THE PUBLISHING HOUSE OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 6, Number 1/2005, pp. 000-000

STARTING FLOW OF AN OLDROYD-B FLUID BETWEEN ROTATING CO-AXIAL CYLINDERS

Constantin FETECĂU

Technical University of Iasi, Romania
E-mail: cfetecau@yahoo.de

Analytical solutions corresponding to the starting flow of an Oldroyd-B fluid between rotating cylinders are established. If the radius of the inner cylinder tends to zero the flow within a circular cylinder is obtained. The well-known solutions for a Navier-Stokes fluid, as well as those corresponding to a Maxwell fluid and to a second grade one, appear as limiting cases of our solutions. The steady state solutions, the same for all types of fluid, are also obtained for $t \to \infty$.

Key words: *Oldroyd-B fluid; Velocity field; Tangential stress; Analytical solutions.*

1. INTRODUCTION

In recent years, the interest for flows of viscoelastic fluids of Oldroyd-B type has considerably increased. Rajagopal [1] has established two simple but elegant solutions corresponding to the flow past an infinite porous plate and to the longitudinal and torsional oscillations of an infinitely long rod. Hayat *et al.* [2] investigated some interesting unsteady problems as: Stokes problem, modified Stokes problem, Poiseuille flow due to an oscillating pressure gradient, etc. Fetecau [3, 4], Fetecau and Fetecau [5, 6] and Fetecau *et al.* [7] have also determined the velocity fields and the associated tangential tensions for different time-dependent flows of an incompressible Oldroyd-B fluid.

The aim of this note is to establish exact solutions for the starting flows of an Oldroyd-B fluid between two rotating co-axial circular cylinders and in a rotating cylinder. The flow between rotating cylinders is one of the most important and most interesting problems of motion near rotating bodies. During recent years quite a lot of papers on this type of flow have been published. The extensive study [8-10] of such flows is motivated by both their fundamental interest and their practical importance. The Cauchy stress $\mathbf{T}$ in such a model is given by [1-7]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^{\mathrm{T}})], \tag{1}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, μ is the dynamic viscosity, λ and λ_r are relaxation and retardation times, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ and the superposed dot indicates the material time derivative. For $\lambda_r = 0$ the Oldroyd-B fluid reduces to the classical model due to Maxwell, for $\lambda = 0$ it becomes of the type of a second grade fluid and for both $\lambda_r = \lambda = 0$ it reduces to the linearly viscous fluid model.

2. GOVERNING EQUATIONS

Consider an incompressible Oldroyd-B fluid at rest in the annular region between two infinite co-axial cylinders located at $r = R_1$ and $r = R_2\ (> R_1)$ of the cylindrical coordinate system r, θ and z. At time $t = 0^+$ the two cylinders start rotating about their common axis $(r = 0)$ with the constant angular velocities

Reccomended by Lazăr DRAGOŞ, member of the Romanian Academy

Ω_1 and Ω_2. By the influence of shear the fluid is gradually moved. We shall consider a velocity field and the stress of the form

$$\mathbf{v} = \omega(r,t)\mathbf{e}_\theta, \quad \mathbf{S} = \mathbf{S}(r,t), \tag{2}$$

where $\mathbf{e}_\theta$ is the unit vector along the θ direction. This flow field automatically satisfies the constraint of incompressibility.

Substituting (2) into (1) and having in mind the initial condition (the fluid being at rest up to the moment $t = 0$)

$$\mathbf{S}(r,0) = \mathbf{0} \tag{3}$$

and Eqs. (21.1-4) of [1], we get $S_{rr} = S_{rz} = S_{\theta z} = S_{zz} = 0$ and

$$(1 + \lambda\partial_t)\tau(r,t) = \mu(1 + \lambda_r\partial_t)(\partial_r - 1/r)\omega(r,t), \tag{4}$$

$$(1 + \lambda\partial_t)\sigma(r,t) - 2\lambda\tau(r,t)(\partial_r - 1/r)\omega(r,t) = -2\mu\lambda_r[(\partial_r - 1/r)\omega(r,t)]^2, \tag{5}$$

where $\tau = S_{r\theta}$ and $\sigma = S_{\theta\theta}$ are the tangential and normal stress components respectively.

