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INSTANTANEOUS INTEREST RATES AND HAZARD RATES

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An instantaneous interest rate (IIR) is a CADLAG function $\delta:[0,\infty) \to 0,\infty)$ which has the meaning that for 1 monetary unit (MU) borrowed at time $t = 0$ costs $\sigma(t_1) = \exp\left(\int_0^{t_1} \delta(x)\mathrm{d}x\right)$ MU at moment $t = t_1$. The mapping $\underline{F}(t) = 1/\sigma(t)$ is the survival function of some lifetime τ. In this framework, δ is the failure rate (FR) of τ. We investigate the analogy IIR – FR in the case of credit reimbursement. We say that a IIR δ is of positive type if for any constant cash flow on the interval $[0,T]$ the flow of principals is non-negative. We prove that δ is of positive type iff τ is a lifetime with decreasing mean rezidual life (DMRL) .

1. CREDIT REIMBURSEMENT. DISCRETE TIME

We deal with two partners: the creditor **C** and the debitor **D**. At moment $t_0 = 0$, **C** lends to **D** a cash amount *C*. After a deal, the two partners agree to a ***reimbursement schedule.*** They agree that the ***instantaneous interest rate*** is δ.

Definition 1.1. *An instantaneous interest rate* (IIR) *is any function* $\delta:[0,\infty) \to [0,\infty)$ *which is right-continuous and has limit to the left. Its meaning is that* 1 MU *borrowed at time* $t_0 = 0$ *costs* $\sigma(t_1) = \exp\left(\int_0^{t_1} \delta(x)\mathrm{d}x\right)$ MU *at time* $t_1 = t$. *We call the function* σ *the* ***fructification factor***. *For any function which is right continuous and with finite limits to the left we shall use the abbreviation* ***CADLAG***.

Definition 1.2. *A* ***reimbursement schedule*** *of the credit C in n installments on the interval* $[0,T]$ *with* IIR δ *is any system* (D,R,C,δ) *where* $D = \{0 = t_0 < t_1 < \ldots < t_n = T\}$, $R = (r(t_1), r(t_2),\ldots, r(t_n))$. *The number* t_j *is the moment of the jth payment and the quantity* R_j *is the value of the jth payment. The reimbursement condition is*

$$r(t_1)\exp\left(-\int_0^{t_1}\delta(u)\mathrm{d}u\right) + r(t_2)\exp\left(-\int_0^{t_2}\delta(u)\mathrm{d}u\right) + \ldots + r(t_n)\exp\left(-\int_0^{t_n}\delta(u)\mathrm{d}u\right) = C. \tag{1.1}$$

The motivation of (1.1) is that an amount of $R(t_j)$ MU paid at moment t_j has the same value as an amount of $r(t_j)\exp\left(-\int_0^{t_j}\delta(u)\mathrm{d}u\right)$ MU at moment $t_0 = 0$. Notice that the first payment is made at moment t_1 : we deal with *posticipated* payments.

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Definition 1.3. *The function* $\underline{F}$:[0,∞) → [0,1] *defined by* $\underline{F}(t) = e^{-\int_0^t \delta(u)\mathrm{d}u} = \dfrac{1}{\sigma(t)}$ *is called the **actualization factor***.

Using this notation, the reimbursement condition becomes

$$r(t_1)\,\underline{F}(t_1) + r(t_2)\,\underline{F}(t_2) + \ldots + r(t_n)\,\underline{F}(t_n) = C \tag{1.2}$$

Notice that we accepted that $\delta \geq 0$. It seems natural to be so: a negative interest rate has no economic meaning. It seems also natural to consider

Definition 1.4. *An* IIR δ *is called **natural** iff* $\int_0^\infty \delta(u)\mathrm{d}u = \infty$.

Thus, for a natural IIR the function $\underline{F}$ has the following properties: $\underline{F}(0) = 1$, $\underline{F}$ is non-increasing and $\underline{F}(\infty) = 1$. In that case the function $F(t) = 1 - \underline{F}(t)$ is non-decreasing, continuous, $F(0) = 0$ and $F(\infty) = 1$. So, F is a distribution function of some non-negative random variable τ. This random variable can be interpreted as being a lifetime – so, F is a *life distribution*. Moreover, since the mapping $t \mapsto \int_0^t \delta(u)\mathrm{d}u$ is right-differentiable, it is absolutely continuous. It has a density $f(t) = \delta(t)\underline{F}(t)$. In this interpretation we can write

$$\delta(t) = \lim_{h\downarrow 0}\frac{\underline{F}(t)-\underline{F}(t+h)}{h\underline{F}(t)} = \lim_{h\downarrow 0}\frac{P(\tau - t \leq h \mid \tau > t)}{h}\;. \tag{1.3}$$

In this form, δ has been intensively studied in reliability theory under the name of *failure rate* ([1],[2],[3], [5]) or *hazard rate* ([2], [6[) and in demography and actuaries under the name of *mortality rate* ([4] or even *mortality force* ([7]) .

