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PIVOT FRICTION IN PLANAR MOTIONS

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This paper presents an anallysis of a frictional planar rigid-body motion. Friction occurs on the contact area between the rigid body and the bearing horizontal plane producing a resultant friction force and a pivot friction moment. The general solution depends on four distinct integrals. In the case of a circular contour only two integrals remain which result in closed forms. Their evaluation is made through an integral representation for the reciprocal distance between two points. The solution of the planar circular disk motion with friction is checked against known results on particular motions.

Key words: planar motion, friction force, pivot friction moment, elliptical integrals

1. INTRODUCTION

In most dynamics texts contact between two rigid bodies is admitted at one point. Constraints at this contact point generally produce a normal reaction force, a friction force, a rolling friction moment and a pivot friction moment [1], [3]. Motions of a circular disc in contact with a horizontal plane and a slope [3] are outstanding examples of this kind. But what happens when the contact between two moving bodies occurs not at a point, but on a surface? How do we evaluate the friction forces that appear on the contact area? In planar motions with friction on the bearing surface a resultant friction force and a pivot friction moment are produced. This paper deals with the frictional planar motion of a flat plate on the horizontal free surface of a half-space. Most equations presented are those commonly found in dynamics texts in one form or another. We are not aware of a similar problem treated in the literature.

2. FORMULATION OF THE PROBLEM FOR A FLAT PLATE OF ARBITRARY CONTOUR

Consider a flat plate of arbitrary contour performing a planar motion in the plane defined by the axes O_1x_1 and O_1y_1 of the fixed frame of reference $O_1x_1y_1z_1$, unit vectors $\vec{i}_1$, $\vec{j}_1$, $\vec{k}_1$. The plate is loaded with an uniform normal pressure γ on the domain of contact D with the plane $z_1 = 0$. Its motion is opposed by coulombian friction of coefficient μ. Let O be the center of mass of the plate and $(x_0,\ y_0,\ 0)$ its co-ordinates in the fixed frame $O_1x_1y_1z_1$. Let $O\,x\,y\,z$ be the frame of reference attached to the moving plate, unit vectors $\vec{i}$, $\vec{j}$, $\vec{k} = \vec{k}_1$, with the axes Ox and Oy in the plane of motion. The angle ψ measures the rotation of the moving frame of reference with respect to the fixed one, figure 1.

Let τ_O be the torsor about point O of the directly applied force system $\vec{F}_i$, $i = \overline{1,n}$, acting on the plane $z_1 = 0$ at the points A_i, $i = \overline{1,n}$ and τ_{fO} the torsor about the same point of the friction forces. If $\vec{v}$ is the velocity of the current point M (Cartesian co-ordinates x and y) and $d\sigma$ the area element on the moving plate, figure 2, then:

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$$\tau_O = \begin{cases} \vec{R} = \sum_{i=1}^{n} \vec{F}_i = X\,\vec{i}_1 + Y\,\vec{j}_1 \\ \vec{M}_O = \sum_{i=1}^{n} \overrightarrow{OA_i} \times \vec{F}_i = M_O \vec{k}_1 \end{cases} \quad , \quad \tau_{fO} = \begin{cases} \vec{F}_f = \iint_D d\vec{F}_f = -\mu\gamma \cdot \iint_D \frac{\vec{v}}{v}\, d\sigma \\ \vec{M}_{fO} = \iint_D \vec{r} \times d\vec{F}_f = -\mu\gamma \cdot \iint_D \vec{r} \times \frac{\vec{v}}{v}\, d\sigma \ . \end{cases} \tag{1}$$