The equations of motion, in the absence of body forces, reduce to (cf. [1], eqs. (22.1-3))

$$\frac{\partial p}{\partial r} + \frac{1}{r}\sigma = \rho\frac{\omega^2}{r}, \quad -\frac{\partial p}{\partial\theta} + \partial_r\tau + \frac{2}{r}\tau = \rho\partial_t\omega, \quad \frac{\partial p}{\partial z} = 0, \tag{6}$$

where ρ is the constant density of the fluid. Due to the rotational symmetry, $\partial_\theta p$ from the second equation has to be zero [1]. Eqs. (5) and $(6)_1$, for σ and p, are not coupled with eqs. (4) and $(6)_2$, meaning that one can firstly solve the system of the latter two and then calculate σ and p. Eliminating τ between eqs. (4) and $(6)_2$ we attain to the next linear partial differential equation

$$\lambda\partial_t^2\omega(r,t) + \partial_t\omega(r,t) = \nu(1 + \lambda_r\partial_t)\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega(r,t); \quad r \in (R_1, R_2), \quad t > 0, \tag{7}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

Assuming that the fluid adheres to the walls, the following boundary conditions

$$\omega(R_1, t) = R_1\Omega_1, \ \omega(R_2, t) = R_2\Omega_2; \quad t > 0, \tag{8}$$

have to be satisfied. As regards the initial conditions we have [2-7]

$$\omega(r,0) = \partial_t\omega(r,0) = 0; \quad r \in (R_1, R_2). \tag{9}$$

3. SOLUTION OF THE PROBLEM

3.1. Calculation of the velocity field

Making the change of unknown function

$$\omega(r,t) = v(r,t) + \Omega(r) \quad \text{with} \quad \Omega(r) = r\,\Omega_2 + \frac{R_1^2(R_2^2 - r^2)}{r(R_2^2 - R_1^2)}(\Omega_1 - \Omega_2), \tag{10}$$

we get the following problem with initial and boundary conditions

$$\lambda\partial_t^2 v(r,t) + \partial_t v(r,t) = \nu(1 + \lambda_r\partial_t)\mathrm{L}v(r,t); \quad r \in (R_1, R_2), \quad t > 0, \tag{11}$$

$$v(R_1,t) = v(R_2,t) = 0; \quad t > 0, \tag{12}$$

$$v(r,0) = -\Omega(r), \ \ \partial_t v(r,0) = 0; \quad r \in (R_1, R_2), \tag{13}$$

where the differential operator $L = \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}$.

In order to solve this problem we shall use, as in [4] and [9], the well-known expansion theorem of Steklov. In view of this theorem our solution, $v(r,t)$ whose partial derivatives $\partial_r v$ and $\partial_r^2 v$ have to be piecewise continuous, can be written, for each $t > 0$, as Fourier-Bessel series absolutely and uniformly convergent in terms of the eigenfunctions

$$\mathrm{B}(rr_n) = A_n\left[\mathrm{J}_1(rr_n) - \frac{\mathrm{J}_1(R_1r_n)}{\mathrm{Y}_1(R_1r_n)}\mathrm{Y}_1(rr_n)\right], \tag{14}$$

of the eigenvalue problem $\mathrm{L}\Phi + a^2\Phi = 0, \ \Phi(R_1) = \Phi(R_2) = 0$, i.e.,

$$v(r,t) = \sum_{n=1}^{\infty} v_n(t)\mathrm{B}(rr_n). \tag{15}$$

Here, $\mathrm{J}_1(\cdot)$ and $\mathrm{Y}_1(\cdot)$ are Bessel functions of order one of the first and second kind, r_n are positive roots of the transcendental equation

$$\mathrm{J}_1(R_2r)\mathrm{Y}_1(R_1r) - \mathrm{Y}_1(R_2r)\mathrm{J}_1(R_1r) = 0$$

and the constants A_n are chosen so that the normalization conditions

$$\int_{R_1}^{R_2} r\,[\mathrm{B}(rr_n)]^2\,\mathrm{d}r = 1, \tag{16}$$

to be satisfied.