Conclusion: one may think of δ as being the hazard rate of a lifetime τ. If $\int_0^\infty \delta(u)\mathrm{d}u < \infty$, then this lifetime τ may also assume the value +∞ with probability $\underline{F}(\infty) = \exp\left(-\int_0^\infty \delta(u)\mathrm{d}u\right)$.

Using this similarity, in the case of natural IIR's, the reimbursement condition (1.2) has a probabilistic interpretation: it is the expectation of some discrete random variable constructed using D. Namely, let us add to D the point $t_{n+1} = \infty$. Let also $R(t_0) = 0$, $R(t_j) = R(t_{j-1}) + r(t_j) = \sum_{k=0}^{j} r(t_k)$ Consider the discretization of τ denoted by τ_D given by

$$\tau_D = \sum_{j=0}^{n} t_j 1_{\{\tau\in[t_j,t_{j+1})\}} = t_0 1_{\{\tau<t_1\}} + t_1 1_{\{t_1\leq\tau<t_2\}} + t_2 1_{\{t_2\leq\tau<t_3\}} + \ldots + t_n 1_{\{t_n\leq\tau<t_{n+1}\}} \tag{1.4}$$

Proposition 1.1. *If an IIR* δ *is natural, then the reimbursement condition* (1.2) *is equivalent to the fact that* $\mathrm{E}R(\tau_D) = C$.

*Proof***.** We have $\mathrm{E}R(\tau_D) = \sum_{j=0}^{n} R(t_j)\,P(t_j \leq \tau < t_{j+1}) = \sum_{j=0}^{n} R(t_j)\,(\underline{F}(t_j) - \underline{F}(t_{j+1})) = \sum_{j=0}^{n} R(t_j)\underline{F}(t_j) - \sum_{j=1}^{n+1} R(t_{j-1})\underline{F}(t_j)) = \sum_{j=1}^{n} R(t_j)\underline{F}(t_j) - \sum_{j=1}^{n} R(t_{j-1})\underline{F}(t_j))$ (as $\underline{F}(t_{n+1}) = \underline{F}(\infty) = R(t_0) = 0$!)

$= \sum_{j=1}^{n} (R(t_j) - R(t_{j-1})\underline{F}(t_j) = \sum_{j=1}^{n} r(t_j)\underline{F}(t_j)$ and by (1.2) the last sum is equally to C.

A payment $R(t_j)$ has two components: *the principal* and *the interest.* The principal, denoted by $a(t_j)$ is the fraction of the debt C which is paid by the installment $R(t_j)$ while the interest, denoted by $d(t_j)$, is an extra pay meaning the cost of the credit. It is accepted that if we denote $S(t_0) = C$, $S(t_1) = C - a(t_1),\ldots, S(t_j) = S(t_{j-1}) - a(t_j)$,… (thus $S(t_n) = 0$!) the *remaining debt after the jth payment,* then the interest produced by $S(t_{j-1})$ on the interval between two successive payments $[t_{j-1}, t_j)$ is equal to $S(t_{j-1})\left(\exp\left(\int_{t_{j-1}}^{t_j}\delta(u)\mathrm{d}u\right)-1\right)$.

If we denote, as usual in banking and accounting,

$$i_k := \exp\left(\int_{t_{k-1}}^{t_k}\delta(u)\mathrm{d}u\right)-1 = \frac{F(t_k)-F(t_{k-1})}{\underline{F}(t_k)} \tag{1.5}$$

then the connection between installments, principals and interests is given by the well know relation (see, for instance [4])

$$r(t_1) = Ci_1 + a(t_1) = S(t_0)\,(1 + i_1) - S(t_1)\,,\ \ldots.,\ r(t_k) = S(t_{k-1})(1 + i_k) - S(t_k)\,,\ \ 1 \le k \le n \tag{1.6}$$

This means that if we know the installment $r(t_j)$ we can compute the principal $a(t_j)$ and conversely, if the credit reimbursement schedule contains the principals $a(t_j)$ only, one can compute the installments.

