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ON VOINEA'S ANALOGY

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R.Voinea previously remarked an analogy between the relativistic displacements acted by a constant force and the deformation of a Bernoulli-Euler bar, in the case of a constant moment couple. In this paper, we extend this analogy to forces varying with the time and variable bending moments and rigidities accordingly, putting into evidence intrinsic equations and setting up the corresponding solutions for arbitrary Cauchy data. We then notice that the intrinsic equation for the beam deflection coincides with that corresponding to the deflection of a relativistic electron beam under a certain magnetic field.

Key words: Bernoulli-Euler bar, relativistic mechanics, Newton's mechanics, REB (relativistic electron beam)

1. INTRODUCTION

In [11] R. Voinea emphasized an analogy between two completely different physical phenomena: the rectilinear displacement in the relativistic frame under a constant force and the large deformations of a straight bar for a constant bending moment and constant rigidity. He showed that the corresponding governing equations differ by a sign and both the solutions for null Cauchy data may be put under a common form of a conic depending on a parameter a. The case $a < 0$ yields a hyperbola and represents the implicit solution of the relativistic Cauchy problem, while $a > 0$ corresponds to an ellipse (or circle) and gives the implicit solution of the standard cantilever bar problem.

In what follows, we consider the relativistic model for time-dependent forces on the one hand, and the Bernoulli-Euler bar acted upon by variable bending moments and rigidities on the other hand. We firstly try to reduce each model class to an intrinsic equation, which does not depend on the physical data, and then find the corresponding solutions for associated Cauchy problems with arbitrary data. It should be mentioned that this problem, solved by the linear equivalence method, introduced by the second author [10], served as a common frame for several typical bar problems: cantilever, simply supported and hyperstatic [6], [9].

In the last section, a third term of comparison is emphasized: the deflection of a relativistic electron beam – REB – under a magnetic field, previously associated to the Bernoulli-Euler bar deflection [7], [8].

2. THE MODELS

The general relativistic movement of a material particle $x = x(t)$ of rest mass m_0 in the case of an arbitrary force $F = F(t)$ is given by the second order ODE [1], [11]

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{F(t)}{m_0}\left[1 - \frac{1}{c^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right]^{\frac{3}{2}}, \tag{1}$$

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where c is the speed of light in vacuum.

The deformation $y = y(x)$ of a Bernoulli-Euler bar [3], [4] is given by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{M(x)}{E(x)I(x)}\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}, \tag{2}$$

where $M = M(x)$ is the bending moment and $E(x)I(x)$ is the rigidity (product between the modulus of longitudinal elasticity and the moment of inertia of the cross section with respect to the neutral axis), both considered as functions of x, varying on an interval $[0, l]$, taken along the rest position of the bar, l being the bar length.

3. THE INTRINSIC EQUATIONS

We get now the corresponding intrinsic equations for the above two models.

Let us first take equation (1). Performing at the beginning the change of variable

$$\tau = ct, \tag{3}$$

we get the equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = f(\tau)\left[1-\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2\right]^{\frac{3}{2}}, \tag{4}$$

where

$$f(\tau) = \frac{1}{m_0 c^2} F\left(\frac{\tau}{c}\right). \tag{5}$$

Now if we denote by

$$z = \frac{\mathrm{d}x}{\mathrm{d}\tau}, \tag{6}$$

equation (4) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = f(\tau)\left(1 - z^2\right)^{\frac{3}{2}}. \tag{7}$$

Introducing in (7) the new variable

$$h(\tau) = \int_{\tau_0}^{\tau} f(\theta)\mathrm{d}\theta, \tag{8}$$

we get a new equation

$$\frac{\mathrm{d}z}{\mathrm{d}h} = \left(1 - z^2\right)^{\frac{3}{2}} \tag{9}$$

We may call this equation *intrinsic*, as it does not formally depend on any physical data.

