PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, **Series A,**
Volume 4, Number 3/2003, pp.00-00

ON THE SEPARABILITY OF PURE STATES

Horia SCUTARU*

Center for Advanced Studies in Physics of the Romanian Academy
Casa Academiei Romane , Calea 13 Septembrie nr. 13, RO-050711, Bucharest 5, Romania
E-mail: scutaru@theor1.theory.nipne.ro

A conjecture of An Min Wang concerning the separability of quantum pure states is proved for bipartite quantum systems.

I. INTRODUCTION

It is well know that the problem of the description of separable states of finite dimensional quantum system is central for the quantum information theory. The aim of the present paper is to point out that the conjecture of An Min Wang [1] concerning the separability of pure states is valid for pure states of bipartite quantum systems. A general condition for separability of a quantum state could be in principle obtained from the measure of entanglement. For a bipartite pure state the natural measure of entanglement is given by the von Neumann entropy of the reduced density matrices. Then, a bipartite state is separable if and only if this measure of entanglement is equal to zero, i.e., if and only if the reduced states are pure states. The purity of any state, described by a density matrix, can be verified directly using the Bloch vector, associated with that state. One of the restrictions is on the value of the squared Euclidean norm of the Bloch vector. Exactly on this restriction is based the An Min Wang criterion for the separability of a bipartite quantum state. The other restrictions concern the symmetric product of the Bloch vector with itself. We shall prove that the equations of the second set of restrictions follows from the An Min Wang restrictions and from the equations fulfilled by the Fano parameters of any pure bipartite state. In order to prove the An Min Wang conjecture we shall use three different parametrizations for the pure states of bipartite quantum systems: the generalized Bloch vector parametrization [2-10]; the Fano parametrization [3-10]; the Schmidt parametrization [11,12], and the relations between them.

2. THE BLOCH PARAMETRIZATIONS

a) The Bloch vector.

Let H be a finite-dimensional Hilbert space of dimension d. We denote by $End(H)$ the vectorial space of the linear operators on H and define on this space the Hilbert-Schmidt inner product by the formula:$(A,B)=Tr(A^*B)$ for any $A,B\in End(H)$ (the operator A^* is the adjoint of the operator A). The Lie algebra $su(d)$ of all selfadjoint operators $A\in End(H)$ with $TrA=0$ is a real subspace of $End(H)$, of dimension $D=d^2-1$. We shall take a base $\{t_j\}_{j=1}^{D}$ of this subspace such that the following relations are satisfied $(t_j,t_k)=2d_{jk}$. Then any density matrix r i.e. any linear selfadjoint and positive definite operator with $Tr\,r=1$ can be described by the formula

* Member of the Romanian Academy

$$r(v) = \frac{1}{d} I + \frac{1}{2} \sum_{j=1}^{D} v_j t_j \tag{2.1}$$

The real vector $v = (v_1, v_2, ..., v_D) \in R^D$ is called the generalized Bloch vector [2-11] and is defined in a unique way by the density matrix $r : v_j = Tr\, r\, t_j = (r, t_j)$. But the converse correspondence is not valid for any vector $v = (v_1, v_2, ..., v_D) \in R^D$. The positivity of the density matrix r imposes severe restrictions on the Bloch vectors [11]. Let us denote by $< v, u >= \sum_{j=1}^{D} v_j u_j$ the Euclidean inner product on R^D and by $\| v \| = \sqrt{< v, v >}$ the corresponding norm.

b) The equations satisfied by the Bloch vector of a pure state.

