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SOLVING A CONJECTURE ABOUT CERTAIN f - EXPANSIONS

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The conjecture asserts that the equivalence of the label sequence of the regular continued fraction (RCF) expansion to the sequence $(\xi_n)_{n\in \mathbf{N}_+}$ associated with it by the basic existence Theorem 1.1.2 from [6], still holds for the label sequence of any f-expansion satisfying conditions (C) and (BD$^{(2)}$). We prove that condition (C) and a stregthening of a Lipschitz condition used in [8] are sufficient to ensure a necessary and sufficient condition under which the asserted equivalence holds. The proof involves processes on several probability spaces and some associated dynamical systems relating the f-expansion considered to r.v.s. on the probability space used in the concluding theorem.

1. INTRODUCTION

Let $\mathbf{N}_+ = \{1,2,...\}$ and $\mathbf{N} = \mathbf{N}_+ \cup \{0\}$. Given an RSCC $\{(W, \mathcal{W}), X, u, P\}$, where X is a countable set, by Theorem 1.1.2 in [6] for any $w \in W$ there exist a probability space (Ω, K, P_w) and a sequence $(\xi_n)_{n\in \mathbf{N}_+}$ of X-valued r.v.s. on (Ω, K) such that

a. $$P_w(\xi_1 = i) = P(w,i),$$
$$P_w(\xi_{n+1} = i \mid \xi_1,...,\xi_n, \zeta_0, \zeta_1,...,\zeta_n) = P(\zeta_n, i), \ \ i \in X, \ n \in \mathbf{N}_+, \tag{1}$$

where

$$\zeta_n = u_{\xi_n ...\xi_1}(w), \ \ n \in \mathbf{N}_+, \ \zeta_0 \equiv w.$$

b. The sequence $(\zeta_n)_{n\in \mathbf{N}}$ is a W-valued homogenous Markov chain.

In particular, this theorem holds for the RSCC associated with the RCF and D-adic expansions. Denoting by $(a_n)_{n\in \mathbf{N}_+}$ the label sequence and by λ the Lebesgue measure, in these two cases the following equations which can be obtained by direct computation do hold:

$$\lambda(a_1 = i) = P(0,i) \ \ ; \ \lambda(a_{n+1} = i \mid a_1 = i_1,...,a_n = i_n) = P\left(u_{i_n ... i_1}(0)\right) \tag{2}$$

In what follows we shall use the notation from [6] where is shown that with any f-expansion satisfying conditions (BD$^{(2)}$) and (C) one can associate an RSCC. Hence one may particularize (1) to such f-expansions.

Let $I = [0,1]$ and denote by B_I the σ-algebra of Borel sets in I. For any $n \in \mathbf{N}_+$ let $I(i^{(n)}), i^{(n)} \in X^n$, be the fundamental intervals of order n and E_n the set of their endpoints. Let be $[\alpha, \beta]$ the interval of definition of f and put $\mathtt{I} := I \setminus \cup_{n\geq 1} E_n$. Clearly, $\lambda(\mathtt{I}) = 1$.

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It is known (see, e.g.,[6]) that when f has a second derivative in $[\alpha,\beta]\setminus \mathbf{N}_+$, and conditions (C) and ($\mathrm{BD}^{(2)}$) are fulfilled, hence an RSCC can be associated, the following properties hold .

c. The terms of the label sequence $(a_n)_{n\in\mathbf{N}_+}$ of the *f*-expansion are r.v. on (I,B_I).

d. The so-called representation map $\varphi(w)=(a_n(w))_{n\in\mathbf{N}_+}$ is well defined in *I*. Hence an *f*-expansion exists for almost every point $w\in I$.

e. The *f*-expansion transformation τ_f defined by $\tau_f(w)=$ fractionary part of $f^{-1}(w), w\in I$, has a unique invariant probability μ on B_I. We have $\mu\equiv\lambda$ while $h:=\mathrm{d}\mu/\mathrm{d}\lambda$ is Lipschitz continuous on *I*.

We are now able to state the conjecture formulated in [5] and [6]. *Consider an f-expansion for which conditions* ($\mathrm{BD}^{(2)}$) *and* (C) *are satisfied. Then the label sequence* (a_n) *under* λ *is equivalent to the sequence* (ξ_n) *under* P_0. Hence the conjecture asserts that equations (2) hold in the general case.

Assume that the *f*-expansions considered are such that $\mathrm{d}f(x)/\mathrm{d}x$ continuously exists in $[\alpha,\beta]$, except perhaps for $x\in\mathbf{N}$, and that the hypotheses called (E) and (C) in [8] hold. Then given an *f*-expansion one may also define an RSCC and the properties c.- e. also hold.

Denote by U the Perron-Frobenius operator of τ_f under μ. For arbitrary $w\in I$ we associate with the *f*-expansion the sequence $(s_n^w)_{n\in\mathbf{N}}$ of *I*-valued r.v.s on (I,B_I) defined recursively by $s_n^w=f(a_n+s_{n-1}^w),\ n\in\mathbf{N}_+\ \ s_0^w=w,\ \ w\in I$. We note that $(a_n)_{n\in\mathbf{N}_+}$ and $(s_n^w)_{n\in\mathbf{N}}$ are strictly stationary under μ.