Let us consider now the Bernoulli-Euler bar equation (2). Here also we can denote by

$$z = \frac{\mathrm{d}y}{\mathrm{d}x}, \tag{10}$$

thus (2) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}x} = f(x)\left(1 + z^2\right)^{\frac{3}{2}}, \tag{11}$$

where

$$f(x) = \frac{M(x)}{E(x)I(x)}. \tag{12}$$

Introducing in (11) the new variable

$$h(x) = \int_{x_0}^{x} f(x')\mathrm{d}x', \tag{13}$$

we get in this case another equation

$$\frac{\mathrm{d}z}{\mathrm{d}h} = \left(1 + z^2\right)^{\frac{3}{2}}, \tag{14}$$

for which again one has any reason to call *intrinsic*, as it also does not depend on the physical data.

So, we see that both physical phenomena, different as they are, have a similar mathematical core, differing by a sign only.

4. THE SOLUTIONS

To get the solutions of the above models, we add some arbitrary Cauchy conditions

$$x(t_0) = \alpha \quad ; \quad \frac{\mathrm{d}x}{\mathrm{d}t}(t_0) = c\beta, \tag{15}$$

to equation (1) and the arbitrary Cauchy conditions

$$y(x_0) = \alpha \quad ; \quad \frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = \beta, \tag{16}$$

to equation (2).

With these specifications, we try to solve the above models, starting from the corresponding intrinsic equations. We begin with the Cauchy problem (1), (15), thus starting from (9), in which we perform the change of function

$$z = \sin u, \tag{17}$$

which leads to the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}h} = \cos^2 u \tag{18}$$

allowing the general solution

$$\tan u = h + k, \tag{19}$$

or, in terms of z,

$$z = \frac{h + k}{\sqrt{1 + (h + k)^2}}. \tag{20}$$

From (15) and (20) we immediately get

$$k = \frac{\beta}{\sqrt{1-\beta^2}}. \tag{21}$$

So, the general solution of the Cauchy problem (1), (14) is expressed as

$$x(t) = \alpha + \int_{ct_0}^{ct} \frac{h(\tau') + k}{\sqrt{1 + [h(\tau') + k]^2}} \, \mathrm{d}\tau'. \tag{22}$$

In [5] we deduced by similar techniques the solution of the second Cauchy problem (2), (16), also starting from the corresponding intrinsic equation (14). Yet in this case, we used no more the trigonometric change of function (17), but the change

$$z = \sinh u, \tag{23}$$

leading to the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}h} = \cosh^2 u, \tag{24}$$

whose general solution is

$$\tanh u = h + k, \tag{25}$$

and for (14) we find

$$z = \frac{h+k}{\sqrt{1-(h+k)^2}}. \tag{26}$$

The constant k results from (16) and (26)

$$k = \frac{\beta}{\sqrt{1+\beta^2}}. \tag{27}$$

Consequently, the general solution of the Cauchy problem (2), (16) is expressed as

$$y(x) = \alpha + \int_{x_0}^{x} \frac{h(x') + k}{\sqrt{1 - [h(x') + k]^2}} \, \mathrm{d}x', \tag{28}$$

with k defined in (27) and α given by (16).

We see that the difference in sign of the two considered models is also reflected in their solutions and even more, in the associated constants. To get more insight into this similarity, we shall write the intrinsic equations and their corresponding solutions under common formulae, by introducing the parameter sign:

$$\mathrm{sign} = \begin{cases} -1 & \text{in the relativistic case} \\ 1 & \text{in the case of the bar} \\ 0 & \text{in the limit case.} \end{cases} \tag{29}$$

Thus, the intrinsic equations (9) and (14) may be written under the common form

$$\frac{\mathrm{d}z}{\mathrm{d}h} = \left(1 + \mathrm{sign}\; z^2\right)^{\frac{3}{2}}. \tag{30}$$

Now, if we define the function:

$$k(\mathrm{sign}) = \frac{\beta}{\sqrt{1 + \mathrm{sign}\,\beta^2}}, \tag{31}$$

we see that both formulae (20) and (26) may be also written in a common frame

$$z = \frac{h + k(\text{sign})}{\sqrt{1 - \text{sign}[h + k(\text{sign})]^2}}. \tag{32}$$

The limit case – sign =0 – fits in for both models.

