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AN EXTENSION OF CARNOT'S THEOREM FOR RIGID BODY COLLISIONS

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This paper derives a solution for the calculus of the energy dissipated in the frictionless collision of two rigid bodies by using the impulse and the velocities that occur on the bodies at the theoretical impact point. Similar formulas are also derived in the cases when impact occurs over a surface (not a point) and for the collision of rigidf body with a fixed obstacle. Representative examples support the theory.

Key words: collision, energy dissipated, contact impulse, momentum, angular momentum

1. INTRODUCTION

Compact solid bodies that collide at low or moderate speeds suffer negligible deformation outside a small region surrounding the point of contact. Each body is assumed to exert a significantly large force on the other at the point of contact. During collision, these forces decrease the sum of the kinetic energies of the two bodies in an initial phase of compression. The elastic strain energy from the deformed region drives the bodies apart and restores some kinetic energy in a succeeding period of restitution. The energy dissipated in collision is the difference between the initial and final kinetic energies.

2. ENERGY DISSIPATED IN RIGID BODY COLLISIONS

For partly elastic collisions of rigid bodies, the kinetic energy that is dissipated (E_d) is equal at any time during collision to the negative of work (L) done by reaction forces ($\vec{F}$) on the colliding bodies:

$$E_d(t) = -L(t) = -\int_0^t \vec{F} \cdot \vec{v}_r dt = -\int_0^{P(t)} \vec{v}_r \cdot d\vec{P}\,, \tag{1}$$

where $\vec{v}_r$ stands for the relative velocity across the small deforming region that surrounds the contact point and

$$\vec{P}(t) = \int_0^t \vec{F}(u)\,du \tag{2}$$

is the contact impulse (percussion) of the reaction forces. In most cases the contact duration is sufficiently small and the contact impulse (finite) is represented by Dirac's delta function:

$$\vec{P} = \lim_{t \to 0} \vec{P}(t). \tag{3}$$

A useful method of calculating energy dissipation in a frictionless collision is through the following: the absolute value of the energy dissipated in a collision between two bodies is equal to the scalar product of

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Recommended by Radu P.VOINEA, member of the Romanian Academy the contact impulse on either body and the mean of the velocity difference between the points of contact before collision and their velocity difference after collision.

Proof:

Consider two rigid bodies labelled $i = 1$ and $i = 2$ which collide at a common point O. Their center of mass (C_i) velocities $\vec{v}_{C\,i}$ and their angular velocities $\vec{\omega}_i$ will change discontinously from their values $\vec{v}'_{C\,i}$ and $\vec{\omega}'_i$ just before collision to their values $\vec{v}''_{C\,i}$ and $\vec{\omega}''_i$ just after collision. We denote by m_i the mass of body i and by $[J]_i$ its central inertia tensor. Let $(-1)^i\vec{P}$ be the contact impulse exerted on body i at the point of contact by the other body during collision. The following relations derive from the linear an angular momentum laws:

$$m_i\left(\vec{v}''_{C\,i} - \vec{v}'_{C\,i}\right) = (-1)^i\vec{P} \quad , \quad [J]_i(\vec{\omega}''_i - \vec{\omega}'_i) = \overrightarrow{C_i O}\times(-1)^i\vec{P} \quad , \quad i = 1, 2\,. \tag{4}$$

The kinetic energy of the two bodies before and after collision will be expressed through Koening's formula:

$$E' = \frac{1}{2}\sum_{i=1}^{2}\left(m_i\vec{v}'^{\,2}_{C\,i} + \vec{\omega}'_i\cdot[J]_i\vec{\omega}'_i\right) \ , \quad E'' = \frac{1}{2}\sum_{i=1}^{2}\left(m_i\vec{v}''^{\,2}_{C\,i} + \vec{\omega}''_i\cdot[J]_i\vec{\omega}''_i\right). \tag{5}$$