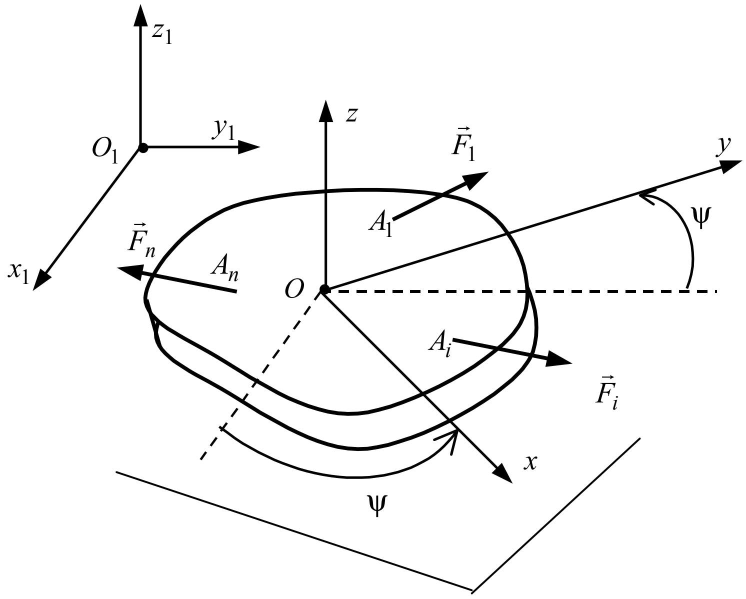

Fig.1 Flat plate in planar motion.

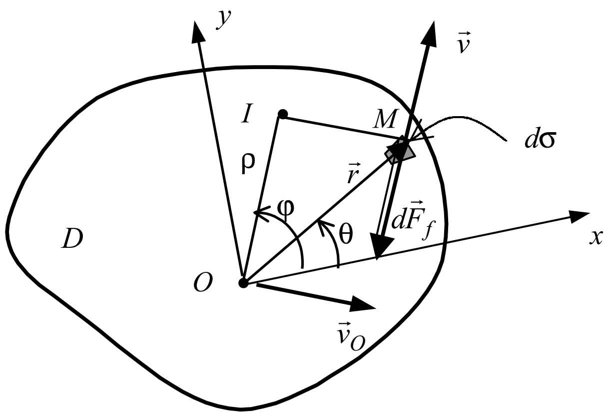

Fig.2 Geometry related to frictional planar motion.

The equations of motion of the plate are:

$$m\dot{\vec{v}}_O = \vec{R} + \vec{F}_f \ ; \qquad J_O\dot{\vec{\omega}} = \vec{M}_O + \vec{M}_{fO} \, . \tag{2}$$

Here we identify m as the mass of the plate, J_O as its central moment of inertia, $\vec{v}_O$ as the velocity of point O and $\vec{\omega}$ as the angular velocity of the plate ($\omega = \dot{\psi}$). Integration of the differential equations (2) requires for the integrals defined in (1) at least a form suitable for numerical evaluation.

Let us first note that $\vec{v}$ and v can be written as

$$\vec{v} = \vec{v}_O + \vec{\omega} \times \vec{r} \qquad , \qquad v = \omega \cdot IM \, , \tag{3}$$

I being the instant rotation centre of the plate. The integrals in (1) become

$$\iint_D \frac{\vec{v}}{v}\,d\sigma = \frac{\vec{v}_O}{\omega}\cdot\iint_D \frac{1}{IM}\,d\sigma + \vec{k}_1 \times \iint_D \frac{\vec{r}}{IM}\,d\sigma\,,$$
$$\iint_D \vec{r}\times\frac{\vec{v}}{v} = -\frac{\vec{v}_O}{\omega}\times\iint_D \frac{\vec{r}}{IM}\,d\sigma + \vec{k}_1\cdot\iint_D \frac{r^2}{IM}\,d\sigma\,. \tag{4}$$

