omega}.$$

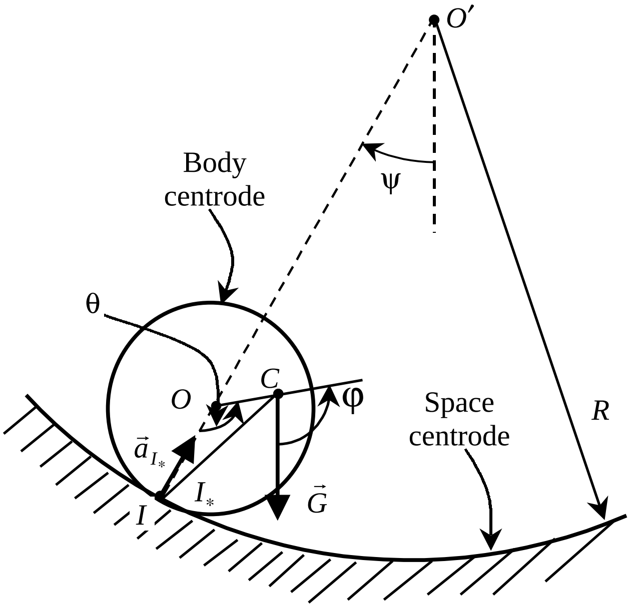

Fig.2 Excentric roller rolling without slipping on the inside of a cylindrical surface

Any of the two equations (15) or (18) yields the following equation of motion

$$\left(k^2 + e^2 + r^2 - 2er\cos\theta\right)\ddot{\varphi} + \frac{erR}{R-r}\dot{\varphi}^2 \sin\theta = -g\left(r\sin\psi + e\sin\varphi\right),$$

which for small amplitudes is simplified to

$$\left[k^2 + (r-e)^2\right]\ddot{\varphi} = -g\frac{r^2 + e(R-r)}{R-r}\varphi.$$

4. REMARKS

4.1. The relative angular momentum about point I_* is

$$\vec{K}'_{I_*} = \left(J_C + m\overrightarrow{IC}^2\right)\vec{\omega}. \tag{20}$$

Despite the fact that $\overrightarrow{IC}^2$ is generally a function of time

$$\dot{\vec{K}}'_{I_*} = \left(J_C + m\overrightarrow{IC}^2\right)\dot{\vec{\omega}}, \tag{21}$$

that is differentiation of $\vec{K}'_{I_*}$ with respect to time is effective only on the angular velocity $\vec{\omega}$ and not on the moment of inertia $J_{I_*} = J_C + m\overrightarrow{IC}^2$. The explanation resides in the fact that in the general case of formula (17) J_A is a constant, A being a point of the rigid. On the contrary, when we refer to I on the space centrode, differentiation of $\vec{K}'_I$ with respect to time is complete since I does not belong to the rigid.

4.2. If relation (7) is written with respect to point I_* we have

$$\vec{K}_{I_*} = \vec{K}'_{I_*}. \tag{22}$$

For planar motions the equality of relative and absolute angular momentums with respect to the I, point on the body centrode is valid not only in the centers of mass, but in the instant centers also.

4.3. The following relation is available from [2]

$$\vec{a}_{I_*} = \vec{u}_I \times \vec{\omega}. \tag{23}$$

Substitution of (23) into (18) yields a slightly modified form for (18):

$$\left(J_C + m\overrightarrow{IC}^2\right)\dot{\vec{\omega}} - m\vec{\omega}\left(\overrightarrow{IC} \cdot \vec{u}_I\right) = \vec{M}_I. \tag{24}$$

This equation is to be used mainly when the velocity of the instant center on the space centrode can be easily calculated.

4.4. If general formula (6) is written with respect to points A and C, that is

$$\vec{K}_A = \vec{K}_C + \overrightarrow{AC} \times \vec{H} \tag{25}$$

and since

$$\vec{V}_C - \vec{V}_A = \dot{\overrightarrow{AC}} \tag{26}$$

relations (7) and (8) will yield

$$\vec{K}'_A = \vec{K}'_C + m\overrightarrow{AC} \times \dot{\overrightarrow{AC}} \tag{27}$$

Relation (27) is the equivalent of relation (25) in terms of relative angular momentums. Differentiation with respect to time of this relation gives

$$\dot{\vec{K}}'_A = \dot{\vec{K}}'_C + m\overrightarrow{AC} \times \ddot{\overrightarrow{AC}}. \tag{28}$$

Substitution of (28) into (9) yields

$$m\overrightarrow{AC} \times \vec{a}_C + \dot{\vec{K}}'_C = \vec{M}_A. \tag{29}$$

Finally, for a planar motion the issue of distinguishing point I being either on the space centrode or on the body centrode does not arise anymore if one replaces in equation (29) A with I:

$$m\overrightarrow{IC} \times \vec{a}_C + J_C\dot{\vec{\omega}} = \vec{M}_I \tag{30}$$

Equation (30) represents a new form for the relative angular momentum theorems with respect to the instant center I.

4.5. Equations (15) and (18) have both been obtained in [4] by other considerations. Together with equation (30) they are the main results of this paper.

5. CONCLUSIONS

This paper derives equations of planar motion by using a generalization of Koenig's angular momentum theorem. The reference point is the instant center. The advantage of using this point for relative angular momentum theorems is the elimination of the unknown reaction forces from the equations of motion. Instant center must be distinguished either on the space centrode or on the body centrode. Specific equations are derived in each case. However, the two forms are unified in equation (30) that makes no difference regarding the centrode to be considered.

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