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ON THE TRANSCENDENCE OF THE TRACE FUNCTION

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The aim of this paper is to show that the results from [APZ2] on the "trace function" $F(x,z)$, namely the transcendence of this function over $\mathbf{C}_p(Z)$, are valid in more general framework. We prove also that, in some conditions, the derivative of the trace function $F(x,z)$ is also trancendental over $\mathbf{C}_p(Z)$.

1. INTRODUCTION

Let p be a prime number, $\mathbf{Q}_p$ the field of *p-adic* numbers and | | the natural *p-adic* module of $\mathbf{Q}_p$ Denote by $\overline{\mathbf{Q}_p}$ a fixed algebraic closure of $\mathbf{Q}_p$ and also by | | the unique extension of | | to $\overline{\mathbf{Q}_p}$ and $\mathbf{C}_p$ the completion of $\overline{\mathbf{Q}_p}$ with respect to | | (see [APZ1]). Denote $G = Gal(\overline{\mathbf{Q}_p} \mid \mathbf{Q}_p)$ endowed with so called Krull topology. The group G is canonicaly identified to $G = Gal_{cant}(\mathbf{C}_p \mid \mathbf{Q}_p)$ the group of all $\mathbf{Q}_p$ – continous automorphisms of $\mathbf{C}_p$. Usualy we shall assume $G = Gal_{cant}(\mathbf{C}_p \mid \mathbf{Q}_p)$.

For an element $x \in \mathbf{C}_p$, denote $H(x) = \{\sigma \in G \mid \sigma(x) = x\}$ and $O(x) = \{\sigma(x) \mid \sigma \in G\}$. According to [APZ1] the map $\sigma \longrightarrow \sigma(x)$ defines a natural homeomorphism to $\left(G/H(x) \right)_s$ endowed to quotient topology, on to $O(x)$ endowed to topology induced by $\mathbf{C}_p$.

For any real number $\varepsilon > 0$ let us denote $H(x,\varepsilon) = \{\sigma \in G \mid |\sigma(x) - x| < \varepsilon\}$. One has $[G : H(x,\varepsilon)] < \infty$, and $\bigcap_{\varepsilon > 0} H(x,\varepsilon) = H(x)$. It is clear that if $\varepsilon' < \varepsilon$, then $H(x,\varepsilon') \subseteq H(x,\varepsilon)$. For any $\sigma \in G$ denote $B(\sigma(x),\varepsilon) = \{y \in O(x) \text{such that } |\sigma(x) - y| < \varepsilon\}$ the "open ball" with radius ε in $O(x)$. Then $N(x,\varepsilon) = [G : H(x,\varepsilon)]$ is just the number of all distinct open balls in $O(x)$ which cover $O(x)$. Further, denote $\overline{H}(x,\varepsilon) = \{\sigma \in G \mid |\sigma(x) - x| \le \varepsilon\}$, $I(x,\varepsilon) = [G : \overline{H}(x,\varepsilon)]$ and $\overline{B}(\sigma(x),\varepsilon) = \{y \in \sigma(x) \mid |\sigma(x) - y| \le \varepsilon\}$, the "closed ball" of radius ε in $O(x)$. $I(x,\varepsilon)$ is just the number of distinct closed boll of radius ε which covers $O(x)$.

According to (APZ2) for any $\varepsilon > 0$ denote $\mu_x(B(x,\varepsilon)) = \dfrac{1}{N(x,\varepsilon)}$ the "Haar measure" induced by usual Haar measure μ on G (normalized such that $\mu(G) = 1$). Also denote π_x the "*p-adic* measure" (which

Reccomended by Nicolae POPESCU, member of the Romanian Academy generaly is not bounded) defined on balls $B(x,\varepsilon)$, $\overline{B}(x,\varepsilon)$ just as μ_x, i.e. $\frac{1}{N(x,\varepsilon)}$ is viewed as a *p-adic* number.