Now, introducing (15) into (11), multiplying then by $r\mathrm{B}(rr_p)$ and integrating with respect to r from R_1 to R_2, we find that

$$\lambda\ddot{v}_n(t) + (1 + \alpha r_n^2)\dot{v}_n(t) + \nu r_n^2 v_n(t) = 0, \quad t > 0, \ \ n = 1, 2,... \tag{17}$$

where $\alpha = \nu\lambda_r$. From (13) it also results

$$v_n(0) = -\Omega_n, \quad \dot{v}_n(0) = 0, \ \ n = 1, 2,... \tag{18}$$

where Ω_n is the modified Hankel transform of $\Omega(r)$.

Solving the ordinary differential equations (17) subject to the initial conditions (18) and having in mind (10) and (15), we find the velocity field $\omega(r,t)$ under forms:

$$\omega(r,t) = \Omega(r) - \exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Omega_n\left[\mathrm{ch}\left(\frac{\alpha_n t}{2\lambda}\right) + \frac{1+\alpha r_n^2}{\alpha_n}\mathrm{sh}\left(\frac{\alpha_n t}{2\lambda}\right)\right]\exp\left(-\frac{\alpha r_n^2}{2\lambda}t\right)\mathrm{B}(rr_n) \quad \text{if } \lambda < \lambda_r, \tag{19}$$

$$\omega(r,t) = \Omega(r) - \sum_{n=1}^{\infty}\frac{\alpha r_n^2\exp(-t/\lambda) - \exp(-\nu r_n^2 t)}{\alpha r_n^2 - 1}\Omega_n\mathrm{B}(rr_n) \qquad \text{if } \lambda = \lambda_r \tag{20}$$

and

$$\omega(r,t)=\Omega(r)-\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Omega_n B_n(t)\exp\left(-\frac{\alpha r_n^2}{2\lambda}t\right)\mathrm{B}(rr_n)\qquad \text{if } \lambda>\lambda_r, \tag{21}$$

where

$$\alpha_n=\sqrt{(1+\alpha r_n^2)^2-4\nu\lambda r_n^2}\ ,$$

$$B_n(t)=\begin{cases}\mathrm{ch}\left(\dfrac{\alpha_n t}{2\lambda}\right)+\dfrac{1+\alpha r_n^2}{\alpha_n}\mathrm{sh}\left(\dfrac{\alpha_n t}{2\lambda}\right) & \text{for } r_n\in(0,\ a)\cup(b,\ \infty),\\ 1+\dfrac{1+\alpha r_n^2}{2\lambda}\ t & \text{for } r_n\in\{a,\ b\},\\ \cos\left(\dfrac{\beta_n t}{2\lambda}\right)+\dfrac{1+\alpha r_n^2}{\beta_n}\sin\left(\dfrac{\beta_n t}{2\lambda}\right) & \text{for } r_n\in(a,\ b),\end{cases}$$

$$a=\frac{1}{\sqrt{\nu}\,(\sqrt{\lambda}+\sqrt{\lambda-\lambda_r}\,)},\quad b=\frac{1}{\sqrt{\nu}\,(\sqrt{\lambda}-\sqrt{\lambda-\lambda_r}\,)}\quad \text{and}\quad \beta_n=\sqrt{4\nu\lambda r_n^2-(1+\alpha r_n^2)^2}.$$

The starting solution, presented under the forms (19), (20) or (21), is the sum of the steady-state solution

$$v_s(r)=\Omega(r)=r\,\Omega_2+\frac{R_1^2(R_2^2-r^2)}{r(R_2^2-R_1^2)}(\Omega_1-\Omega_2) \tag{22}$$

and of the transient solution $v_t(r,t)$ given by the second terms. It describes the motion of the fluid for some time after its initiation. After that time, in which the transients disappear, this solution tends to the steady-state solution.