The positions of points I and M will be represented by the polar co-ordinates $(\rho\,,\ \varphi)$ and $(r\,,\ \theta)$ respectively. F_f and M_{fO} can be consequently expressed by means of four independent integrals:

$$I_1(\rho\,,\varphi) = \iint_D \frac{1}{IM}\,d\sigma = \int_0^{2\pi}\int_0^{a(\theta)} \frac{r\,dr\,d\theta}{\sqrt{\rho^2 + r^2 - 2\rho r\cos(\varphi-\theta)}}\,,$$
$$I_{2x}(\rho\,,\varphi) = \iint_D \frac{x}{IM}\,d\sigma = \int_0^{2\pi}\int_0^{a(\theta)} \frac{r^2\cos\theta\,dr\,d\theta}{\sqrt{\rho^2 + r^2 - 2\rho r\cos(\varphi-\theta)}}\,,$$
$$I_{2y}(\rho\,,\varphi) = \iint_D \frac{y}{IM}\,d\sigma = \int_0^{2\pi}\int_0^{a(\theta)} \frac{r^2\sin\theta\,dr\,d\theta}{\sqrt{\rho^2 + r^2 - 2\rho r\cos(\varphi-\theta)}}\,, \tag{5}$$
$$I_3(\rho\,,\varphi) = \iint_D \frac{r^2}{IM}\,d\sigma = \int_0^{2\pi}\int_0^{a(\theta)} \frac{r^3\,dr\,d\theta}{\sqrt{\rho^2 + r^2 - 2\rho r\cos(\varphi-\theta)}}\,,$$

$r = a(\theta)$ being the equation of the boundary of D in polar co-ordinates.

The following equations slightly modified from [4]

$$\rho\cos\varphi = \frac{\dot{x}_O}{\dot{\psi}}\sin\psi - \frac{\dot{y}_O}{\dot{\psi}}\cos\psi \quad,\quad \rho\sin\varphi = \frac{\dot{x}_O}{\dot{\psi}}\cos\psi + \frac{\dot{y}_O}{\dot{\psi}}\sin\psi\,, \tag{6}$$

link the co-ordinates of I in the moving frame to the velocity of O and the angular velocity of the plate.

The expressions of the resultant friction force and of the resultant friction moment are

$$\vec{F}_f = -\mu\gamma\cdot\left[\rho\, I_1(\rho\,,\ \varphi)\sin(\varphi+\psi) - I_{2x}(\rho\,,\ \varphi)\sin\psi - I_{2y}(\rho\,,\ \varphi)\cos\psi\right]\cdot\vec{i}_1 +$$
$$+\,\mu\gamma\cdot\left[\rho\, I_1(\rho\,,\ \varphi)\cos(\varphi+\psi) - I_{2x}(\rho\,,\ \varphi)\cos\psi + I_{2y}(\rho\,,\ \varphi)\sin\psi\right]\cdot\vec{j}_1\,, \tag{7}$$
$$\vec{M}_{fO} = -\mu\gamma\cdot\left\{I_3(\rho\,,\ \varphi) - \rho\left[I_{2x}(\rho\,,\ \varphi)\cos\varphi + I_{2y}(\rho\,,\ \varphi)\sin\varphi\right]\right\}\cdot\vec{k}_1\,.$$

The equations of motion (2) in the fixed frame of reference result as

$$m\ddot{x}_O = X - \mu\gamma\cdot\left[\rho\, I_1(\rho\,,\ \varphi)\sin(\psi+\varphi) - I_{2x}(\rho\,,\ \varphi)\sin\psi - I_{2y}(\rho\,,\ \varphi)\cos\psi\right],$$
$$m\ddot{y}_O = Y + \mu\gamma\cdot\left[\rho\, I_1(\rho\,,\ \varphi)\cos(\psi+\varphi) - I_{2x}(\rho\,,\ \varphi)\cos\psi + I_{2y}(\rho\,,\ \varphi)\sin\psi\right], \tag{8}$$
$$J_O\ddot{\psi} = M_O - \mu\gamma\cdot\left\{I_3(\rho\,,\ \varphi) - \rho\left[I_{2x}(\rho\,,\ \varphi)\cos\varphi + I_{2y}(\rho\,,\ \varphi)\sin\varphi\right]\right\},$$

and they can be integrated numerically in arbitrary initial conditions along with the equations (6).

