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ON THE PSEUDO-PURE STATES OF TWO QUBITS

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Pseudo-pure states are defined as the maximally degenerate mixed states. The density matrices of such states have only two distinct degenerate eigenvalues. We shall show that, in the particular case of the two qubits systems, these states are completely determined by the one of the two Bloch vectors of the corresponding qubits.

I. INTRODUCTION

It is well know that the pure states are described by maximally degenerate density matrices which have only two distinct eigenvalue: a non-degenerate eigenvalue equal to one and a degenerate eigenvalue equal to zero. In other words the density matrix is a projector, i.e., is an idempotent operator. By definition such an operator ρ satisfies a second order algebraic equation $\rho^2 = \rho$. In the papers [1-5] it was shown that the two-qubits density matrices of pure states are completely determined by the Bloch vector of one of the qubits. In the present paper we shall show that this situation is valid also in the case of the maximally degenerate (pseudo-pure) states of the two-qubits systems. For these states their density matrices satisfy second order algebraic equations $\rho^2 - s\rho + pI = 0$. For $s = 1$ and $p = 0$ we have the case of pure states. We shall prove that the equations fulfilled by the Fano parameters of any maximally degenerate mixed state of a two qubits system determine in a unique way the correlation matrix and one of the Bloch vector as functions of the other Bloch vector. In order to obtain the equations fulfilled by the Fano parameters we shall use two different parametrizations for the states of two-qubits quantum systems: the generalized Bloch vector parametrization [1-10] and the Fano parametrization [3-9], [12-17] and the relations between them.

2. THE BLOCH PARAMETRIZATIONS

a) The Bloch vector.

Let H be a finite-dimensional Hilbert space with dimension equal to d. We denote by $End(H)$ the vector space of the linear operators on H and define on this space the Hilbert-Schmidt inner product by the formula: $(A,B) = Tr(A^* B)$ for any $A, B \in End(H)$ (the operator A^* is the adjoint of the operator A). The Lie algebra $su(d)$ of all selfadjoint operators $A \in End(H)$ with $TrA = 0$ is a real subspace of $End(H)$, with dimension equal to $D = d^2 - 1$. We shall take a basis $\{\tau_j\}_{j=1}^{D}$ of this subspace such that the following relations are valid $(\tau_j, \tau_k) = 2\delta_{jk}$. Then any density matrix ρ i.e. any linear selfadjoint and positive definite operator with $Tr\rho = 1$ can be described by the following formula:

$$\rho(v) = \frac{1}{d} I + \frac{1}{2} \sum_{j=1}^{D} v_j \tau_j \tag{2.1}$$

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The real vector $v=(v_1,v_2,...,v_D)\in R^D$ is called the generalized Bloch vector [1-10] and is defined in a unique way by the density matrix ρ : $v_j = Tr\rho\tau_j = (\rho,\tau_j)$. But the converse correspondence is not valid for any vector $v=(v_1,v_2,...,v_D)\in R^D$. The fact that the density matrix ρ is positive definite imposes severe restrictions on the Bloch vectors [10]. Let us denote by $<v,u>=\sum_{j=1}^{D} v_j u_j$ the Euclidean inner product on R^D and by $\| v \| = \sqrt{<v,v>}$ the corresponding norm.

b) The equations satisfied by the Bloch vector of a pseudo-pure state.

The Lie brackets of the generators $\{\tau_j\}_{j=1}^{D}$ of the Lie algebra $su(d)$ are given by the structure constants $\{f_{jkl}\}_{j,k,l=1}^{D}$:

$$[\tau_j,\tau_k] = 2i\sum_{l=1}^{D} f_{jkl}\tau_l \tag{2.2}$$

These structure constants are the components of a totally anti-symmetric tensor and fulfill the Jacoby identity:

$$\sum_{m=1}^{D}(f_{klm}f_{mpq}+f_{plm}f_{mkq}+f_{kpm}f_{mlq})=0 \tag{2.3}$$

A remarkable fact, specific to the Lie algebra $su(d)$, is the existence of a symmetric bracket:

$$\tau_j\tau_k+\tau_k\tau_l=\frac{4}{d}\delta_{jk}I+2\sum_{l=1}^{D} d_{jkl}\tau_l \tag{2.4}$$

Here d_{jkl} are the components of a totally symmetric tensor. With the aid of anti-symmetric and symmetric tensors we define an anti-symmetric and a symmetric product on the Euclidean space R^D. The anti-symmetric product is defined by:

$$(x\bigcap y)_j=\sum_{k=1}^{D}\sum_{l=1}^{D} f_{jkl}\, x_k\ y_l \tag{2.5}$$

The symmetric product is defined by:

$$(x\bigcup y)_j=\sum_{k=1}^{D}\sum_{l=1}^{D} d_{jkl}\, x_k\ y_l \tag{2.6}$$

Then the commutators and anticommutators becomes respectively:

$$\left[<x,\tau>,<y,\tau>\right]=2i<(x\bigcap y),\tau> \tag{2.7}$$

$$\left\{<x,\tau>,<y,\tau>\right\}=\frac{4}{d}<x,y>I+2<(x\bigcup y),\tau> \tag{2.8}$$