3.2. Calculation of the tangential tension

In view of (4) and (3) we have

$$\tau(r,t)=\frac{\mu}{\lambda}\exp\left(-\frac{t}{\lambda}\right)\int_0^t\exp\left(\frac{\tau}{\lambda}\right)(1+\lambda_r\partial_\tau)\left(\partial_r-\frac{1}{r}\right)\omega(r,\tau)d\tau. \tag{23}$$

By substituting (19)-(21) in (23) we find that

$$\tau(r,t)=T(r,t)-\frac{2\mu}{r}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\frac{\Omega_n}{\alpha_n}\mathrm{sh}\left(\frac{\alpha_n t}{2\lambda}\right)\exp\left(-\frac{\alpha r_n^2}{2\lambda}t\right)[rr_n\mathrm{B}'(rr_n)-\mathrm{B}(rr_n)] \quad \text{if } \lambda<\lambda_r, \tag{24}$$

$$\tau(r,t)=T(r,t)-\frac{\mu}{r}\sum_{n=1}^{\infty}\frac{\exp(-t/\lambda)-\exp(-\nu r_n^2 t)}{\alpha r_n^2-1}\Omega_n[rr_n\mathrm{B}'(rr_n)-\mathrm{B}(rr_n)]\qquad \text{if } \lambda=\lambda_r\ , \tag{25}$$

respectively,

$$\tau(r,t)=T(r,t)-\frac{2\mu}{r}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Omega_n T_n(t)\exp\left(-\frac{\alpha r_n^2}{2\lambda}t\right)[rr_n\mathrm{B}'(rr_n)-\mathrm{B}(rr_n)]\ \text{ if } \lambda>\lambda_r, \tag{26}$$

where

$$T(r,t) = \frac{2\mu R_1^2 R_2^2(\Omega_2 - \Omega_1)}{r^2(R_2^2 - R_1^2)}\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right],$$

$$T_n(t) = \begin{cases} \frac{1}{\alpha_n}\,\mathrm{sh}\left(\frac{\alpha_n t}{2\lambda}\right) & \text{for} \quad r_n \in (0, a) \cup (b, \infty), \\ C_n + D_n \exp\left(\frac{-1 + \alpha r_n^2}{2\lambda} t\right) & \text{for} \quad r_n \in \{a, b\}, \\ \frac{1}{\beta_n}\sin\left(\frac{\beta_n t}{2\lambda}\right) & \text{for} \quad r_n \in (a, b), \end{cases} \tag{27}$$

$$C_n = -\frac{1}{1-\alpha r_n^2}\left[\lambda_r\left(\frac{1+\alpha r_n^2}{2\lambda}\right)^2\left(t - \frac{2\lambda}{1-\alpha r_n^2}\right) - \frac{1+\alpha r_n^2}{2\lambda}t + \frac{2\alpha r_n^2}{1-\alpha r_n^2}\right] \quad \text{and}$$

$$D_n = \frac{2\alpha r_n^2}{(1-\alpha r_n^2)^2} - \frac{\lambda_r}{2\lambda}\left(\frac{1+\alpha r_n^2}{1-\alpha r_n^2}\right)^2.$$

By letting $t \to \infty$ in Eqs. (24)-(26) we get

$$\tau_s(r) = \frac{2\mu R_1^2 R_2^2(\Omega_2 - \Omega_1)}{r^2(R_2^2 - R_1^2)}, \tag{28}$$

which represents the tangential tension corresponding to the steady state. The velocity field $\omega(r,t)$ and the shear stress $\tau(r,t)$ being determined, we can determine the normal stress $\sigma(r,t)$ by means of (3) and (5). The hydrostatic pressure p can be also obtained from (6) up to an arbitrary function of t. If $\Omega_1 = \Omega_2 = \Omega$, the relations (22) and (28) become

$$v_s(r) = r\Omega \qquad \text{and} \qquad \tau_s(r) = 0. \tag{29}$$

Consequently, after a long enough time of rotation of the two cylinders with the same constant angular velocity Ω, the whole system is rotating as a rigid-body with the same angular velocity.