The energy dissipated during the impact is

$$E_d = E' - E'' = \frac{1}{2}\left[\sum_{i=1}^{2}m_i\left(\vec{v}'^{\,2}_{C\,i} - \vec{v}''^{\,2}_{C\,i}\right) + \sum_{i=1}^{2}\left(\vec{\omega}'_i\cdot[J]_i\vec{\omega}'_i - \vec{\omega}''_i\cdot[J]_i\vec{\omega}''_i\right)\right]. \tag{6}$$

Based on (4) one can write

$$\sum_{i=1}^{2}m_i\left(\vec{v}'^{\,2}_{C\,i} - \vec{v}''^{\,2}_{C\,i}\right) = \sum_{i=1}^{2}m_i\left(\vec{v}'_{C\,i} - \vec{v}''_{C\,i}\right)\cdot\left(\vec{v}'_{C\,i} + \vec{v}''_{C\,i}\right) = \vec{P}\cdot\sum_{i=1}^{2}(-1)^{i+1}\cdot\left(\vec{v}'_{C\,i} + \vec{v}''_{C\,i}\right). \tag{7}$$

The second term in formula (6) will be computed using (4) and the simetry of the inertia tensor

$$\vec{\omega}'\cdot[J]\vec{\omega}'' = \vec{\omega}''\cdot[J]\vec{\omega}'\,. \tag{8}$$

Thus

$$\begin{aligned}\sum_{i=1}^{2}\left(\vec{\omega}'_i\cdot[J]_i\vec{\omega}'_i - \vec{\omega}''_i\cdot[J]_i\vec{\omega}''_i\right) &= \sum_{i=1}^{2}\left(\vec{\omega}'_i\cdot[J]_i\vec{\omega}'_i - \vec{\omega}'_i\cdot[J]_i\vec{\omega}''_i + \vec{\omega}''_i\cdot[J]_i\vec{\omega}'_i - \vec{\omega}''_i\cdot[J]_i\vec{\omega}''_i\right) = \\ &= \sum_{i=1}^{2}(\vec{\omega}'_i + \vec{\omega}''_i)\cdot[J]_i(\vec{\omega}'_i - \vec{\omega}''_i) = \sum_{i=1}^{2}(\vec{\omega}'_i + \vec{\omega}''_i)\cdot\left(\overrightarrow{C_i O}\times(-1)^{i+1}\vec{P}\right) = \\ &= \vec{P}\cdot\sum_{i=1}^{2}(-1)^{i+1}(\vec{\omega}'_i + \vec{\omega}''_i)\times\overrightarrow{C_i O}\ .\end{aligned} \tag{9}$$

Taking into account that the velocities of the contact point on each body are

$$\vec{v}'_{O\,i} = \vec{v}'_{C\,i} + \vec{\omega}'_i\times\overrightarrow{C_i O} \quad , \quad \vec{v}''_{O\,i} = \vec{v}''_{C\,i} + \vec{\omega}''_i\times\overrightarrow{C_i O}\ , \tag{10}$$

the energy dissipation in the collision will result as

$$E_d = \frac{1}{2}\vec{P}\cdot\sum_{i=1}^{2}(-1)^{i+1}\left(\vec{v}'_{O\,i} + \vec{v}''_{O\,i}\right), \tag{11}$$

which proves the assertion. This is a generalization of Kelvin's theorem [1] for rigid body collisions.

Remarks:

a). In some cases after collision the velocities of the contact point on both rigid bodies result perpendicular on the contact impulse. Here the energy dissipated during impact will be equal to the kinetic energy of lost velocities (Carnot's theorem) [1], [2] :

$$E_d = E' - E'' = \frac{1}{2}\vec{P}\cdot\sum_{i=1}^{2}(-1)^{i+1}\vec{v}'_{O\,i} = E_p \ , \tag{12}$$