According to (APZ2) an element $x \in C_p$ is called to be Lipschitzian if $\lim_{\varepsilon \to 0} \frac{\varepsilon}{|N(x,\varepsilon)|} = 0$. For example an element $x = \lim a_n$, $a_n \in \overline{\mathbf{Q}_p}$ such that (*) $\frac{|\alpha_{n+1} - \alpha_n|}{\max(|d_n|,|d_{n+1}|)} \to 0$ where $d_n = \deg \alpha_n$ is Lipschitzian. Also in [APZ2] it is proved that Lipschitzian functions $f : O(x) \to C_p$, where x is a Lipschitzian element are integrable with respect to *p-adic* "measure" π_x. Particulary, for $z \in C_p \cup \{\infty\} \setminus O(x)$, the functions $\frac{1}{1-zx}$, $\frac{1}{z-x}$ are Lipschitzian, and one can speak that the functions $F(x,x) = \int_{O(x)} \frac{1}{1-zt} d\pi_x(t)$, $\overline{F}(x,z) = \int_{O(x)} \frac{1}{1-zt} d\pi_x(t)$ are analytical on $z \in C_p \cup \{\infty\} \setminus O(x^{-1})$ (respectively on $z \in C_p \cup \{\infty\} \setminus O(x)$). Moreover the elements of $O(x^{-1})$ (respectively $O(x)$) are the singular points of $F(x,z)$ (respectively of $\overline{F}(x,z)$). In [APZ2] the function $F(x,z)$ is called "trace function" and the fact that the elements of $O(x^{-1})$ are just the singular points for $F(x,z)$ is proved only in the case where the element x verify the stronger condition (*). In fact, in [APZ2] is proved that for an element x with condition (*) there exists a sequence $(z_n)_n$, $z_n \to x^{-1}$ such that $|F(x,z_n)| \to \infty$. By this remark (which is proved in [APZ2], but not so easy) it follows that $F(x,z)$ is transcendental over $\mathbf{C}_p(z)$ if x is transcendental over $\mathbf{Q}_p$. Since the functions $F(x,z)$ and $\overline{F}(x,z)$ are related by the formula $\overline{F}(x,z) = zF(x, 1/z)$, $z \in C_p \cup \{\infty\} - O(x)$ there rezult that is enough to study the the behavior of one of there functions around of it singular points.

2. THE TRANSCENDENCE OF THE TRACE FUNCTION

The aim of this paper is to show that the results from [APZ2] with respect to the function $F(x,z)$ when x is a (*) element, are also valid in more general framework, namely, if for the element x is defined the "trace".

$Tr(x) = \int_{O(x)} t \, d\pi_x(t)$ is defined (see (PVZ)) and get one additional assumption. Particulary this is valid for all Lipschitzian points x.

In what follows we should denote by $F(x,z) = \int_{O(x)} \frac{1}{z-t} d\pi_x(t)$.

According to [APZ1] let $\{\overline{\alpha_n}\}_n$ be a distinguished sequence of x (i.e. $x = \lim \overline{\alpha_n}$, $\overline{\alpha_n} \in \overline{\mathbf{Q}_p}$) and denote $D_n = \deg \overline{\alpha_n}$, and $\varepsilon_n = |x - \overline{\alpha_n}|$, $n \geq 0$. One has $\varepsilon_n \to 0$.

According to [APZ2] and [PVZ], if $(\varepsilon_n)_n$ is a decreasing sequence of positive real numbers with limit zero and if S_n is a systeme of representatives of left cosets of G modulo $H(x,\varepsilon_n)$ or modulo $\overline{H}(x,\varepsilon_n)$, then one has: $F(x,z) = \lim_n \frac{1}{d_n} \left[\sum_{\sigma \in S_n} \frac{1}{z - \sigma(x)} \right]$ (where $d_n = |S_n| = [G : H(x,\varepsilon_n)]$), the limit being taken for any $z \in C_p \cup \{\infty\} \setminus O(x)$. Also it is clear that the convergence of the above limit is uniform on the complement of any neighbourhood of $O(x)$.