The Lie brackets of the generators $\{t_j\}_{j=1}^{D}$ of the Lie algebra $su(d)$ are described by the structure constants $\{f_{jkl}\}_{j,k,l=1}^{D}$:

$$[t_j, t_k] = 2i \sum_{l=1}^{D} f_{jkl} t_l \tag{2.2}$$

These structure constants are the components of a totally anti-symmetric tensor and fulfill the Jacoby identity:

$$\sum_{m=1}^{D} (f_{klm} f_{mpq} + f_{plm} f_{mkq} + f_{kpm} f_{mlq}) = 0 \tag{2.3}$$

A remarkable fact, specific to the Lie algebra $su(d)$, is the existence of a symmetric bracket:

$$t_j t_k + t_k t_l = \frac{4}{d} d_{jk} I + 2 \sum_{l=1}^{D} d_{jkl} t_l \tag{2.4}$$

Here d_{jkl} are the components of a totally symmetric tensor. With the aid of anti-symmetric and symmetric tensors we define an anti-symmetric and a symmetric product on the Euclidean space R^D. The anti-symmetric product is defined by:

$$(x \bigcap y)_j = \sum_{k=1}^{D} \sum_{l=1}^{D} f_{jkl} x_k \; y_l \tag{2.5}$$

The symmetric product is defined by:

$$(x \bigcup y)_j = \sum_{k=1}^{D} \sum_{l=1}^{D} d_{jkl} x_k \; y_l \tag{2.6}$$

Then the commutators and anticommutators becomes respectively:

$$\left[< x, t >, < y, t >\right] = 2i < (x \bigcap y), t > \tag{2.7}$$

$$\left\{< x, t >, < y, t >\right\} = \frac{4}{d} < x, y > I + 2 < (x \bigcup y), t > \tag{2.8}$$

For any density matrix (2.1) we have:

$$r(v)^2 = \left(\frac{1}{d^2} + \frac{1}{2d} < v, v >\right) I + < \left[\frac{1}{d} v + \frac{1}{4}(v \bigcup v)\right], t > \tag{2.9}$$

The quantum state described by the density matrix r is a pure state if and only if:

$$r^2(v) = r(v) \tag{2.10}$$

Then, from (2.9) it follows that the Bloch vector v describes a pure state if and only if the squared Euclidean norm of v is given by [11]:

$$< v, v >= 2\left(1 - \frac{1}{d}\right) \tag{2.11}$$

and the symmetric product of v with v is in the one- dimensional subspace generated by v:

$$v \bigcup v = 4\left(\frac{1}{2} - \frac{1}{d}\right) v \tag{2.12}$$

3. THE FANO PARAMETRIZATION

a) The Fano parameters.

The density matrix corresponding to a state of a bipartite quantum system which is composed from two subsystems of dimensions d_1 and d_2 can be parametrized by the Fano parameters [3-7]:

$$\rho = \frac{1}{\mathrm{d}_1 \mathrm{d}_2}(I_1 \otimes I_2) + \frac{1}{2\mathrm{d}_2} < x, \tau > \otimes I_2 + \frac{1}{2\mathrm{d}_1} I_1 \otimes < y, \tau > + \frac{1}{4}\sum_{k=1}^{\mathrm{d}_1^2-1}\sum_{l=1}^{\mathrm{d}_2^2-1} K_{kl}(\tau_k \otimes \tau_l) \tag{3.1}$$

b) The equations satisfied by the Fano parameters of a pure state.

The purity condition (3.3) gives us the following equations for the Fano parameters:

$$\left(1 - \frac{2}{\mathrm{d}_1 \mathrm{d}_2}\right) x_j = \frac{1}{2\mathrm{d}_2}(x \bigcup x)_j + \frac{1}{\mathrm{d}_1}(Ky)_j + \frac{1}{4}\sum (KK^T)_{sp}\, \mathrm{d}^{(1)}{}_{spj} \tag{3.2}$$

$$\left(1 - \frac{2}{\mathrm{d}_1 \mathrm{d}_2}\right) y_j = \frac{1}{2\mathrm{d}_1}(y \bigcup y)_j + \frac{1}{\mathrm{d}_2}(K^T y)_j + \frac{1}{4}(K^T K)_{sp}\, \mathrm{d}^{(2)}{}_{spj} \tag{3.3}$$

$$\begin{aligned}\left(1 - \frac{2}{\mathrm{d}_1 \mathrm{d}_2}\right) K_{jl} = {} & \frac{2}{\mathrm{d}_1 \mathrm{d}_2} x_j y_l + \frac{1}{\mathrm{d}_2}\sum K_{sl}\, \mathrm{d}^{(1)}{}_{jps}\, x_p + \frac{1}{\mathrm{d}_1}\sum K_{jt}\, \mathrm{d}^{(2)}{}_{ltq}\, y_q + \\ & + \frac{1}{4}\sum K_{st} K_{pq}\, \mathrm{d}^{(1)}{}_{spj}\, \mathrm{d}^{(2)}{}_{tql} - \frac{1}{4}\sum K_{st} K_{pq}\, f^{(1)}{}_{spj}\, f^{(2)}{}_{tql}\end{aligned} \tag{3.4}$$