Equation (1.6) do not have an immediate probabilistic meaning. However, we can state

Proposition 1.2. *If* τ_D *has the same meaning as in Proposition* 1.1. *then* (1.6) *becomes*

$$\mathrm{E}(R(\tau_D)\,;\,\tau_D \le t_{m-1}) = C - (S(t_m) + r(t_m))P(\tau > t_m),\ \ 1 \le m \le n \tag{1.7}$$

Proof. Write (1.6) under the form

$$(R(t_k) - R(t_{k-1})\underline{F}(t_k) = S(t_{k-1})\underline{F}(t_{k-1}) - S(t_k)\underline{F}(t_k)\,,\ \ 1 \le k \le n \tag{1.8}$$

and add them for $k = 1$ to $k = m$. As $R(t_0) = 0$, $\underline{F}(t_0) = 1$ and $S(t_0) = C$, we get $R(t_1)(\underline{F}(t_1) - \underline{F}(t_2)) + R(t_2)(\underline{F}(t_2) - \underline{F}(t_3)) + \ldots + R(t_{m-1})(\underline{F}(t_m) - \underline{F}(t_{m-1})) + R(t_m)\underline{F}(t_m) = C - S(t_m)\underline{F}(t_m)$ or $R(t_1)P(\tau_D = t_1) + R(t_2)\,P(\tau_D = t_2) + \ldots + R(t_{m-1})\,P(\tau_D = t_{m-1}) = C - S(t_m)\underline{F}(t_m) - R(t_m)\underline{F}(t_m)$; this is (1.7).

2. CONTINUOUS CASH FLOW. ANALOGY IIR – FAILURE RATE

Now, we shall assume that the reimbursement is made by a *cash flow*.

Definition 2.1. *A* ***cash flow*** *is any CADLAG function* r: $[0,\infty) \to [0,\infty)$. *Notice that the function* $R(t) = \int_0^t r(s)\mathrm{d}s$ *does exist and is right-differentiable. Moreover, if* R' *is its right derivative , then* $R' = r$.

The meaning is that **D** and **C** accept a *continuous reimbursement schedule* using r, given an IIR δ. If **C** lends to a **D** capital amount of C MU, the reimbursement condition is that

$$\int_0^\infty r(s)e^{-\int_0^s \delta(u)\mathrm{d}u}\,\mathrm{d}s = C. \tag{2.1}$$

Definition 2.2. *We denote such a* ***reimbursement schedule*** *by* (r,C,δ). *If* $R(\infty) < \infty$ *then r is called* ***proper.*** *If* $r(t) = 0$ *for t greater than some T, then* (r,C,δ) *will be called* ***natural.***

If $\underline{F}$ and τ have the same meaning as in the first section, the reimbursement condition is

$$\int_0^\infty r(s)\underline{F}(s)\,\mathrm{d}s = C. \tag{2.2}$$

The analog of Proposition 1.1. is

Proposition 2.1. *The reimbursement condition* (2.2) *is equivalent to* $\mathrm{E}R(\tau) = C$.
Proof. Remark that $R(0) = 0$ and use integration by parts:

$$\mathrm{E}R(\tau) = \mathrm{E}R(\tau) - R(0) = \int_0^\infty R'(s)P(\tau > s)\,\mathrm{d}s = \int_0^\infty r(s)\underline{F}(s)\,\mathrm{d}s$$

The cash flow has two components: the *flow of principals* and *the flow of interests.* The first one will be denoted by a and the second by d.

Mathematically, $r(t) = a(t) + d(t)$, where $d(t)$ is the interest paid for the remaining debt and $a(t)$ is the flow of principals.

The condition for a to be a flow of principals for the credit C is that $\int_0^\infty a(s)ds = C$. Let, as before, $S(t)$ $= S(t) = \int_t^\infty a(t)\mathrm{d}s = C - \int_0^t a(t)\mathrm{d}s$ denote the remaining debt at moment t.

We want to find the relationship between r and a. Let us accept that a debt of $S(t)$ MU left unpaid in the interval $[t,t+h)$ yields an interest $d(t,t+h)$ $= \left(\exp\left(\int_t^{t+h}\delta(u)\mathrm{d}u\right) - 1\right)S(t)$ MU. If we let $h \to 0$ and use the right continuity of δ we infer that

$$\lim_{h\downarrow 0}\frac{\mathrm{d}(t,t+h)}{h} = \delta(t)S(t). \tag{2.3}$$

Using this fact we get the following result.