In case of the Cauchy problem (1), (15) we admit that $f(\tau) \neq 0$, i.e. $F(t) \neq 0$ for $t \geq t_0$. Analogously, in case of the B. – E. bar we admit that $f(x) \neq 0$, hence $M(x) \neq 0$, thus excluding a point of inflexion of the deformed bar axis, that would have required a piecewise calculus. We observe that both functions $h(\tau)$ and $h(x)$ are dimensionless; this follows from the geometric and mechanic signification of the functions $f(\tau)$, $f(x)$ accordingly, as well as from those of their primitives (8) and (13). As a consequence, the function z is also dimensionless, and this is an outstanding property that should also be expected from the intrinsic character of the equations (9) and (14).

Introducing the velocity $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, we can also write

$$v = c \tanh \theta, \tag{33}$$

where $\theta = \theta(t)$ is a dimensionless function; the second condition (15) leads to

$$v_0 = v(t_0) = c\beta = c \tanh \theta_0, \quad \theta_0 = \theta(t_0). \tag{34}$$

We can "translate" the Cauchy conditions (15) for the intrinsic equation (9), putting $\tau_0 = ct_0$. We have

$$\begin{aligned} z\big|_{h=0} &= z\big|_{\tau=\tau_0} = \left.\frac{dx}{d\tau}\right|_{\tau=\tau_0} = \tanh \theta_0, \\ \left.\frac{dz}{dh}\right|_{h=0} &= \left(1 - z^2\right)^{\frac{3}{2}}\Big|_{h=0} = \operatorname{sech}^3 \theta_0. \end{aligned} \tag{35}$$

From (20) we have also

$$h + k = \frac{z}{\sqrt{1 - z^2}}, \tag{36}$$

hence the integration constant is

$$k = \sinh \theta_0. \tag{37}$$

Relationships (20) and (36) can also be written in the form

$$\frac{1}{z^2} - \frac{1}{(h+k)^2} = 1, \tag{38}$$

representing a rectangular hyperbola with respect to the variables $\frac{1}{z}$ and $\frac{1}{h+k}$. We can write

$$z = \tanh \theta, \quad h + k = \sinh \theta, \tag{39}$$

which should be expected. We also notice that

$$\begin{aligned} -1 < z = \frac{v}{c} = \tanh \theta < 1, \\ -\infty < h + k = \sinh \theta < \infty, \end{aligned} \tag{40}$$

so that from the hyperbola only two half-branches correspond, except for the points $(\pm 1, 0)$ (Fig.1, thick line).

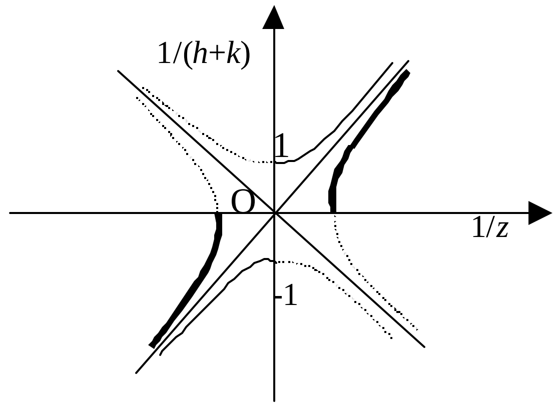

Fig. 1

In the case of the B.-E. bar we can write

$$z = \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta, \tag{41}$$

where the dimensionless function $\theta = \theta(x)$ represents the rotation of the cross section of the bar. The Cauchy conditions for the equation (14)are of the form

$$\begin{gathered} z\big|_{h=0} = z\big|_{x=x_0} = \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x=x_0} = \tan\theta_0, \quad \theta_0 = \theta(x_0), \\ \frac{\mathrm{d}z}{\mathrm{d}h}\bigg|_{h=0} = \left(1+z^2\right)^{\frac{3}{2}}\bigg|_{h=0} = \sec^3\theta_0. \end{gathered} \tag{42}$$