4. COUETTE FLOW THROUGH A CIRCULAR CYLINDER

Taking the limit of Eqs. (14) and (16) when $R_1 \to 0$ and $R_2 \to R$ we find the eigenfunctions $\sqrt{2}\,\mathrm{J}_1(rr_n)/[R\mathrm{J}_0(Rr_n)]$ corresponding to the Couette flow through an infinite circular cylinder. Further, $\Omega_n = -R\Omega\sqrt{2}/r_n$ where r_n are the positive roots of the equation $\mathrm{J}_1(Rr) = 0$, while the boundary conditions (8) becomes

$$|\omega(0,t)| < \infty, \qquad \omega(R,t) = R\Omega; \qquad t > 0. \tag{30}$$

From (19)-(21) and (24)-(26) it easily results

$$\omega(r,t) = r\Omega + 2\Omega\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\left[\mathrm{ch}\left(\frac{\alpha_n t}{2\lambda}\right) + \frac{1+\alpha r_n^2}{\alpha_n}\mathrm{sh}\left(\frac{\alpha_n t}{2\lambda}\right)\right]\exp\left(-\frac{\alpha r_n^2}{2\lambda}t\right)\frac{\mathrm{J}_1(rr_n)}{r_n\mathrm{J}_0(Rr_n)} \tag{31}$$

if $\lambda < \lambda_r$,

$$\omega(r,t) = r\Omega + 2\Omega\sum_{n=1}^{\infty}\frac{\alpha r_n^2\exp(-t/\lambda) - \exp(-\nu r_n^2 t)}{\alpha r_n^2 - 1}\,\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \quad \text{if} \quad \lambda = \lambda_r\,, \tag{32}$$

$$\omega(r,t) = r\Omega + 2\Omega\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty} B_n(t)\exp\left(-\frac{\alpha r_n^2}{2\lambda}t\right)\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \quad \text{if} \quad \lambda > \lambda_r\,, \tag{33}$$

respectively,

$$\tau(r,t) = -4\mu\Omega\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\,\text{sh}\left(\frac{\alpha_n t}{2\lambda}\right)\exp\left(-\frac{\alpha r_n^2}{2\lambda}t\right)\frac{J_2(rr_n)}{J_0(Rr_n)} \quad \text{if} \quad \lambda < \lambda_r\,, \tag{34}$$

$$\tau(r,t) = -2\mu\Omega\sum_{n=1}^{\infty}\frac{\exp(-t/\lambda) - \exp(-\nu r_n^2 t)}{\alpha r_n^2 - 1}\,\frac{J_2(rr_n)}{J_0(Rr_n)} \quad \text{if} \quad \lambda = \lambda_r \tag{35}$$

and

$$\tau(r,t) = -4\mu\Omega\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty} T_n(t)\exp\left(-\frac{\alpha r_n^2}{2\lambda}t\right)\frac{J_2(rr_n)}{J_0(Rr_n)} \quad \text{if} \quad \lambda > \lambda_r\,. \tag{36}$$

5. LIMITING CASES

1. Taking the limits of Eqs. (19), (24), (31) and (34) as $\lambda \to 0$, we attain to the similar solutions

$$\omega(r,t) = \Omega(r) - \sum_{n=1}^{\infty}\Omega_n\exp\left(-\frac{\nu r_n^2}{1+\alpha r_n^2}t\right)B(rr_n)\,, \tag{37}$$

$$\tau(r,t) = T(r,t) - \frac{\mu}{r}\sum_{n=1}^{\infty}\frac{\Omega_n}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2}{1+\alpha r_n^2}t\right)[rr_n B'(rr_n) - B(rr_n)]\,, \tag{38}$$

$$\omega(r,t) = r\Omega + 2\Omega\sum_{n=1}^{\infty}\exp\left(-\frac{\nu r_n^2}{1+\alpha r_n^2}t\right)\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \tag{39}$$

and

$$\tau(r,t) = -2\mu\Omega\sum_{n=1}^{\infty}\frac{1}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2}{1+\alpha r_n^2}t\right)\frac{J_2(rr_n)}{J_0(Rr_n)}\,, \tag{40}$$

for a second grade fluid. The expression (39) of $\omega(r,t)$ can be also obtained from [9], eq. (13), for $\Omega(t) = r\Omega$.