In what follows we shall use the following result proved in [APP]. If x is a Lipschitzian element, there exists a sequence $(\alpha_n)_n$, $\alpha_n \in \overline{\mathbf{Q}_p[x]} \cap \overline{\mathbf{Q}_p}$ such that $\lim_n \alpha_n = x$, and $\frac{|x-\alpha_n|}{|d_n|} \to 0$ when $d_n = \deg \mathbf{Q}_p(\alpha_n) = [G : H(x, \varepsilon_n)]$, and $\mathbf{Q}_p(\alpha_n) = \mathrm{Fix} H(x, \varepsilon_n)$ (of course, $\varepsilon_n \geq |x - \alpha_n|$ and $\varepsilon_n = 0$). By this result, there result that $F(x,z) = \lim_n \frac{1}{d_n}\left[\sum_{\sigma \in S_n} \frac{1}{z - \sigma(\alpha_n)}\right] = \lim_n \frac{1}{d_n} \cdot \frac{f_n'(z)}{f^n(z)}$, where $f_n(x) = Irr_{\mathbf{Q}_p}(\alpha_n)$. Of course, one has $d_n \mid d_{n+1}$, for all $n \geq 1$.

It is clear that

$$H(x, \varepsilon_n) \supseteq H(\overline{\alpha_n}) = \{\sigma \in G \mid \sigma(\overline{\alpha_n}) = (\overline{\alpha_n})\},$$

and so $\mathbf{Q}_p(\alpha_n) \subseteq \mathbf{Q}_p(\overline{\alpha_n})$ and thus d_n divides D_n, $D_n = d_n q_n$. Let us denote $\overline{f_n}(X)$ the minimal polynomial of $\overline{\alpha_n}$ over $\mathbf{Q}_p$. Also denote $\overline{S_n}$ a system of representatives of left cosets of G with respect to $H(\overline{\alpha_n})$. One can assume $S_n \subseteq \overline{S_n}$. Moreover one can assume that the elements of $\overline{S_n}$ are of the form $\{\sigma_i g_i\}$, $\sigma_i \in S_n$, and g_j runs a system of left cosets of $H(x, \varepsilon_n)$ with respect to $H(\overline{\alpha_n})$ (remind that the number of these cosets is just q_n). It is clear that $B(x, \varepsilon_n) \cap O(x)$ contains just q_n conjugates of $\overline{\alpha_n}$ of the form $g(\overline{\alpha_n})$ where g belongs to the choosed system of left representatives of the $H(x, \varepsilon_n)$ with respect to $H(\overline{\alpha_n})$.

By the above considerations one know that: $F(x,z) = \lim_n \frac{1}{d_n}\left[\sum_{\sigma \in S_n} \frac{1}{z - \sigma(\alpha_n)}\right]$, $z \in C_p \cup \{\infty\} \setminus O(x)$. Now one have:

Remark. One has: $F(x,z) = \lim_n \frac{1}{D_n}\left[\sum_{\sigma \in \overline{S_n}} \frac{1}{z - \sigma(\overline{\alpha_n})}\right]$, $z \in C_p \cup \{\infty\} \setminus O(x)$.

Proof. Indeed, for any fixed $z \in C_p \cup \{\infty\} \setminus O(x)$ and a suitable ball $B(z, \delta)$ which do not intersect $O(x)$, and using the norm of uniform convergence on $B(z, \delta)$ one has:

$$\left\|\frac{1}{d_n}\left(\sum_{\sigma \in S_n} \frac{1}{z - \sigma(\overline{\alpha_n})}\right) - \frac{1}{d_n}\left(\sum_{\sigma \in S_n} \frac{1}{z - \sigma(\alpha_n)}\right)\right\| = \left\|\frac{1}{d_n}\sum_{\sigma \in S_n} \frac{\sigma(g(\overline{\alpha_n})) - \sigma(\alpha_n)}{(z - \sigma(\overline{\alpha_n}))(z - \sigma(\alpha_n))}\right\| \leq \frac{\varepsilon_n}{|d_n|} \cdot \frac{1}{\delta^2} \to 0$$

when $n \to \infty$.