Proposition 2.2. *Suppose that a is a CADLAG flow of principals for the debt C and* δ *is a natural IIR. Then the reimbursement schedule is*

$$r(t) = a(t) + \delta(t)S(t) \tag{2.4}$$

Therefore the analog of (1.7) *is*

$$\mathrm{E}(R(\tau);\ \tau < t) = C - (R(t) + S(t))\,P(\tau > t) \tag{2.1}$$

Moreover, the mapping $t \to (R(t) + S(t))\,P(\tau > t)$ *is non-increasing.*

Proof. We have to check that the reimbursement condition $\int_0^\infty (a(t) + \delta(t)S(t))\underline{F}(t)\mathrm{d}t = C$ holds. But, by our assumptions, S and $\underline{F}$ are right-differentiable and $S' = -a$, $\underline{F}' = -\delta\underline{F}$. This means that

$$\int_0^\infty (a(t) + \delta(t)S(t))\underline{F}(t)\mathrm{d}t = -\int_0^\infty (S'(t)\underline{F}(t) + S(t)\underline{F}'(t))\mathrm{d}t = -\int_0^\infty (S\underline{F})'(t)\mathrm{d}t = \underline{F}(0)S(0) - \underline{F}(\infty)S(\infty).$$

As $\underline{F}(0) = 1$, $S(0) = C$ and $S(\infty) = 0$, it follows that $\int_0^\infty r(t)\,\underline{F}(t)\mathrm{d}t = C$. Moreover, replacing the integration limits by t_1 and t_2, $t_1 < t_2$, we get the formula

$$\int_{t_1}^{t_2} r(t)\underline{F}(t)\mathrm{d}t = \underline{F}(t_1)S(t_1) - \underline{F}(t_2)S(t_2)\,, \tag{2.6}$$

which implies in particular that

$$\int_0^t r(t)\underline{F}(t)\mathrm{d}t = C - \underline{F}(t)S(t)\ . \tag{2.7}$$

If we use again the integration by parts formula in the form

$$\mathrm{E}f(\tau) = f(0) + \int_0^\infty r(t)\underline{F}(t)\mathrm{d}t, \tag{2.8}$$

which holds for any continuous right-differentiable function f (see, for instance, [8]), for the particular function $f(x) = R(x \wedge t)$ equation (2.7) becomes

$$\mathrm{E}(R(\tau \wedge t) = C - S(t)\, P(\tau > t), \tag{2.9}$$

which is the same as the claim (2.5).

Finally, the last claim is obvious from (2.6) : the cash flow r is non-negative.

The above result can be used in two ways: the first problem is to find r knowing a while the second one is to find a knowing r.

Proposition 2.3. *Suppose one has a reimbursement schedule for the principals, i.e. a CADLAG mapping a from* $[0,\infty)$ to $[0,\infty)$ *such that* $\int_0^\infty a(t)\mathrm{d}t < \infty$. *Let* $S(t) = \int_0^\infty r(t)\mathrm{d}t$. *Then r is proper if*

$$E\left(\frac{S(\tau)}{\underline{F}(\tau)}\right) < \infty\ . \tag{2.10}$$

Proof. By (2.5), $R(\infty) = \int_0^\infty \left[a(t) + \delta(t)S(t)\mathrm{d}r = C + \int_0^\infty \delta(t)S(t)\mathrm{d}r \right]$. Thus, $R(\infty) < \infty$ is the same as $\int_0^\infty \delta(t)S(t)\mathrm{d}r < \infty$. Since $\delta = \frac{f}{\underline{F}}$, where $f = -\underline{F}'$ is the density of F, $\int_0^\infty \delta(t)S(t)\mathrm{d}t = \int_0^\infty \frac{S(x)}{\underline{F}(x)} f(x)\mathrm{d}x = E\left(\frac{S(\tau)}{\underline{F}(\tau)}\right)$. We proved (2.10) and, moreover, the equality $R(\infty) = C + E\left(\frac{S(\tau)}{\underline{F}(\tau)}\right)$.

In the second case, one knows r and wants to find a. If we suppose that R and δ are differentiable, then (2.5) involves an integral equation with one unknown function a. It is possible that this equation have no acceptable solution.