The kinetic energy lost instantaneously by the moving plate due to friction is

$$P_f = -\left(\vec{v}_O\cdot\vec{F}_f + \vec{\omega}\cdot\vec{M}_{fO}\right) = \mu\gamma\dot{\psi}\cdot\left\{\rho^2 I_1(\rho,\quad \varphi) -\right.$$
$$\left. -\,2\rho\left[I_{2x}(\rho,\quad \varphi)\cos\varphi + I_{2y}(\rho,\quad \varphi)\sin\varphi\right] + I_3(\rho,\quad \varphi)\right\} = \mu\gamma\dot{\psi}\cdot\iint_D IM\,d\sigma \tag{9}$$

and results from (5), (6), (7) and (8) along with the values of x_O , y_O , ψ , ρ , φ .

3. CLOSED FORM SOLUTION FOR THE CIRCULAR DISK.

For a circular plate of radius a the integrals (5) can be evaluated in closed forms. The approach is through the integral representation for the reciprocal distance between two points [5]:

$$\frac{1}{\left[\rho^2+r^2-2\rho r\cos(\varphi-\theta)\right]^{\frac{1+\varepsilon}{2}}}=\frac{2}{\pi}\cos\frac{\pi\varepsilon}{2}\int_0^{m(r)}\frac{\lambda\left(\frac{u^2}{\rho r},\ \varphi-\theta\right)}{\left[(\rho^2-u^2)(r^2-u^2)\right]^{\frac{1+\varepsilon}{2}}}u^{\varepsilon}\,du \qquad (10)$$

with $0\le\varepsilon<1$ and

$$m(r)=\min(\rho\,,r)=\frac{1}{2}\left[r+\rho-|r-\rho|\right]=\begin{cases}r\ , & r<\rho\\ \rho\ , & \rho\le r\le a\end{cases}, \qquad (11)$$

$$\lambda(k\ ,\ \alpha)=\frac{1-k^2}{1+k^2-2\,k\cos\alpha}\ . \qquad (12)$$

For $\varepsilon=0$ we have

$$\frac{1}{\sqrt{\rho^2+r^2-2\rho r\cos(\varphi-\theta)}}=\frac{2}{\pi}\int_0^{m(r)}\frac{\lambda\left(\frac{u^2}{\rho r},\ \varphi-\theta\right)}{\sqrt{(\rho^2-u^2)(r^2-u^2)}}\,du\ . \qquad (13)$$

3.1. Evaluation of integral $I_1(\rho\ ,\ \varphi)$

Substitution of (13) into the first integral (5) yields after changing the order of integration (appendix 1)

$$\begin{aligned}I_1(\rho\,,\varphi)&=\frac{2}{\pi}\int_0^{2\pi}d\theta\int_0^a r\,dr\int_0^{m(r)}\frac{\lambda\left(\frac{u^2}{\rho r},\varphi-\theta\right)}{\sqrt{(\rho^2-u^2)(r^2-u^2)}}du=\\&=\frac{2}{\pi}\int_0^{m(a)}\frac{du}{\sqrt{\rho^2-u^2}}\int_u^a\frac{r\,dr}{\sqrt{r^2-u^2}}\int_0^{2\pi}\lambda\left(\frac{u^2}{\rho r},\varphi-\theta\right)dr.\end{aligned} \qquad (14)$$

The first iterate integral (see appendix 2) is $L_0\left(u^2/\rho r\ ,\ \varphi\right)=2\pi$ so that the result depends only of ρ:

$$I_1(\rho\,,\varphi)=I_1(\rho)=4\int_0^{m(a)}\frac{\sqrt{a^2-u^2}}{\sqrt{\rho^2-u^2}}du=\begin{cases}4a\cdot E\left(\frac{\rho}{a}\right) & \text{if }\ \rho<a\\ 4\rho\cdot\left[E\left(\frac{a}{\rho}\right)+\left(\frac{a^2}{\rho^2}-1\right)\cdot K\left(\frac{a}{\rho}\right)\right] & \text{if }\ \rho\ge a\end{cases}, \qquad (15)$$

where

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-k^2\sin^2 t}}=\frac{\pi}{2}\left\{1+\sum_{p=1}^{\infty}\left[\frac{(2p-1)!!}{(2p)!!}\right]^2\cdot k^{2p}\right\},\quad 0\le k<1, \qquad (16)$$

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2 t}\,dt=\frac{\pi}{2}\left\{1-\sum_{p=1}^{\infty}\left[\frac{(2p-1)!!}{(2p)!!}\right]^2\cdot\frac{k^{2p}}{2p-1}\right\},\quad 0\le k\le 1,$$

stand for Legendre's elliptical integrals [6].