Furthermore, one can write:

$$F(x,z) = \frac{1}{q_n} \cdot \sum_{\overline{g} \in H(x, \varepsilon_n)/H(\overline{\alpha_n})} \left(\lim_n \frac{1}{d_n}\left(\sum_{\sigma \in S_n} \frac{1}{z - \sigma(g(\overline{\alpha_n}))}\right)\right) = \lim_n \frac{1}{D_n} \sum_{\sigma \in \overline{S_n}} \left(\frac{1}{z - \sigma(\overline{\alpha_n})}\right) = \lim_n \frac{1}{D_n} \cdot \frac{\overline{f}_n'(z)}{\overline{f}_n(z)}$$

By a repeated use of the algorithm of division with remainder there result that any polynomial $g(x) \in Q_p[x]$ can be uniquely written as: $g(x) = \sum_{s \geq 0} a_s M_s(x)$ (finite sum) where $\deg M_s(x) = s$, and $M_s(x) = \overline{f_1}^{e_1}(x) \cdot \ldots \cdot \overline{f_q}^{e_q}(x)$, $0 \leq e_i \leq D_i$, $i = 1,\ldots, q$., $0 \leq e_i \leq D_i$, $i = 1,\ldots, q$.

Moreover one has $|g(x)| = \sup_{s \geq 0} |a_s M_s(x)|$, see [APZ1.]

Proposition 1. One has:

$$\sup_n \left| \frac{\overline{f_n}'(x)}{\overline{f_n}(x)} \right| = \infty$$

Proof. Assume, that the Proposition is not true. Then there exist a real number $k > 0$, such that

$$\left| \frac{\overline{f_n}'(x)}{\overline{f_n}(x)} \right| < k \text{, for all } n \geq 1 \text{ or equivalently} \left| \overline{f_n}'(x) \right| < k \cdot \left| \overline{f_n}(x) \right|, \forall\ n \geq 1 .$$

But then one must have : $\left| M_s'(x) \right| < k \cdot \left| M_s(x) \right|$ for all $s \geq 1$. Indeed, $\frac{M_s'(x)}{M_s(x)}$ is a sum of terms of form $\sum_{\sigma(x) \in S_{\alpha_i}} \frac{1}{x - \sigma(\overline{\alpha_i})}$, where S_{α_i} is the set of all conjugates of $\overline{\alpha_i}$ over $\mathbf{Q}_p$. Now since:

$$\left| \sum_{\sigma(x) \in S_{\alpha_i}} \frac{1}{x - \sigma(\alpha_i)} \right| = \left| \frac{\overline{f_i}'(x)}{\overline{f_i}(x)} \right| < k$$

then results $\left| \frac{M_s'(x)}{M_s(x)} \right| < k$, $s \geq 1$.

Now let $D : \mathbf{Q}_p[x] \longrightarrow \mathbf{Q}_p[x]$ the map defined by: $D(P(x)) = P'(x)$, $P(x) \in \mathbf{Q}_p[x]$, when $P'(x)$ means the "formal derivative". One knows that D is a linear map and one has: $D(P(x)Q(x)) = P(x)DQ(x) + Q(x)DP(x)$. Since $\left| M_s'(x) \right| < k \cdot \left| M_s(x) \right|$ for all $s \geq 1$ then by previous Remark, there rezults that D is continuous, and so it can be extended by continuity to a map $\widetilde{D} : \overline{\mathbf{Q}_p[x]} \longrightarrow \overline{\mathbf{Q}_p[x]}$, such that $\widetilde{D}(PH) = P\widetilde{D}(H) + H\widetilde{D}(P)$ for all $P, H \in \overline{\mathbf{Q}_p[x]}$.