4. THE SCHMIDT PARAMETRIZATION

The Schmidt decomposition of an arbitrary bipartite pure state $|\Psi>$ of a $d_1 \times d_2$ system is given by:

$$|\Psi>=\sum_{j=1}^{M}\sqrt{a_j}\,|f_j>\otimes|j_j>=\sum_{j=1}^{M}\sqrt{a_j}U_1|e_j>\otimes U_2|f_j> \tag{4.1}$$

Here $M=\min(d_1,d_2)$ and the squared Schmidt coefficients a_j are the eigenvalues of the reduced density operators of the two subsystems:

$$r_1=Tr_2|\Psi><\Psi| \tag{4.2}$$

$$r_2=Tr_1|\Psi><\Psi| \tag{4.3}$$

The vectors $\{|f_j>\}$ and $\{|j_j>\}$ are the orthonormal eigenvectors of these reduced density operators. Both new bases are connected to the fiducial orthonormal bases $\{|e_j>\}$ and $\{|f_j>\}$ by unitary transformations U_1 and U_2 respectively. The state $|\Psi>$ is specified by the Schmidt numbers $\{a_j\}_{j=1}^{M}$ and the unitary operators U_1 and U_2.

5. THE AN MIN WANG'S CONJECTURE

The conjecture of the An Min Wang [1] concerning the separability of a pure state of a bipartite quantum system can be formulated in the following way: *a bipartite pure state is separable if and only if the Bloch vectors of the reduced density matrices fulfill the following equations:*

$$<x,x>=2(1-\frac{1}{d_1}) \tag{5.1}$$

$$<y,y>=2(1-\frac{1}{d_2}) \tag{5.2}$$

6. THE BLOCH VECTORS OF THE REDUCED DENSITY MATRICES

It was proved in [5] that the Bloch vectors of the reduced density matrices are given by:

$$x_j=\sum_{s=1}^{M}a_s w^{(1)s}{}_j \tag{6.1}$$

$$y_k=\sum_{s=1}^{M}a_s w^{(2)s}{}_k \tag{6.2}$$

The coefficients $w^{(1)s}{}_j$ and $w^{(2)s}{}_k$ are defined by the following relations, for $t=1,2$:

$$w^{(t)s}{}_l=\sqrt{\frac{2}{l(l+1)}}\ ,\quad 1\le s\le l \tag{6.3}$$

$$w^{(t)(l+1)}{}_l=-l\sqrt{\frac{2}{l(l+1)}}\ ,\quad w^{(t)s}{}_l=0\ ,\ l+1\le s\le d_t \tag{6.4}$$

The following sum rules are valid:

$$\sum_{l=1}^{d_t-1} w^{(t)m}{}_l w^{(t)n}{}_l = 2(d_{mn} - \frac{1}{d_t}) \ , \quad t = 1,2 \tag{6.5}$$

Then, the squared Euclidean norms of the Bloch vectors of the reduced density matrices are given by:

$$< x, x >= \sum_{l=1}^{d_1} x_l{}^2 = 2(\sum_{s=1}^{M} a_s{}^2 - \frac{1}{d_1}) \tag{6.6}$$

$$< y, y >= \sum_{j=1}^{d_2} y_j{}^2 = 2(\sum_{s=1} a_s{}^2 - \frac{1}{d_2}) \tag{6.7}$$