3.2. Evaluation of integrals $I_{2x}(\rho,\ \varphi)$ and $I_{2y}(\rho,\ \varphi)$

Introduce the complex $\left(i^2=-1\right)$ integral:

$$I_2^c(\rho,\varphi)=I_{2x}(\rho,\ \varphi)+i\cdot I_{2y}(\rho,\ \varphi)=\frac{2}{\pi}\int_0^{2\pi}e^{i\theta}d\theta\int_0^a r^2dr\int_0^{m(r)}\frac{\lambda\left(\frac{u^2}{\rho r},\varphi-\theta\right)}{\sqrt{\left(\rho^2-u^2\right)\left(r^2-u^2\right)}}du=$$

$$=\frac{2}{\pi}\int_0^{m(a)}\frac{du}{\sqrt{\rho^2-u^2}}\int_u^a\frac{r^2dr}{\sqrt{r^2-u^2}}\int_0^{2\pi}\lambda\left(\frac{u^2}{\rho r},\varphi-\theta\right)e^{i\theta}\,d\theta. \tag{17}$$

From appendix 2 the first iterate integral results as $L_1\left(u^2/\rho r,\ \varphi\right)=2\pi\, e^{i\varphi}\,u^2/\rho r$. Therefore

$$I_2^c(\rho,\varphi)=e^{i\varphi}\cdot I_2(\rho), \tag{18}$$

where

$$I_2(\rho)=\left|I_2^c(\rho,\varphi)\right|=\frac{4}{\rho}\int_0^{m(a)}\frac{u^2\sqrt{a^2-u^2}}{\sqrt{\rho^2-u^2}}du=\begin{cases}4a\rho\cdot B\left(\dfrac{\rho}{a}\right) & if\ \ \rho<a\\ 4\dfrac{a^4}{\rho^2}\cdot C\left(\dfrac{a}{\rho}\right) & if\ \ \rho\ge a\end{cases} \tag{19}$$

with

$$B(k)=\int_0^{\frac{\pi}{2}}\sin^2 t\sqrt{1-k^2\sin^2 t}\,dt\quad,\quad C(k)=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t\cos^2 t}{\sqrt{1-k^2\sin^2 t}}dt\quad,\quad 0\le k\le 1. \tag{20}$$

In order to compute integrals $B(k)$ and $C(k)$ we introduce a new elliptical integral:

$$A(k)=\int_0^{\frac{\pi}{2}}\cos^2 t\sqrt{1-k^2\sin^2 t}\,dt\quad,\quad 0\le k\le 1\ . \tag{21}$$

Note that

$$A(k)+B(k)=E(k)\ ,\ \ A(k)-B(k)=k^2\,C(k)\ ,\ \ \frac{d}{dk}A(k)=-kC(k). \tag{22}$$

Therefore $A(k)$ satisfies the first order linear differential equation

$$\frac{d}{dk}A(k)+\frac{2}{k}A(k)=\frac{1}{k}E(k), \tag{23}$$

whose general solution is

$$A(k)=\frac{c}{k^2}+\frac{1}{3k^2}\left[\left(1+k^2\right)E(k)-\left(1-k^2\right)K(k)\right], \tag{24}$$

c being an arbitrary constant. As $A(1)=2/3$ it results $c=0$. Consequently we have

$$A(k)=\frac{1}{3k^2}\left[\left(1+k^2\right)E(k)-\left(1-k^2\right)K(k)\right] \quad , \quad B(k)=\frac{1}{3k^2}\left[\left(2k^2-1\right)E(k)+\left(1-k^2\right)K(k)\right] \ ,$$
$$C(k)=\frac{1}{3k^4}\left[\left(2-k^2\right)E(k)-2\left(1-k^2\right)K(k)\right]. \tag{25}$$