where

$$E_p = \frac{1}{2}\left[\sum_{i=1}^{2} m_i\left(\vec{v}'_{C\,i} - \vec{v}''_{C\,i}\right)^2 + \sum_{i=1}^{2}\left(\vec{\omega}'_i - \vec{\omega}''_i\right)\cdot[J]_i\left(\vec{\omega}'_i - \vec{\omega}''_i\right)\right]. \tag{13}$$

b). Some recent developments in collision theory [3], [4] tend to introduce a moment impulse $\vec{M}$ in addition to the resultant impulse at the contact point. This is justified by the fact that in real collisions the impact forces occur over a surface, not a point. More, the contact impulse is the resultant of surface forces whose exact location may not only be accurately know, but that usually changes during the contact duration. Consequently in some cases uncertainty in the selection of the contact point demands an equivalent force system which should include a moment. In such cases the law of angular momentum will change to

$$[J]_i\left(\vec{\omega}''_i - \vec{\omega}'_i\right) = \overrightarrow{C_i O}\times(-1)^i\vec{P} + (-1)^i\vec{M} \qquad i = 1, 2. \tag{14}$$

Energy dissipation in the collision will result in the form

$$E_d = \frac{1}{2}\vec{P}\cdot\sum_{i=1}^{2}(-1)^{i+1}\left(\vec{v}'_{O\,i} + \vec{v}''_{O\,i}\right) + \frac{1}{2}\vec{M}\cdot\sum_{i=1}^{2}(-1)^{i+1}\left(\vec{\omega}'_i + \vec{\omega}''_i\right) . \tag{15}$$

Particular case - Collision of a rigid body with a fixed obstacle.

In the case of collision between a rigid body ($i = 1$) and a fixed obstacle ($i = 2$), $\vec{v}'_{O\,2} = \vec{v}''_{O\,2} = 0$ and $\vec{\omega}'_2 = \vec{\omega}''_2 = 0$. Energy dissipation during impact will be

$$E_d = -\vec{P}\cdot\frac{\vec{v}'_{O1} + \vec{v}''_{O1}}{2} - \vec{M}\cdot\frac{\vec{\omega}'_1 + \vec{\omega}''_1}{2} \ . \tag{16}$$

3. APPLICATIONS

In order to illustrate the theory the following examples will be considered.

3.1. Collision between a rigid cylinder and a sharp edge.

Figure 1 shows the cylinder and the associated coordinates. The velocity of impact point A before and after collision is

$$v'_{A\,n} = -v'_O\cos\alpha \quad , \quad v'_{A\,t} = v'_O\sin\alpha - r\omega',$$
$$v''_{A\,n} = v''_{O\,n} \quad , \quad v''_{A\,t} = v''_{O\,t} - r\,\omega'' = 0\,.$$

The slipless collision equations are:

$$m\,v''_{O\,n} + m\,v'_O\cos\alpha = P_n \ , \quad m\,v''_{O\,t} - m\,v'_O\sin\alpha = P_t \ , \quad P_t \le \mu\,P_n,$$

$$J_O\cdot\left(\omega'' - \frac{v'_O}{r}\right) = -r\,P_t \ , \quad e = \frac{v''_{O\,n}}{v'_O\cos\alpha} \ , \quad v''_{O\,t} - r\omega'' = 0 \ ,$$

These equations give:

$$P_n = m v'_O (1+e)\cos\alpha \quad , \qquad P_t = \frac{m}{3}\left(r\omega' - v'_O \sin\alpha\right) ,$$

$$v''_{On} = e\, v'_O \cos\alpha \quad , \qquad v''_{Ot} = \frac{2}{3} v'_O \sin\alpha + \frac{1}{3} r\omega' \quad , \qquad \omega'' = \frac{2}{3}\frac{v'_O}{r}\sin\alpha + \frac{1}{3}\omega' \ ,$$

where e and μ represent the restitution and the friction collision coefficients between the cylinder and the sharp corner.