Denote $L = \overline{\mathbf{Q}_p[x]} \cap \overline{\mathbf{Q}_p}$. Since L_x is dens in $\overline{\mathbf{Q}_p[x]}$, then rezults that $\widetilde{D}$ is not identical, zero on L_x. Hence let $\alpha \in L_x$ such that $\widetilde{D}(\alpha) \neq 0$, and $q(X) = Irr_{\mathbf{Q}_p}(\alpha)$.

There one has: $0 = \tilde{D}\, q(\alpha) = q'(\alpha)\tilde{D}(\alpha)$ and so $q'(\alpha) = 0$, a contradiction.

Hence one must have $\sup_n \left|\frac{\overline{f_n}'(x)}{\overline{f_n}(x)}\right| = \infty$, as claimed.

Now using the same argument as in ([APZ2], pag 44) there rezults that if z is near by $O(x)$, the values of the function $F(x,z) = \lim_n \frac{1}{d_n} \cdot \frac{\overline{f_n}'(x)}{\overline{f_n}(x)}$, can be as big as we want in absolute values. But then (see also [APZ2]) from this there rezults that the function $F(x,z)$ is trancedental over $\mathbf{C}_p[Z]$.

Theorem 1. With the previous notation assume that $|d_n| \to 0$. Then $F'(x,z)$ is not bounded where z is near by $O(x)$. Then $F'(x,z)$ is a trancedental over $\mathbf{C}_p[Z]$. Moreover by transcendence of $F'(x,z)$ follows the trancendency of $F(x,z)$.

Proof. One has:

$$F(x,z) = \lim_n \frac{1}{d_n} \cdot \left[\frac{\overline{f_n}'(x)}{\overline{f_n}(x)}\right]' = \lim_n \frac{1}{d_n} \cdot \left[\frac{\overline{f_n}^{(2)}(z)\overline{f_n}(z) - \left(\overline{f_n}'(z)\right)^2}{\overline{f_n}^2(z)}\right] =$$

$$= \lim_n \left[\frac{1}{d_n} \cdot \frac{\overline{f_n}^{(2)}(z)}{\overline{f_n}(z)} - \frac{1}{d_n} \cdot \left(\frac{\overline{f_n}'(z)}{\overline{f_n}(z)}\right)^2\right]$$

Now since $\frac{1}{d_n^2} \cdot \left(\frac{\overline{f_n}'(z)}{\overline{f_n}(z)}\right)^2 \longrightarrow F^2(x,z)$ it is clear that $\frac{1}{d_n} \cdot \left(\frac{\overline{f_n}'(z)}{\overline{f_n}(z)}\right)^2 \to 0$ and so one obtains:

$$F'(x,z) = \lim_n \frac{1}{d_n} \cdot \frac{\overline{f_n}^{(2)}(z)}{\overline{f_n}(z)}$$

On another part one has:

$$\left(\frac{\overline{f_n}'(x)}{\overline{f_n}(x)}\right)' = \frac{\overline{f_n}^{(2)}(x)}{\overline{f_n}(x)} - \left(\frac{\overline{f_n}'(x)}{\overline{f_n}(x)}\right)^2$$

and so if would be a real number $k > 0$ such that $\left|\left(\frac{\overline{f_n}'(x)}{\overline{f_n}(x)}\right)'\right| < k$ and $\left|\frac{\overline{f_n}^2(x)}{\overline{f_n}(x)}\right| < k$, then one would obtain: $\left|\frac{\overline{f_n}'(x)}{\overline{f_n}(x)}\right| < \sqrt{k}$, for all $n \geq 1$.

Now since this is not possible according to above conditions, one must have:

$$\sup_n \left|\left(\frac{\overline{f_n}'(x)}{\overline{f_n}(x)}\right)'\right| = \infty \text{ or } \sup_n \left|\frac{\overline{f_n}^{(2)}(x)}{\overline{f_n}(x)}\right| = \infty .$$

But there rezults that $F'(x, z)$ is not bounded around of $O(x)$, and so it is transcendental over $\mathbf{C}_p[Z]$ see the arguments of [APZ2].

For the derivations of higher order one can obtains similar results.

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