2. By letting now $\lambda_r \to 0$ into Eqs. (21), (26), (33) and (36) it results the analogues solutions for a Maxwell fluid

$$\omega(r,t) = \Omega(r) - \exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Omega_n C_n(t)B(rr_n), \tag{41}$$

$$\tau(r,t) = T(r,t) - \frac{2\mu}{r}\exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty}\Omega_n S_n(t)[rr_n B'(rr_n) - B(rr_n)], \tag{42}$$

$$\omega(r,t) = r\Omega + 2\Omega \exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty} C_n(t)\frac{J_1(rr_n)}{r_n J_0(Rr_n)}\,, \tag{43}$$

respectively,

$$\tau(r,t) = -4\mu\Omega \exp\left(-\frac{t}{2\lambda}\right)\sum_{n=1}^{\infty} S_n(t)\frac{J_2(rr_n)}{J_0(Rr_n)}\,, \tag{44}$$

where

$$C_n(t) = \begin{cases} ch\left(\frac{\gamma_n t}{2\lambda}\right) + \frac{1}{\gamma_n} sh\left(\frac{\gamma_n t}{2\lambda}\right) & \text{for} \quad r_n \in \left(0, \frac{1}{2\sqrt{\nu\lambda}}\right), \\ 1 + \frac{t}{2\lambda} & \text{for} \quad r_n = \frac{1}{2\sqrt{\nu\lambda}}, \\ \cos\left(\frac{\delta_n t}{2\lambda}\right) + \frac{1}{\delta_n}\sin\left(\frac{\delta_n t}{2\lambda}\right) & \text{for} \quad r_n \in \left(\frac{1}{2\sqrt{\nu\lambda}}, \infty\right), \end{cases}$$

$$S_n(t) = \begin{cases} \frac{1}{\gamma_n} \text{sh}\left(\frac{\gamma_n t}{2\lambda}\right) & \text{for} \quad r_n \in \left(0, \frac{1}{2\sqrt{\nu\lambda}}\right), \\ \frac{t}{2\lambda} & \text{for} \quad r_n = \frac{1}{2\sqrt{\nu\lambda}}, \\ \frac{1}{\delta_n}\sin\left(\frac{\delta_n t}{2\lambda}\right) & \text{for} \quad r_n \in \left(\frac{1}{2\sqrt{\nu\lambda}}, \infty\right), \end{cases}$$

$$\gamma_n = \sqrt{1 - 4\nu\lambda r_n^2} \quad \text{and} \quad \delta_n = \sqrt{4\nu\lambda r_n^2 - 1}.$$

The expression (43) of $\omega(r,t)$ is identical to that obtained in [8], eq. (3.4).

2. In the special case when both λ and λ_r tend to zero, our solutions (19)-(21), (24)-(26), (31)-(36) as well as (37)-(40) for $\lambda_r \to 0$, respectively, (41)-(44) for $\lambda \to 0$, take the classical forms

$$\omega(r,t) = r\,\Omega_2 + \frac{R_1^2(R_2^2 - r^2)}{r(R_2^2 - R_1^2)}(\Omega_1 - \Omega_2) - \sum_{n=1}^{\infty}\Omega_n \exp(-\nu r_n^2 t)\mathrm{B}(rr_n), \tag{45}$$

$$\tau(r,t) = \frac{2\mu R_1^2 R_2^2(\Omega_2 - \Omega_1)}{r(R_2^2 - R_1^2)} - \frac{\mu}{r}\sum_{n=1}^{\infty}\Omega_n \exp(-\nu r_n^2 t)[rr_n \mathrm{B}'(rr_n) - \mathrm{B}(rr_n)], \tag{46}$$

$$\omega(r,t) = r\Omega + 2\Omega\sum_{n=1}^{\infty} \exp(-\nu r_n^2 t)\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \tag{47}$$

and

$$\tau(r,t) = -2\mu\Omega\sum_{n=1}^{\infty} \exp(-\nu r_n^2 t)\frac{J_2(rr_n)}{J_0(Rr_n)}\,, \tag{48}$$

corresponding to a Navier-Stokes fluid. The expression (47) of $\omega(r,t)$, identical to (5.1) of [8], was recently obtained in [10] (eq. (21)) by a different technique. Finally, by making $t \to \infty$ in anyone of the above relations we get the steady state solutions. They are the same for all types of fluids.