Definition 2.3. *Call a reimbursement schedule* (r,C,δ) ***realistic*** *if the integral equation* (2.4.) *has a non-negative solution, a, with the property that* $S(0) = \int_0^\infty a(s)\mathrm{d}s = C$.

Proposition 2.4.

(i) *If r is continuous and δ is differentiable, then a formal solution of* (2.4.) *is*

$$a(t) = r(t) - \frac{\delta(t)}{\underline{F}(t)}\left(C - \int_0^t r(s)\underline{F}(s)\mathrm{d}s\right) = r(t) - \frac{\delta(t)}{\underline{F}(t)}\int_t^\infty r(s)\underline{F}(s)\mathrm{d}s = r(t) - \delta(t)S(t) \tag{2.11}$$

(ii) *The equation*

$$S(t) = \mathrm{E}(R(\tau) - R(t) \mid \tau > t) \tag{2.12}$$

always holds. Moreover, if r is proper or if $\lim_{t\to\infty}\dfrac{r(t)}{\delta(t)} = 0$ *then* $S(\infty) = 0$ *thus* $S(t) = \int_t^\infty a(s)\mathrm{d}s$.

(iii) *If r is proper or if* $\lim_{t\to\infty}\dfrac{r(t)}{\delta(t)} = 0$, *then* (r,C,δ) *is realistic iff the map* $t \mapsto \mathrm{E}(R(\tau) - R(t)\,|\,\tau > t)$ *is non increasing. An equivalent condition is that*

$$\int_t^\infty r(x)\underline{F}(x)\mathrm{d}x \le \frac{r(t)\underline{F}(t)}{\delta(t)}\ \forall\, t \ge 0\,. \tag{2.13}$$

Proof.

(i) By our assumptions, a is continuous, hence S is differentiable. Moreover, $S' = -a$ hence (2.5) becomes $S'(t) = \delta(t)S(t) - r(t)$ with the initial condition $S(0) = C$. This is a linear differential equation. If one solves it using the method of variation of constants, one gets $S(t) = \dfrac{\int_t^\infty r(x)\underline{F}(x)\mathrm{d}x}{\underline{F}(t)}$ which, by (2.2), implies that $S(0) = C$. Taking the derivative of S one get (2.11).

(ii). The integral equation $a(t) = r(t) - \delta(t)\int_t^\infty a(s)\mathrm{d}s$, $\int_0^\infty a(s)\mathrm{d}s = C$ implies that $S'(t) = \delta(t)S(t) - r(t)$, $S(0) = C$, but they are not equivalent. Remark that $\int_0^\infty a(s)\mathrm{d}s = S(0) - S(\infty)$. If we want a to be a real reimbursement schedule then we should add the condition $S(\infty) = 0$. The equality $S(t) = \dfrac{\int_t^\infty r(x)\underline{F}(x)\mathrm{d}x}{\underline{F}(t)}$ always holds. By L'Hospital rule, $S(\infty) = \lim_{t\to\infty}\dfrac{-r(t)\underline{F}(t)}{\underline{F}'(t)} = \lim_{t\to\infty}\dfrac{r(t)}{\delta(t)}$ provided that the last limit exist. Thus a condition in order that $S(\infty) = 0$ would be $\lim_{t\to\infty}\dfrac{r(t)}{\delta(t)} = 0$. However, it is possible that this last limit does not exist and still $S(\infty) = 0$. That might happen if r is proper. In that case we need another proof.

First, we check (2.12). Remark that $\mathrm{E}(R(\tau) - R(t)\,|\,\tau > t) = \dfrac{\int (R(\tau)-R(t))1_{(\tau>t)}\mathrm{d}P}{\underline{F}(t)} = \dfrac{\int\left(\int r(x)1_{(t,\tau)}(x)\mathrm{d}\lambda(x)\right)1_{(\tau>t)}\mathrm{d}P}{\underline{F}(t)}$ (here λ is Lebesgue measure) $= \dfrac{\int\left(r(x)1_{[t,\infty)}(x)\left(\int 1_{(\tau>t)}\mathrm{d}P\right)\mathrm{d}\lambda(x)\right)}{\underline{F}(t)}$ (by Fubini !)$= \dfrac{\int_t^\infty r(x)\underline{F}(x)\mathrm{d}x}{\underline{F}(t)}$, hence we checked the claimed equality. If r is proper, then $R(\infty) < \infty$ so $0 \le S(\infty) = \lim_{t\to\infty}\mathrm{E}(R(\tau) - R(t)\,|\,\tau > t) \le \lim_{t\to\infty}\mathrm{E}(R(\infty) - R(t)\,|\,\tau > t) = \lim_{t\to}(R(\infty) - R(t)) = 0$, hence $S(\infty) = 0$.