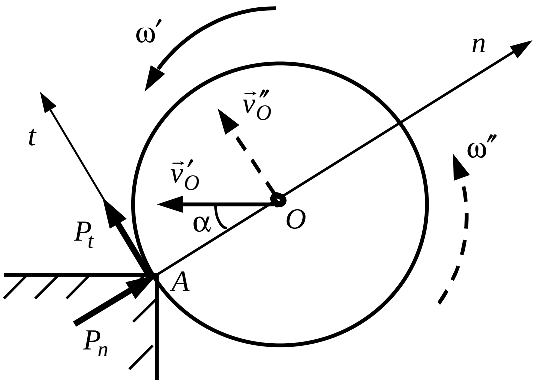

Fig. 2. Rigid cylinder impacting a sharp edge; mass – m, radius – r, central moment of inertia $J_O = mr^2/2$.

In order to prevent sliding at point A it is required that

$$\mu \geq \frac{P_t}{P_n} = \frac{r\omega' - v'_O \sin\alpha}{3(1+e)\cdot v'_O \cos\alpha} \ .$$

Energy dissipation during impact is

$$E_d = -\frac{1}{2}\left[P_t\left(v'_{At} + v''_{At}\right) + P_n\left(v'_{An} + v''_{An}\right)\right] = \frac{m\, v'^{\,2}_O}{2}\left[\frac{1}{3}\left(\frac{r\omega'}{v'_O} - \sin\alpha\right)^2 + \left(1 - e^2\right)\cos^2\alpha\right] .$$

One should note in this example the contribution of both P_n and P_t to E_d. This is a generalization of a known problem [4], [5].

3.2. Rigid pendulum striking a massive plane surface.

The collision is shown in figure 2. The impulse and the moment impulse at the theoretical impact point C result along with ω'' from [6]:

$$\omega'' = -e\,\omega' \ , \ P_{Ot} = P_{On} = 0 \ , \ P_C = \frac{1}{2} m\omega'(1+e)(2l+h) \ , \ M_{PC} = P_C \frac{h^2}{6(2l+h)} = \frac{1}{12} m\omega'(1+e)h^2 \ .$$

As

$$v'_C = \frac{1}{2}\omega'(2l+h) \quad , \qquad v''_C = -\frac{1}{2} e\omega'(2l+h)$$

the dissipated energy during impact is

$$E_d = \frac{1}{2} P\left(v'_C + v''_C\right) + \frac{1}{2} M\left(\omega' + \omega''\right) = \frac{1}{2} m\cdot\left(1 - e^2\right)\cdot\left(l^2 + lh + \frac{h^2}{3}\right)\cdot \omega'^2 \ ,$$

which can be easily verified by computing the difference of the kinetic energies of the pendulum before and after collision. Note that in the particular case of this example, formula (15) proves the corectness of the chosen impulse distribution on the impact surface shown in [6], as only a corect impulse distribution is consistent with the energy loss during collision.

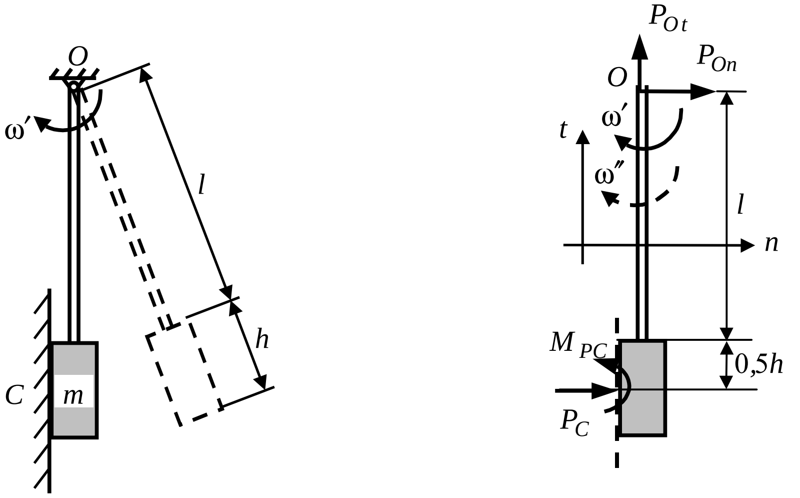

Fig. 2. Rigid pendulum striking a massive plane surface

3.3. Sudden fastening of a point of a rigid body.

Consider a rigid body moving freely in the space prior to its sudden fastening of its point O. The collision is shown in figure 3a. The velocity constraint on the rigid body requires that the velocity of point O after collision be zero ($\vec{v}''_O = 0$). Linear and angular momentum equations are:

$$m\left[\vec{\omega}''\times\overrightarrow{OC}-\left(\vec{v}'_O+\vec{\omega}'\times\overrightarrow{OC}\right)\right]=\vec{P}\ ,\quad [J]_O\vec{\omega}''-\left([J]_O\vec{\omega}'+m\,\overrightarrow{OC}\times\vec{v}'_O\right)=0\ .$$

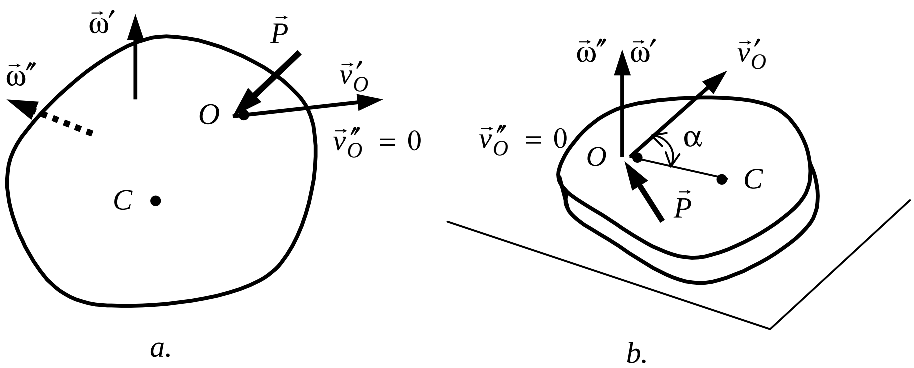

Fig. 3. Rigid body in motion with one of its points suddenly fixed; a.- tridimensional, b. – planar; mass-m, inertia tensor about point O – $[J]_O$.

It results that

$$\vec{\omega}''=\vec{\omega}'+m\,[J]_O^{-1}\,\overrightarrow{OC}\times\vec{v}'_O\ ,\quad \vec{P}=-m\vec{v}'_O-m\,\overrightarrow{OC}\times[J]_O^{-1}\left(\overrightarrow{OC}\times\vec{v}'_O\right).$$

Energy dissipation during collision will be

$$E_d = \frac{1}{2} m\vec{v}_O'^{\,2} + \frac{1}{2} m\vec{v}_O' \cdot \left[\vec{OC} \times [J]_O^{-1} \left(\vec{OC} \times \vec{v}_O' \right) \right].$$

If the rigid body is a plate performing a planar motion (figure 3b) whose point O will be impacted then

$$\omega'' = \omega' - \frac{m v_O' l \sin\alpha}{J_O} \quad , \quad E_d = \frac{1}{2} m v_O'^{\,2} \left(1 - \frac{m l^2 \sin^2\alpha}{J_O} \right).$$

One should note in this example the efficiency of formula (11) compared to the commonly used method $E_d = E' - E''$.

4. CONCLUSION

The paper presented an extention of Carnot's theorem regarding the energy dissipated during a frictionless collision between two rigid bodies. If the collision occurs at a point the theorem is ilustrated by relation (11); if the collision occurs over a surface the theorem is ilustrated by formula (15). In such cases a suitable impulse distribution over the contact area is induced by the impact velocities and must be compatible with the loss of kinetic energy during collision. In some particular cases when after collision the velocities of the contact point on both rigid bodies result perpendicular on the contact impulse this theorem gives the same results as Carnot's theorem.

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