From (26) we also have

$$h + k = \frac{z}{\sqrt{1+z^2}}, \tag{43}$$

and thus the integration constant is

$$k = \sin\theta_0. \tag{44}$$

Relationships (26) and (43) lead to

$$\frac{1}{(h+k)^2} - \frac{1}{z^2} = 1, \tag{45}$$

i.e. another rectangular hyperbola, this time with respect to the variables $\frac{1}{h+k}$ and $\frac{1}{z}$, conjugate to the hyperbola (38). Observing that

$$z = \tan\theta, \quad h + k = \sin\theta, \tag{46}$$

we also notice that

$$-1 < h + k < 1, \quad -\infty < z < \infty, \tag{47}$$

which means $-\pi/2 < \theta < \pi/2$; hence

$$h < 1 - k = 1 - \sin\theta_0 . \qquad (48)$$

So, in this case too, there correspond two half branches of the hyperbola, but the points $(0,\pm 1)$ (Fig. 1, thin lines).

Another outstanding property of the hyperbolae is that their graphs do not change, no matter z, h and k; thus, they are invariants for each corresponding problem.

5. GRAPHICAL APPROACH

The graph of the function (20) is represented in Fig. 2, where we emphasize its remarkable points and the horizontal asymptotes.

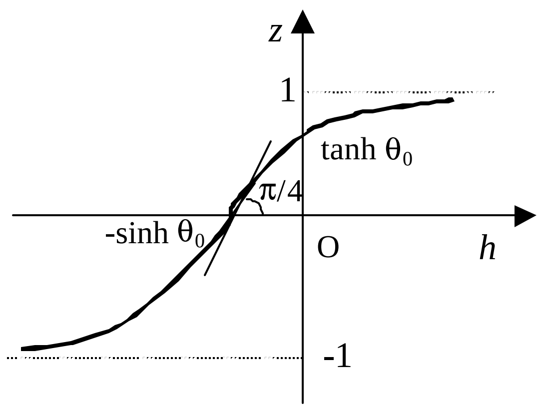

Fig. 2

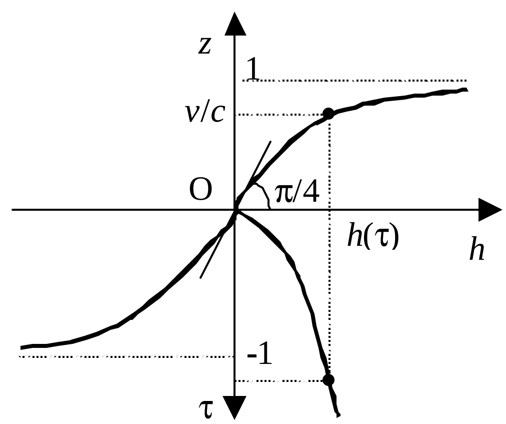

Fig. 3

Given the initial velocity v_0, therefore $\tanh\theta_0$, and consequently θ_0, the corresponding graph allows to get the velocity v as a function of the new variable h. For homogeneous initial conditions $v_0 = 0$, we have $\theta_0 = 0$, involving $\sinh\theta_0 = 0$, $\tanh\theta_0 = 0$ and $k = 0$ and, obviously, the function (20) becomes

$$z = \frac{h}{\sqrt{1 + h^2}} . \qquad (49)$$

Its graph is represented in Fig. 3. Admitting that $F(t) > 0$, it follows $f(\tau) > 0$, whence $h(\tau) > 0$. Joining the graph of the function $h(\tau)$ with respect to an $O\tau$ – axis along the Oz – axis and of opposite sense (the function $h(\tau)$ is obtained by quadrature), we obtain a graphical approach for $\tanh\theta$ for an arbitrary $\tau = \text{const}$. Hence we know the velocity v at any moment t. We observe that for $t > 0$ we get $h > 0$ and $z > 0$, hence $\theta > 0$ and $v > 0$. For $t \to \infty$ we get $v \to c$. It is convenient to use a dimensionless co-ordinate $\frac{\tau}{\tau_1} = \frac{t}{t_1}$, ($t_1$ being arbitrarily chosen), so that the two graphs be compatible (the co-ordinates be dimensionless in both cases).