(iii) We want that the function $S(t) = \mathrm{E}(R(\tau) - R(t)\,|\,\tau > t) = \dfrac{\int_t^\infty r(x)\underline{F}(x)\mathrm{d}x}{\underline{F}(t)}$ be non increasing $\Leftrightarrow S' \le 0$. But the condition $S' \le 0$ is exactly (2.13.)

Example 2.1. Suppose that $\delta(t) = \delta = \text{const}$. In this case $\underline{F}(t) = e^{-\delta t} \Leftrightarrow \tau \sim \text{Exponential}(\delta)$. If a is known, then $r(t) = \delta S(t)$; by (2.10) we get $R(\infty) = C + \mathrm{E}(e^{\delta\tau}S(\tau)) = C + \delta\int_0^\infty S(t)\mathrm{d}t$. If r is known and r is proper or $\lim_{t\to\infty} r(t) = 0$, then the flow of principals is given by $a(t) = r(t) - \delta S(t)$. The schedule (r, C, δ) is realistic iff $S(t) \le r(t) / \delta$.

Counterexample 2.1′. Consider the same IIR as before. Suppose that $r(t) = C\delta$, thus $R(t) = C\delta t$. *This is not a proper schedule.* As $R(\infty) = C\delta\mathrm{E}\tau = C$, the reimbursement condition (2.2) is fulfilled. However, this is not a realistic schedule: $S(t) = C\delta\mathrm{E}(\tau - t \mid \tau > t)$ is always equal to C, implying that $a = 0$. No matter how much **D** pays to **C** the debt remains the same ! On the contrary, if $r(t) = 2mt1_{[0,T]}(t)$ hence $R(t) = m(t\wedge T)^2$ with some constant m such that $m\mathrm{E}(\tau\wedge T) = C$, then $a(t) = r(t) - \delta S(t)$ implies $a(0) = -\delta S(0) = -\delta C < 0$. Now, r is proper, but not realistic.

Example 2.1″. Consider the same IIR. Let $r : [0,\infty) \to [0,\infty)$ be *non increasing* and suppose that $r(\infty) = 0$. Then (r, C, δ) is realistic. Indeed, we check that $a(t) \ge 0 \Leftrightarrow \delta S(t) \le r(t)$. Indeed,

$$\delta S(t) = \delta \frac{\int_t^\infty r(x)e^{-\delta x}\mathrm{d}x}{e^{-\delta t}} = \delta\int_t^\infty r(x)e^{-\delta(x-t)}\mathrm{d}x \le \delta\int_t^\infty r(t)e^{-\delta(x-t)}\mathrm{d}x \text{ (since } x \ge t \Rightarrow r(x) \le r(t)\text{ !)} = r(t).$$

Definition 2.4. (*see* [1],[2],[5]). *A lifetime* τ *is called a* DMRL (*Decreasing Mean Rezidual Life*) *iff the mapping* $\mathrm{E}(t) := \mathrm{E}(\tau - t \mid \tau > t)$ *is non increasing. If its failure rate* $\delta_\tau = f_\tau / \underline{F}_\tau$ *is nondecreasing, then* τ *is called an* IFR (*Increasing failure rate i*).

It is easy to see that if τ is an IFR, then τ also is a DMRL (see for instance [1]).

Definition 2.5. *Let* δ *be an* IIR. *We call* δ ***of positive type*** *iff for any non-increasing r such that* $\lim_{t\to\infty} \frac{r(t)}{\delta(t)} = 0$ *and for any credit C the reimbursement schedule* (r, C, δ) *is realistic.*

Now, we characterize positive type IIR's.

Proposition 2.5. *Let* δ *be an* IIR *and let* τ *be a lifetime with the property that the failure rate of* τ *is* δ. *Then*

(i) δ *is of positive type iff* τ *is a* DMRL;

(ii) *If* δ *is non decreasing then* δ *is of positive type*;

(iii) *If* δ *is periodic then* δ *is of positive type if and only if it is constant.*

Proof.