It can be easily cheched that $A(0)=B(0)=\pi/4$, $B(1)=C(1)=1/3$, $C(0)=\pi/16$. Finally we have

$$I_2(\rho)=\begin{cases}\dfrac{4}{3}\dfrac{a^3}{\rho}\left[\left(2\dfrac{\rho^2}{a^2}-1\right)\cdot E\left(\dfrac{\rho}{a}\right)+\left(1-\dfrac{\rho^2}{a^2}\right)\cdot K\left(\dfrac{\rho}{a}\right)\right] & if \ \rho<a \\ \dfrac{4}{3}\rho^2\left[\left(2-\dfrac{a^2}{\rho^2}\right)\cdot E\left(\dfrac{a}{\rho}\right)-2\left(1-\dfrac{a^2}{\rho^2}\right)\cdot K\left(\dfrac{a}{\rho}\right)\right] & if \ \rho\geq a\end{cases} \ , \tag{26}$$

$$I_{2x}(\rho,\varphi)=I_2(\rho)\cdot\cos\varphi \quad , \quad I_{2y}(\rho,\varphi)=I_2(\rho)\cdot\sin\varphi \ . \tag{27}$$

3.3. Evaluation of integral $I_3(\rho,\ \varphi)$

It follows form the substitution of (13) into the last integral (5) and the change of order of integration that

$$\begin{aligned} I_3(\rho,\varphi)&=\frac{2}{\pi}\int_0^{2\pi}d\theta\int_0^a r^3dr\int_0^{m(r)}\frac{\lambda\left(\dfrac{u^2}{\rho r},\varphi-\theta\right)}{\sqrt{\left(\rho^2-u^2\right)\left(r^2-u^2\right)}}du= \\ &=\frac{2}{\pi}\int_0^{m(a)}\frac{du}{\sqrt{\rho^2-u^2}}\int_u^a\frac{r^3dr}{\sqrt{r^2-u^2}}\int_0^{2\pi}\lambda\left(\frac{u^2}{\rho r},\varphi-\theta\right)d\theta= \\ &=\frac{4}{3}a^2\int_0^{m(a)}\frac{\sqrt{a^2-u^2}}{\sqrt{\rho^2-u^2}}du+\frac{8}{3}\int_0^{m(a)}\frac{u^2\sqrt{a^2-u^2}}{\sqrt{\rho^2-u^2}}du=\frac{1}{3}\left[a^2I_1(\rho)+2\rho I_2(\rho)\right].\end{aligned} \tag{28}$$

Note that in the case of the circular disk $I_3(\rho,\ \varphi)$ is not independent, but a combination of $I_1(\rho)$ and $I_2(\rho)$.

3.4. Resultant friction force, pivot friction moment, equations of motion, energy loss through friction.

Substitution of (27) and (28) into (7), (8) and (9) provide the following compact forms for:

1. The resultant friction force ($\vec{F}_f$ opposes $\vec{v}_O$):

$$\vec{F}_f=-\mu\gamma\cdot\left[\rho\, I_1(\rho)-I_2(\rho)\right]\cdot\frac{\vec{v}_O}{v_O}. \tag{29}$$

2. The pivot friction moment:

$$\vec{M}_{fO}=-\frac{1}{3}\mu\gamma\cdot\left[a^2I_1(\rho)-\rho I_2(\rho)\right]\cdot\vec{k}_1. \tag{30}$$

3. The equations of motion:

$$m\ddot{x}_O = X - \mu\gamma\cdot[\rho\, I_1(\rho) - I_2(\rho)]\cdot\frac{\dot{x}_O}{\sqrt{\dot{x}_O^2+\dot{y}_O^2}},$$

$$m\ddot{y}_O = Y - \mu\gamma\cdot[\rho\, I_1(\rho) - I_2(\rho)]\cdot\frac{\dot{y}_O}{\sqrt{\dot{x}_O^2+\dot{y}_O^2}}, \tag{31}$$

$$J_O\ddot{\psi} = M_O - \frac{1}{3}\mu\gamma\cdot[a^2 I_1(\rho) - \rho I_2(\rho)].$$

Note that the first two motion equations are similar to those studied in [2], page 28. However, they are more general and their integration can be done only numerically. A prime integral, $\psi + \varphi = \psi_0 + \varphi_0$ (ψ_0 and φ_0 being the initial values of the angles ψ and φ), exists only if $X = Y = 0$.