From the equations (6.5) and (6.6) it follows that the conditions (5.1) and (5.2) from the An Min Wang conjecture are fulfilled if and only if

$$\sum_{s=1}^{M} a_s{}^2 = 1 \tag{6.8}$$

Because of the fact that the squared Schmidt coefficient are the eigenvalues of the reduced density matrices we have also:

$$\sum_{s=1}^{M} a_s = 1 \tag{6.9}$$

The equations (6.8) and (6.9) are simultaneously valid only for one a_s, say a_1, equal to 1and the rest equal to zero. **Hence the conditions imposed by the An Min Wang conjecture (i.e. the equations (5.1) and (5.2)) are valid if and only if the state $|\Psi>$ is separable.**

7. ON THE PURITY OF THE REDUCED DENSITY MATRICES

If the reduced states described by the density matrices r_1 and r_2 are pure states then the equations (5.1) and (5.2) are valid. But we need also the validity of the equations:

$$x \bigcup x = 4\left(\frac{1}{2} - \frac{1}{d_1}\right) x \tag{7.1}$$

$$y \bigcup y = 4\left(\frac{1}{2} - \frac{1}{d_2}\right) y \tag{7.2}$$

Let us suppose that the pure state $|\Psi>$ is separable, i.e. let us suppose that:

$$|\Psi><\Psi| = r_1 \otimes r_2 \tag{7.3}$$

Then the equations (7.1) and (7.2) result from the An Min Wang conditions (5.1) and (5.2) and from the equations (3.1), (3.2) and (3.4) which are satisfied by the Fano parameters of any pure states. Indeed, from the equations (7.3) it follows that

$$K = xy^T \tag{7.4}$$

Let us introduce the notations:

$$A = \frac{1}{2} < x, x > + \frac{1}{d_1} \tag{7.5}$$

$$B = \frac{1}{2} < y, y > + \frac{1}{d_2} \tag{7.6}$$

The equations (3.1), (3.2) and (3.4) are equivalent with the following equations respectively:

$$x \bigcup x = 4\left(\frac{1}{2B} - \frac{1}{\mathrm{d}_1} \right) x \tag{7.7}$$

$$y \bigcup y = 4\left(\frac{1}{2A} - \frac{1}{\mathrm{d}_2} \right) y \tag{7.8}$$

$$AB = 1 \tag{7.9}$$

But for any density matrix the following equation is valid:

$$\boldsymbol{r}^2 \leq \boldsymbol{r} \tag{7.10}$$

From (7.8) it follows that

$$A \leq 1 \ , \ \ B \leq 1 \tag{7.11}$$

Hence, from (7.7) and (7.9) we must have

$$A = 1 \ , \ \ B = 1 \tag{7.12}$$

Then the relations (7.1) and (7.2) are fulfilled. In this way, we have proved that for pure separable states the Bloch vectors of the reduced density matrices fulfill the equations (2.11) and (2.12) which are the necessary and sufficient conditions for the purity of the reduced density matrices.

REFERENCES

1. WANG, AN, MIN, *Separability criterion for pure states in multipartite high dimensional systems.* quant–phys/0207136.
2. HIOE, F. T., EBERLY, J., Phys. Rev. Lett. **47**, 838, (1981).
3. FANO,U., Rev. Mod. Phys. **55**, 855, (1983).
4. KELLER, M., MAHLER, G., Journal of Modern Optics **41**, 2537, (1994).
5. SCHLIENTZ, J., MAHLER, G., Phys. Rev. A **52**, 4396, (1995).
6. MAHLER, G., WEBERRUS, W. A., *Quantum Networks*, Springer-Verlag, Berlin-Heidelberg, 1995.
7. SCHLIENTZ, J., MAHLER, G., Phys. Lett. A **224**, 39, (1996).
8. ZANARDI, P., J. Math. Phys. **29**, 742, (1998).
9. ALICKI, R., LENDI, K., *Quantum Dynamical Semi-groups and Applications,* Springer-Verlag, Lecture Notes in Physics **286,** Berlin-Heidelberg, 1987.
10. WEIGERT, S., J. Phys. A: Math. Gen. **30**, 8739 (1997).
11. SINOLECKA, M.M., ZYCZKOWSKI, K., KUS, M. Acta Phys. Polon. **33**, 2081, (2002).

Received Julay 21, 2003