(i) Suppose that τ is a DMRL. Then $\mathrm{E}(t) = \frac{\int_t^\infty \underline{F}(x)\mathrm{d}x}{\underline{F}(t)}$ is non increasing $\Leftrightarrow E'(t) \le 0$. By differentiating one finds the equivalent condition

$$\tau \text{ is a DMRL} \Leftrightarrow \int_t^\infty \delta(t)\underline{F}(x)\mathrm{d}x \le \underline{F}(t) \quad \forall\, t \ge 0\,. \tag{2.14}$$

We want to prove that δ is a IIR of positive type. Let $r:[0,\infty) \to [0,\infty)$ be non increasing and $\lim_{t\to\infty} \frac{r(t)}{\delta(t)} = 0$.

Our task is to prove that $a(t) \ge 0 \Leftrightarrow \delta(t)S(t) \le r(t)$. As $S(t) = \frac{\int_t^\infty r(x)\underline{F}(x)\mathrm{d}x}{\underline{F}(t)}$ this is the same as

$$\int_t^{\infty} \delta(t)r(x)\underline{F}(x)\mathrm{d}x \le r(t)\underline{F}(t) \ \ \forall\, t \ge 0 . \qquad (2.15)$$

But $x \ge t \Rightarrow r(x) \le r(t) \Rightarrow \int_t^{\infty} \delta(t)r(x)\underline{F}(x)\mathrm{d}x \le \int_t^{\infty} \delta(t)r(t)\underline{F}(x)\mathrm{d}x = r(t)\int_t^{\infty} \delta(t)\underline{F}(x)\mathrm{d}x \le r(t)\underline{F}(t)$ because of (2.14). We checked (2.15).

Conversely, suppose that δ is of positive type. Choose $r(t) = r1_{[0,T]}(t)$. Then $R = C/I$, where $I = \int_0^T \underline{F}(t)\mathrm{d}t$. We know that $a(t) = C/I - \frac{\delta(t)}{\underline{F}(t)}\left(C - C/I\int_0^t \underline{F}(s)\mathrm{d}s\right) \ge 0 \ \forall\, C,T > 0$. It follows that if δ is of positive type then $\underline{F}(t) - \delta(t)\left(I - \int_0^t \underline{F}(s)\mathrm{d}s\right) \ge 0 \ \forall\, T \Leftrightarrow \underline{F}(t) - \delta(t)\int_t^T \underline{F}(s)\mathrm{d}s \ge 0 \ \forall\, T > 0$. Letting $T \to \infty$, (2.14) follows.

(ii) Obvious. Any IFR is a DMRL.

(iii) If δ is periodic (say, $\delta(t + p) = \delta(t) \ \forall\, t \ge 0$ for some $p > 0$) then $E(t)$ is periodic, too, since the failure rate of the rezidual lifetime $(\tau\text{-}t \mid \tau>t)$ is $\delta_t(x) = \delta(t+x)$. Then $\mathrm{E}(t)$ should be some constant: $\mathrm{E}(t) = \frac{1}{\alpha}$ for some α. Thus $\frac{\int_t^{\infty}\underline{F}(x)\mathrm{d}x}{\underline{F}(t)} = \frac{1}{\alpha} \Leftrightarrow \alpha\int_t^{\infty}\underline{F}(x)\mathrm{d}x = \underline{F}(t) \ \forall\, \alpha \Leftrightarrow \tau$ ~Exponential(α).

As a byproduct we notice

Corollary 2.6. *If* τ *is a* DMRL *and* $R{:}[0,\infty) \to [0,\infty)$ *is concave and increasing then* $R(\tau)$ *is a* DMRL, *too.*

Proof. As R is continuous and one-to-one, we just have to remark that $\mathrm{E}(R(\tau) - R(t) \mid R(\tau) > R(t)) = \mathrm{E}(R(\tau) - R(t) \mid \tau > t)$ and apply Proposition 2.5 (i).

Examples 2.2. The constant simple interest rate (i.e. $\delta(t) = \frac{i}{1+it}$, corresponding to a Pareto distribution) and the usual one (i.e. $\delta(t) = (1+i)^{[t]}\frac{i}{1+i\{t\}}$ with $[t]$ and $\{t\}$ denoting the integer and the fractionary parts of t) are ***not*** of positive type. Here i is the yearly interest rate, supposed to be constant. For the first case the computations are easy and left to the reader. For the second one apply Proposition 2.5 (iii).

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