2. AN INFINITE ORDER CHAIN REPRESENTATION

We define the natural extension *T* of τ_f by $T(\theta,\omega)=(\tau_f(\theta),s_1^\omega(\theta)),\ \theta,\omega\in I$. This is a one-to-one transformation of $I\times I$ with inverse $T^{-1}(\theta,\omega)=(s_1^\theta(\omega),\tau_f(\omega))$. Let us denote $\overline{i^{(n)}}:=(i_n,\dots,i_1)\in X^n, n\in\mathbf{N}_+$.

We now define constructively a *T*-invariant measure. Let $n,s\in\mathbf{N}_+$, $i^{(n)}\in X^n, v^{(s)}\in X^s$.Since $\left(\overline{i^{(n)}}v^{(s)}\right)=(i_n,\dots,i_1,v_1,\dots,v_s)=\overline{\left(\overline{v^{(s)}}i^{(n)}\right)}$, we have

$$T^s\left(I\left(\overline{v^{(s)}}i^{(n)}\right)\times I\right)=I\left(i^{(n)}\right)\times I\left(v^{(s)}\right)\equiv T^{-n}\left(I\times I\left(\overline{i^{(n)}}v^{(s)}\right)\right).$$

Denote by Σ_n the σ-algebra generated by the fundamental intervals of order $n\in\mathbf{N}_+$.

Let $I_n(\Lambda):=\cup_{i^{(n)}\in\Lambda}I\left(i^{(n)}\right), V_s(\Lambda')\equiv\cup_{v^{(s)}\in\Lambda'}I\left(v^{(s)}\right), \overline{I}_n(\Lambda):=\cup_{i^{(n)}\in\Lambda}I\left(\overline{i^{(n)}}\right)$ for any $\Lambda\subset X^n, \Lambda'\subset X^s$. Clearly, $I_n(\Lambda)$ and $V_s(\Lambda')$ are typical elements of Σ_n and Σ_s, respectively. We define a set-function $\overline{\mu}$ on $\Sigma_n\times\Sigma_s$ by setting

$$\overline{\mu}\left(I_n(\Lambda)\times V_s(\Lambda')\right)\equiv\mu\left(\overline{I_n}(\Lambda)\cap\left(\tau_f^s\in V_s(\Lambda')\right)\right) \tag{3}$$

$\overline{\mu}$ so defined is uniquely determined. Clearly, $\{\Sigma_n\times\Sigma_s,\ n,s\in\mathbf{N}_+\}$ generates the Borel σ-algebra on $I\times I$. One can extend the function $\overline{\mu}$ to a measure on $B_I\times B_I$, which we also denote $\overline{\mu}$. By Caratheodory's theorem such an extension exists and is unique. Let us denote by $\overline{\lambda}$ the Lebesgue measure on $I\times I$.

Theorem 1 (Properties of $\bar{\mu}$).

i. $\bar{\mu}$ *is invariant under T and* T^{-1} ;
ii. $\bar{\mu}$ *has marginal distributions equal to* μ ;
iii. $\bar{\mu}$ *is a symmetric measure*.

To prove the last assertion, we use the ergodic Theorem 5 in [2] or [9].
Theorem 1 implies that one can replace (3) by the symmetric relation in the definition of $\bar{\mu}$.

By the Radon-Nikodym theorem there uniquely exists a measurable nonnegative random variable $\bar{\alpha}$ on $\left(I\times I, B_I\times B_I, \bar{\lambda}\right)$ such that

$$\bar{\mu}\left(\hat{A}\right)=\iint_{\hat{A}}\bar{\alpha}\mathrm{d}\bar{\lambda}\,,\ \hat{A}\in B_I\times B_I\,.$$

In the sequel the Hölder condition has the meaning defined in [4] while the kernel associated with a piecewise monotonic transformation has the meaning defined in [7].

Proposition 2 (Properties of $\bar{\alpha}$).

i. $\int_0^1\bar{\alpha}(x,y)\mathrm{d}y=h(x)$ *for any* $x\in I$, *and* $\bar{\alpha}$ *is symmetric*;
ii. $\bar{\alpha}$ *satisfies a Hölder condition of order* 1;
iii. $\bar{\alpha}$ *is a kernel* .

We can now define the infinite order chains involving T and $\bar{\mu}$. Our definitions here are formally identical with those for the RCF expansion (see [4]).

We define the X-valued r.v.s. $\bar{a}_n,\ n\in\mathbf{Z}$, on $\left(I\times I, B_I\times B_I\right)$ by

$$\bar{a}_n(\theta,\omega)=a_n(\theta), n\in\mathbf{N}_+,\ \bar{a}_0(\theta,\omega)=a_1(\omega),\ \bar{a}_{-l}(\theta,\omega)=a_{l+1}(\omega),\ l\in\mathbf{N}_+\,.$$

Hence $\bar{a}_n=\bar{a}_{n-1}(T),\ n\in\mathbf{Z}$.