4. The energy loss through friction:

$$P_f = \mu\gamma\dot{\psi}\cdot\left[\left(\rho^2 + \frac{a^2}{3}\right)I_1(\rho) - \frac{4}{3}\rho I_2(\rho)\right]. \tag{32}$$

3.5. Particular cases.

It is of interest to see if the solutions obtained in the previous paragraphs are checked against known results.

1. *The disk rotates about its centre of mass*. In this case instant centre I overlaps point O $(\rho = 0)$.

$$\lim_{\rho\to 0} I_1(\rho) = 2\pi a\ ,\ \lim_{\rho\to 0} I_2(\rho) = 0\ ,\ F_f = 0\ ,\ M_{fO} = \frac{2}{3}\mu\gamma\pi a^3\ ,\ P_f = \dot{\psi} M_{fO}. \tag{33}$$

2. *The disk translates in the direction of* $\vec{v}_O$. In this case $\rho \to \infty$, $\dot{\psi} \to 0$, $\rho\dot{\psi} = v_O$.

$$\lim_{\rho\to\infty} I_1(\rho) = 0\ ,\quad \lim_{\rho\to\infty} \rho I_1(\rho) = \pi a^2,\quad \lim_{\rho\to\infty} I_2(\rho) = 0\ ,\quad \lim_{\rho\to\infty} \rho I_2(\rho) = 0,$$
$$F_f = \mu\gamma\pi a^2\ ,\quad M_{fO} = 0\ ,\quad P_f = v_O F_f. \tag{34}$$

4. SUMMARY AND CONCLUSIONS

This paper derives solutions for frictional planar rigid-body motions when friction occurs on the whole contact area between the body and the bearing surface. For a plate of arbitrary contour the solution includes four independent integrals which have to be evaluated numerically. For a circular disk there are only two independent integrals which result in closed forms of Legendre's elliptical integrals. The values of the resultant friction force and of the pivot friction moment are checked against known results on particular cases of motion: central rotation and translation. The problem of the frictional planar motion itself appears to be new.

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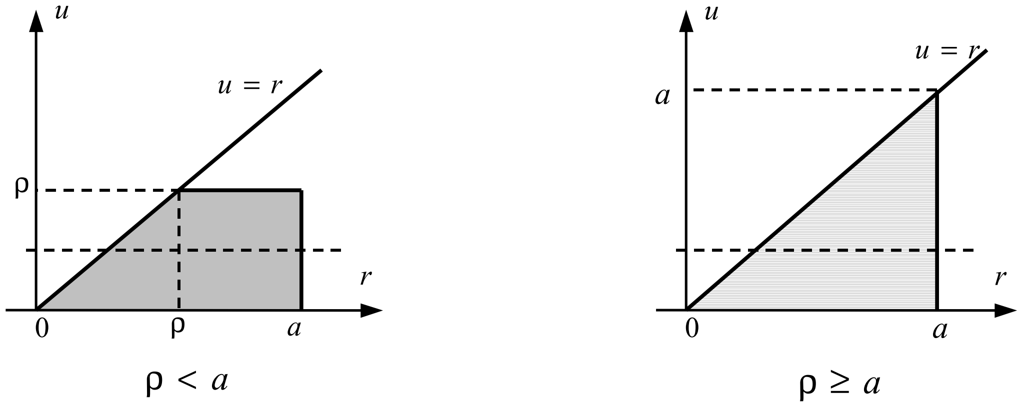
APPENDIX 1.