For any density matrix (2.1) we have

$$\rho(v)^2=(\frac{1}{d^2}+\frac{1}{2d}<v,v>)I+<\left[\frac{1}{d}v+\frac{1}{4}(v\bigcup v)\right],\tau> \tag{2.9}$$

The quantum state described by the density matrix ρ is a maximally degenerate state if and only if:

$$\rho^2(v) - s\rho(v) + pI = 0 \tag{2.10}$$

Then, from (2.9) it follows that the Bloch vector v describes a maximally degenerate state if only if the squared Euclidean norm of v is given by:

$$<v,v>= 2(s - pd - \frac{1}{d}) \tag{2.11}$$

Also it follows that the symmetric product of v with v must lives in the one- dimensional subspace generated by v:

$$v \bigcup v = 4(\frac{s}{2} - \frac{1}{d})v \tag{2.12}$$

We introduce the following notations $\mu = 2s - \frac{4}{d}$ and $\gamma = 2s - 2pd - \frac{2}{d} =<v,v>$.

c) The positivity of pseudo-pure density matrices

In the case of the density matrices ρ for pseudo-pure quantum states we have the spectral decomposition:

$$\rho = \lambda_1 P_1 + \lambda_2 P_2 \tag{2.13}$$

where the positive eigenvalues $\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1\delta_1 + \lambda_2\delta_2 = 1$, are degenerate with multiplicities:

$$\begin{aligned} TrP_1 &= \frac{d}{2}(1 - \frac{1}{\sqrt{1 + \frac{8<v,v>}{d\mu^2}}}) = \delta_1 \\ TrP_2 &= \frac{d}{2}(1 + \frac{1}{\sqrt{1 + \frac{8<v,v>}{d\mu^2}}}) = \delta_2 \end{aligned} \tag{2.14}$$

respectively. Let us suppose that the multiplicities δ_1, δ_2 are given and $\delta_2 - \delta_1 > 0$. Then we have

$$<v,v>= \frac{\delta_1\delta_2(\delta_1 + \delta_2)}{2(\delta_1 - \delta_2)^2}\mu^2 = \frac{\delta_1(d - \delta_1)d}{2(d - 2\delta_1)^2} \tag{2.15}$$

i.e., for such degenerate density matrices the all unitary invariants are expressible as functions of the invariant μ^2. The eigenvalues λ_1, λ_2 are given by:

$$\lambda_{1,2} = \frac{1}{d} + \frac{\mu}{4} \pm \sqrt{\frac{\mu^2}{16} + \frac{1}{2d}<v,v>} \tag{2.16}$$

Then replacing the value of $<v,v>$ given by the equation (2.15) in the equations (2.13) we obtain:

$$\begin{aligned} \lambda_1 &= \frac{1}{d} + \frac{\mu\delta_2}{2(\delta_2 - \delta_1)} \\ \lambda_2 &= \frac{1}{d} - \frac{\mu\delta_1}{2(\delta_2 - \delta_1)} \end{aligned} \tag{2.17}$$

The positivity condition for these eigenvalues gives:

$$\mu \leq \frac{2(\delta_2 - \delta_1)}{d\delta_1} = \frac{2(d - 2\delta_1)}{d\delta_1} \tag{2.18}$$

There are two families of pseudo-pure quantum states which are more carefully studied in the literature; the Werner ρ_W states and the Horodecki states ρ_H which are defined by:

$$\begin{aligned}\rho_W &= \frac{F+1}{\sqrt{d}(\sqrt{d}+1)}\frac{I+V}{2} + \frac{1-F}{\sqrt{d}(\sqrt{d}-1)}\frac{I-V}{2} \\ \rho_H &= fP_+ + \frac{1-f}{d-1}(I - P_+)\end{aligned} \tag{2.19}$$

Here the operators V and P_+ are defined by the following properties:

$$V^2 = I \quad ; \quad TrV = d \quad ; \quad {P_+}^2 = P_+ \quad ; \quad TrP_+ = 1 \quad ; \quad VP_+ = P_+V = P_+ \tag{2.20}$$

For Horodecki states we have $\delta_1 = \frac{\sqrt{d}(\sqrt{d}-1)}{2}, \delta_2 = \frac{\sqrt{d}(\sqrt{d}+1)}{2}$ and for Werner states we have $\delta_1 = 1, \delta_2 = d - 1$. Hence the following restriction follows for the parameter μ :

$$\mu_W \leq \frac{4}{d(\sqrt{d}-1)} \quad ; \quad \mu_H \leq \frac{2(d-2)}{d} \tag{2.21}$$

In the case of two qubits $d = 4$ and $\frac{4}{d(\sqrt{d}-1)} = \frac{2(d-2)}{d} = 1$ because for both classes of states we have $\delta_1 = 1, \delta_2 = 3$.