We also consider the I-valued random variables $\bar{s}_l,\ l\in\mathbf{Z}$, defined by

$$\bar{s}_{-l}(\theta,\omega)=\tau_f^l(\omega),\ l\in\mathbf{N},\ \bar{s}_n(\theta,\omega)=s_n^{\omega}(\theta),\ n\in\mathbf{N}_+\,.$$

The doubly infinite sequences $\left(T^n\right)_{n\in\mathbf{Z}}, \left(\bar{a}_n\right)_{n\in\mathbf{Z}}$ and $\left(\bar{s}_n\right)_{n\in\mathbf{Z}}$ are respectively $I\times I$-, X- and I- valued strictly stationary symmetric processes under $\bar{\mu}$. In other words, they are infinite order chains on $\left(I\times I, B_I\times B_I, \bar{\mu}\right)$.

The introduction in the next section of probability measures $\mu_w, w\in I$, will allow us to complete the description of probabilistic properties of our infinite order chains. It is based on classical notions as given in [3].

3. CONDITIONAL PROBABILITY MEASURES

Whatever $w\in I$ define

$$\mu_w(A):\equiv\int_A\frac{\bar{\alpha}(x,w)}{h(w)}\mathrm{d}x,\ \ A\in B_I$$

Then $\mu_w(\cdot)$ such defined is a probability on B_I. In the RCF case $\mu_w(\ )$ can be expressed in closed form and coincides with the function denoted by $\gamma_a(\ )$ in [4]; $\mu_w(\cdot)$ has most of its properties.

Theorem 3 (Properties of μ_w).

i. $\bar{\mu}(\bar{a}_1 = i \mid \bar{a}_0, \bar{a}_{-1}, \ldots) = \mu_{\bar{s}_0}(I(i)) = P(\bar{s}_0, i) \quad \bar{\mu}$ - *a.s.*, $i \in X$;

ii. $\bar{\mu}\left(A \times I \mid \bar{s}_0\right) = \mu_{\bar{s}_0}(A) \quad \bar{\mu}$ - *a.s.*, $A \in B_I$;

iii. $(\bar{s}_n)_{n\in\mathbf{Z}}$ *is an I-valued Markov chain on* $\left(I \times I,\, B_I \times B_I,\, \bar{\mu}\right)$.

We now return to the random variables on (I, B_I) involved in the conjecture. By the next theorem we see the impact of probabilistic properties of infinite order chains on the sequence $\left(s_n^w\right)_{n\in\mathbf{N}}$ defined on $\left(I, B_I, \mu_w\right)$. Below E_w denotes the mean under P_w, $w \in I$.

Theorem 4 (Properties of the distribution of s_n^w).

i. $\lambda \equiv \mu_w, w \in I$;

ii. $\mu(A) = \int_0^1 \mu_w\left(s_n^w \in A\right)\mu(\mathrm{d}w), \quad A \in B_I, \quad n \in \mathbf{N}_+$;

iii. $\mu_w\left(s_n^w \in A\right) = E_w\left(\chi_A\left\{s_n^w\right\}\right) \equiv U^n \chi_A(w), \ w \in I, \ n \in \mathbf{N}_+$.

Hence, whatever $w \in I$, $\left(s_n^w\right)_{n\in\mathbf{N}}$ is an I -valued Markov chain on $\left(I, B_I, \mu_w\right)$ with transition operator U .

Using the results above we can prove the next result which is essentially used in the proof of Theorem 6 below.

Theorem 5. *We have* $\lambda = \mu_0$ *and* $\bar{\alpha}(w,0) = h(0) = \bar{\alpha}(0,w), \ w \in I$.

4. THE SOLUTION

As we already said, our result confirming the conjecture stated in Section 1 concerns *f*-expansion satisfying Rényi's condition on distortion and a strengthened Lipschitz condition. More precisely, we have

Theorem 6 (Main result). *Consider an f-expansion for which conditions* (E) *and* (C) *hold.*

i. *Equations* (2) *are valid. Hence the label sequence* $(a_n)_{n\in\mathbf{N}_+}$ *under* $\lambda = \mu_0$ *is equivalent to the sequence* $(\xi_n)_{n\in\mathbf{N}}$ *under* P_0.

ii. *The sequence* $\left(s_n^0\right)_{n\in\mathbf{N}}$ *on* $\left(I, B_I, \lambda\right)$ *is an I-valued homogenous Markov chain equivalent to the Markov chain* $(\zeta_n)_{n\in\mathbf{N}}$ *on* (Ω, K, P_0).

iii. *The representation map* φ *settles an isomorphism of measure spaces between* $\left(I, B_I, \lambda\right)$ *and* (Ω, K, P_0).

The statement of conditions (E) and (C) can be found in [1],[8],[9].

Note that conditions (E) and (C) imply a condition slightly weaker than $(\mathrm{BD}^{(2)})$ (esssup replaces sup in $(\mathrm{BD}^{(2)})$) so that one can say that the conjecture is proved as formulated.

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