Similarly, the graph of the function (26) is represented in Fig. 4, where its remarkable points are put into evidence as well as the vertical asymptotes. Given the rotation θ_0 at the bar left end, the corresponding graph allows us to get the rotation of the bar cross section (i.e., $\tan\theta$) as a function of the new variable h. In the case of a cantilever bar, $\theta_0 = 0$ (Fig. 5), hence $k = 0$, and thus the function (26) becomes

$$z = \frac{h}{\sqrt{1 - h^2}}, \qquad h = \sin\theta < 1, \qquad (50)$$

and its graph takes the form of Fig. 5. Admitting $M(x) > 0$, it follows that $f(x) > 0$, whence $h(x) > 0$. Joining the graph of the function $h(x)$ with respect to an Ox – axis along the Oz – axis and of opposite sense (the function $h(x)$ is obtained by quadrature), we obtain a graphical approach for $\tan\theta$ for an arbitrary cross – section x, hence for its rotation θ. We observe that for $x > 0$ we get $h > 0$ and $z = \tan\theta > 0$, hence $\theta > 0$. For $x = l$ (l is the bar length), we get $\theta(l) = \theta_{max}$. In this case too, it is convenient to use a dimensionless co-ordinate x/l, so that the two graphs be compatible (the co-ordinates be dimensionless in both cases).

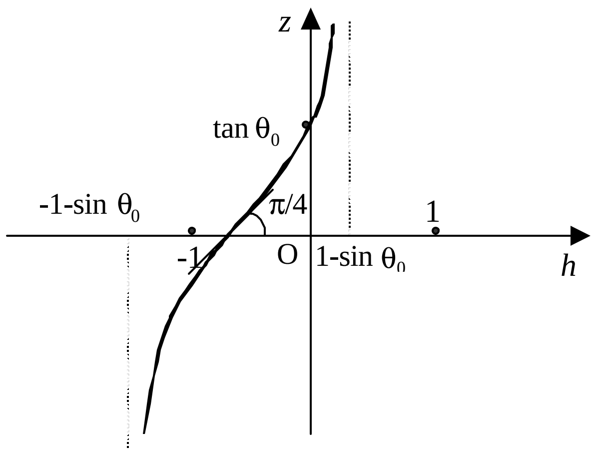

Fig. 4

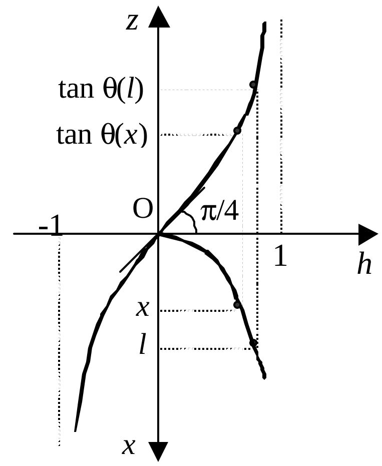

Fig. 5

We thus emphasize a connection between the deformed neutral axis of a straight B.-E. bar, of variable rigidity, acted upon by an arbitrary bending moment, in the nonlinear case, in an Euclid-ean space, and the world line of a particle in relativistic rectilinear movement, acted upon by a time-dependent force, in a Mikowkij pseudo-Euclidean space.

5. THE THIRD TERM OF THE ANALOGY

In some previous papers [7], [8], we put into evidence another analogy, between the Bernoulli-Euler bar deformation in the nonlinear case and the deflection of a REB under certain magnetic fields.