Domains of integration related to integrals $I_1(\rho\,,\,\varphi),\ I_2^c(\rho\,,\,\varphi),\ I_3(\rho\,,\,\varphi)$; Change of order of integration

Computation of integrals given at (14), (17) and (28) requires changes of order of integration. The shaded areas in the figure below represent the domains of integration. If $f(u\,,\,r)$ is an arbitrary integrable function on these domains then in both cases it results that

$$\int_{r=0}^{a}\left(\int_{u=0}^{m(r)} f(u\,,\,r)du\right)dr = \int_{u=0}^{m(a)}\left(\int_{r=u}^{a} f(u\,,\,r)dr\right)du\ .$$



Two-dimensional domains of integration

Case I. $\rho < a\ ,\ m(r) = \rho\ .$

$$\int_{r=0}^{a}\left(\int_{u=0}^{m(r)} f(u\,,\,r)du\right)dr = \int_{r=0}^{\rho}\left(\int_{u=0}^{r} f(u\,,\,r)du\right)dr + \int_{r=\rho}^{a}\left(\int_{u=0}^{\rho} f(u\,,\,r)du\right)dr =$$

$$= \int_{u=0}^{\rho}\left(\int_{r=u}^{\rho} f(u\,,\,r)dr\right)du + \int_{u=0}^{\rho}\left(\int_{r=\rho}^{a} f(u\,,\,r)dr\right)du = \int_{u=0}^{\rho=m(a)}\left(\int_{r=u}^{a} f(u\,,\,r)dr\right)du\ .$$

Case II. $\rho \geq a\ ,\ m(r) = a\ .$

$$\int_{r=0}^{a}\left(\int_{u=0}^{m(r)} f(u\,,\,r)du\right)dr = \int_{u=0}^{a=m(a)}\left(\int_{r=u}^{a} f(u\,,\,r)dr\right)du\ .$$

APPENDIX 2.

Evaluation of integral $L_n(k\,,\ \varphi)$, $n \in N$;

$$L_n(k\,,\ \varphi) = \int_0^{2\pi} \lambda(k\,,\ \varphi - \theta) e^{in\theta} d\theta \ , \ n \in N$$

Function $\lambda(k\,,\ \alpha)$ has been defined at (12). With the change of variable $\zeta = e^{i(\theta-\varphi)}$ the integral can be computed as a complex integral along the unit circle:

$$L_n(k\,,\varphi) = i\,(1-k^2)e^{in\varphi} \int_{|\zeta|=1} \frac{\zeta^n}{\zeta^2 k - \zeta\,(1+k^2) + k} d\zeta \,.$$

The roots of the equation $\zeta^2 k - \zeta\,(1+k^2) + k = 0$ are $\zeta_1 = k$, $\zeta_2 = \dfrac{1}{k}$.

If $|k| < 1$ then

$$L_n(k\,,\varphi) = i\,(1-k^2)e^{in\varphi}\ 2\pi i \operatorname{Re} z\left\{\frac{\zeta^n}{\zeta^2 k - \zeta\,(1+k^2) + k}\,,\ k\right\} =$$

$$= -2\pi(1-k^2)e^{in\varphi} \cdot \lim_{\zeta \to k}\left[(\zeta - k)\frac{\zeta^n}{k(\zeta-k)\left(\zeta - \frac{1}{k}\right)}\right] = 2\pi e^{in\varphi} k^n \ .$$

If $|k| > 1$ then

$$L_n(k\,,\varphi) = i\,(1-k^2)e^{in\varphi}\ 2\pi i \operatorname{Re} z\left\{\frac{\zeta^n}{\zeta^2 k - \zeta\,(1+k^2) + k}\,,\ \frac{1}{k}\right\} =$$

$$= -2\pi(1-k^2)e^{in\varphi} \cdot \lim_{\zeta \to \frac{1}{k}}\left[\left(\zeta - \frac{1}{k}\right)\frac{\zeta^n}{k(\zeta-k)\left(\zeta - \frac{1}{k}\right)}\right] = -2\pi e^{in\varphi}\,\frac{1}{k^n} \,.$$

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