6. CONCLUSIONS

The velocity fields and the adequate tangential tensions corresponding to a starting flow of an incompressible Oldroyd-B fluid between two infinite co-axial cylinders are determined by means of the well-known expansion theorem of Steklov. The flow is produced by the two cylinders, which at time $t = 0^+$ start suddenly rotating about their common axis with constant angular velocities.

Direct computations show that $\omega(r,t)$ and $\tau(r,t)$, given by (19)-(21) and (24)-(26) satisfy both the associate partial differential equations and all imposed initial and boundary conditions, the differentiation term by term of the respective series being clearly permissible. If the radius of the inner cylinder tends to zero, the solutions for the flow within a circular cylinder are immediately obtained.

In the special case, when λ_r or $\lambda \to 0$, the solutions that have been obtained reduce to those corresponding to a Maxwell fluid or to a second grade one, respectively. If both λ_r and $\lambda \to 0$ these solutions tend to those for a Navier-Stokes fluid. The steady state solutions, resulting as a limiting case too, for $t \to \infty$, are the same for all types of fluid. Consequently, the non-Newtonian parameters λ and λ_r do not involve in the steady solutions for an Oldroyd-B fluid.

Finally, we remark that the solutions (41)-(44) corresponding to a Maxwell fluid as well as those for an Oldroyd-B fluid (21), (26), (33) and (36), the case $\lambda > \lambda_r$, contain sine and cosine terms. This indicates that in contrast to the Navier-Stokes and second grade fluids, whose solutions do not contain such terms, oscillations are set up in the fluid. The amplitudes of these oscillations decay exponentially in time, the damping being proportional to $\exp(-t/2\lambda)$.

ACKNOWLEDGEMENT

The author would like to thank to the referee for some suggestions regarding an earlier version of the paper.

REFERENCES

1. RAJAGOPAL, K. R., BHATNAGAR, R. K., *Exact solutions for some simple flows of an Oldroyd-B fluid,* Acta Mech. **113**, pp. 233-239, 1995.
2. HAYAT, T., SIDDIQUI, A. M., ASGHAR, S., *Some simple flows of an Oldroyd-B fluid,* Int. J. Engng. Sci. **39**, pp. 135-147, 2001.
3. FETECAU, C., *The Rayleigh-Stokes problem for an edge in an Oldroyd-B fluid,* C. R. Acad. Sci. Paris, Series **I 335**, pp. 979-984, 2002.
4. FETECAU, C., *Analytical solutions for non-Newtonian fluid flows in pipe-like domains,* Int. J. Non-Linear Mech. **39,** pp. 225-231, 2004.
5. FETECAU, C., FETECAU, CORINA, *The first problem of Stokes for an Oldroyd-B fluid,* Int. J. Non-Linear Mech. **38**, pp. 1539-1544, 2003.
6. FETECAU, C., FETECAU, CORINA, *Decay of a potential vortex in an Oldroyd-B fluid,* Int. J. Engng. Sci. (in press).
7. FETECAU, C., PRASAD, S. C., RAJAGOPAL, K. R., *A note on the flow induced by a constantly accelerating plate in an Oldroyd-B fluid,* (sent for publication).
8. SRIVASTAVA, P. N., *Non-steady helical flow of a visco-elastic liquid,* Arch. Mech. Stos. **18** (2), pp. 145-150, 1966.
9. FETECAU, C., FETECAU, CORINA, *On the uniqueness of some helical flows of a second grade fluid*, Acta Mech. **57**, pp. 247-252, 1985.
10. ERDOGAN, M. E., *On the flows produced by sudden application of a constant pressure gradient or by impulsive motion of a boundary*, Int. J. Non-Linear Mech. **38**, pp. 781-797, 2003.

Received October 15, 2004.