A REB is obtained e.g. in a linear accelerator, the magnetic field being produced by a deflecting coil. One may therefore consider the magnetic induction **B** as defined by three components with respect to a Cartesian co-ordinate system $xOyz$:

$$B_x(x, y, 0)) \equiv B(x, y), \quad B_y = 0, \quad B_z = 0. \tag{51}$$

Without restrictions, we may assume that the electronic beam is contained in the plane xOy. Under these hypotheses, the parametric equations of the plane trajectories [2], together with the components v_x, v_y of the velocity, lead to the following first order ODS, where differentiation is taken with respect to time

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= -\frac{e}{m} B(x, y) v_y \\ \dot{v}_y &= \frac{e}{m} B(x, y) v_x . \end{aligned} \tag{52}$$

In (52), e is the electron charge and m the relativistic mass.

In [8] it was proved that the system (52) may be written in the form (2). Indeed, by eliminating v_x, v_y we find

$$\ddot{x} = -\frac{e}{m} B(x, y)\, \dot{y} \quad ; \quad \ddot{y} = \frac{e}{m} B(x, y)\, \dot{x}. \tag{53}$$

If we consider y as a function of x, then

$$\dot{y} = \dot{x}\frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \ddot{y} = \dot{x}^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \ddot{x}\frac{\mathrm{d}y}{\mathrm{d}x} \tag{54}$$

and this yields

$$\frac{e}{m} B(x, y)\dot{x} = \dot{x}^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{e}{m} B(x, y)\dot{y}\frac{\mathrm{d}y}{\mathrm{d}x}, \tag{55}$$

or

$$\dot{x}^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{e}{m} B(x, y)\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]. \tag{56}$$

From (53) we get straightforwardly the classic prime integral

$$\frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) = K, \tag{57}$$

which, combined with (54), gives

$$\dot{x} = \sqrt{\frac{2K}{m}}\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{-\frac{1}{2}}. \tag{58}$$

Introducing this in (56), it is immediately obtained

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{e}{2K} B(x, y)\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{\frac{3}{2}}. \tag{59}$$

If the induction B depends at most on x, then (59) is exactly of the form (2). Yet, from the physical point of view, one only has either B effectively dependent on both x and y, or B constant. A constant induction is obtained only for an infinite deflection coil; however difficult, the tendency is for such an induction in the case of an industrial accelerator.

Putting in (59) $z = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ and

$$h(x) = \frac{e}{2K}\int_{x_0}^{x} B(x', y(x'))\mathrm{d}x', \tag{60}$$

we get again (14). But in this case, the intrinsic equation gives no more directly the solution, as h depends on y.

The Cauchy conditions are handled somewhat more complicated, because of the constant K and the time- dependence of the unknown functions in (52). From physical reasons, the following initial conditions should be associated to the system (52)

$$x(0) = 0,\ y(0) = 0, \qquad v_x(0) = v_0,\ v_y(0) = 0. \tag{61}$$

Thus, in this case, due to model restrictions, we cannot take arbitrary Cauchy data and must take $t_0 = 0$, which yields $x_0 = x(t_0) = 0$.

So far, we did not discuss the constant K in formula (57). From (61) we get

$$K = \frac{1}{2} m v_0^2 \tag{62}$$

and the initial conditions (61) become, by the first formula (54), for y thought as a function of x

$$y(0) = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x}(0) = \frac{\dot{y}(0)}{\dot{x}(0)} = \frac{0}{v_0} = 0 . \tag{63}$$

In the ideal case of a constant B, the problem is completely solved, either straightforwardly, or starting from the associated intrinsic equation (30), written for $\text{sign} = 1$, whose solution (32) becomes

$$z = \frac{\frac{e}{2K} Bx}{\sqrt{1 - \left(\frac{e}{2K} Bx\right)^2}} , \tag{64}$$

as $k = 0$ and $h(x) = \frac{e}{2K} Bx$, for $x_0 = 0$. Consequently, the solution of (59) under the null Cauchy conditions (63) may be written under the implicit form of a circle $x^2 + y^2 = \frac{4K^2}{e^2}$.

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