3. THE FANO PARAMETRIZATION

a) The Fano parameters.

The density matrix corresponding to a state of a bipartite quantum system composed from two subsystems of dimensions d_1 and d_2 can be parametrized by the Fano parameters [2-9], [12-17]:

$$\rho = \frac{1}{d_1 d_2}(I_1 \otimes I_2) + \frac{1}{2d_2} < x, \tau > \otimes I_2 + \frac{1}{2d_1} I_1 \otimes < y, \tau > + \frac{1}{4}\sum_{k=1}^{d_1^2-1}\sum_{l=1}^{d_2^2-1} K_{kl}(\tau_k \otimes \tau_l) \tag{3.1}$$

b) The equations satisfied by the Fano parameters of a pseudo-pure state.

The pseudo-purity condition (2.10) gives us the following equations for the Fano parameters [17]:

$$\left(s - \frac{2}{d_1 d_2}\right)x_j = \frac{1}{2d_2}(x \bigcup x)_j + \frac{1}{d_1}(Ky)_j + \frac{1}{4}\sum (KK^T)_{sp} d^{(1)}{}_{spj} \tag{3.2}$$

and

$$\left(s - \frac{2}{d_1 d_2}\right)y_j = \frac{1}{2d_1}(y \bigcup y)_j + \frac{1}{d_2}(K^T x)_j + \frac{1}{4}(K^T K)_{sp} d^{(2)}{}_{spj} \tag{3.3}$$

and

$$\left(s-\frac{2}{d_1d_2}\right)K_{jl}=\frac{2}{d_1d_2}x_jy_l$$
$$+\frac{1}{d_2}\sum K_{sl}d^{(1)}{}_{jps}x_p+\frac{1}{d_1}\sum K_{jt}d^{(2)}{}_{ltq}y_q+\frac{1}{4}\sum K_{st}K_{pq}d^{(1)}{}_{spj}d^{(2)}{}_{tql}-\frac{1}{4}\sum K_{st}K_{pq}f^{(1)}{}_{spj}f^{(2)}{}_{tql} \tag{3.4}$$

and

$$\frac{1}{2d_2}<x,x>+\frac{1}{2d_2}<y,y>+\frac{1}{4}TrK^TK=s-d_1d_2p-\frac{1}{d_1d_2} \tag{3.5}$$

In the particular case of two qubits we have $d_1=d_2=2$, $d_{ijk}=0$, for all values of indices and $f_{ijk}=\varepsilon_{ijk}$ for all values of indices. Also we have $\mu=2s-1$ and $\gamma=2s-8p-\frac{1}{2}$. Hence the above equations (3.1)-(3.3) become:

$$\mu x=Ky \tag{3.6}$$

and

$$\mu y=K^Tx \tag{3.7}$$

and

$$\mu K_{jl}=x_jy_l-\frac{1}{2}\sum\varepsilon_{spj}\varepsilon_{tql}K_{st}K_{pq}=x_jy_l-((adjK)^T)_{jl} \tag{3.8}$$

(where $adjK=K^{-1}\det K$), and

$$<x,x>+<y,y>+TrK^TK=2\gamma \tag{3.9}$$

4. THE TRANSFORMATIONS BETWEEN THE FANO AND BLOCH PARAMETERS

In the following we shall restrict our considerations to the case of two-qubits quantum systems for which we have $d_1=d_2=2$. We shall denote by $\{u_1,u_2,...,u_d\}$ a basis in the $d-$dimensional Hilbert space H. The operators E_{jk} are defined by:

$$E_{jk}u_l=\delta_{kl}u_j \tag{4.1}$$

The basis of the Lie algebra of $su(2)$ defined in the first section is defined by:

$$\begin{aligned}\tau_1&=E_{12}+E_{21}\\ \tau_2&=\sqrt{-1}(E_{21}-E_{12})\\ \tau_3&=E_{11}-E_{22}\end{aligned} \tag{4.2}$$

The basis of the Lie algebra $su(3)$ is given by (4.2) and by:

$$\tau_4 = E_{13} + E_{31} \quad ; \quad \tau_5 = \sqrt{-1} \ \ (E_{31} - E_{13}) \quad ; \quad \tau_6 = E_{23} + E_{32} \quad ;$$
$$\tau_7 = \sqrt{-1} \ \ \left(E_{32} - E_{23}\right) \quad ; \quad \tau_8 = \frac{1}{\sqrt{3}}(E_{11} + E_{22} - 2E_{33}) \tag{4.3}$$

Analogously the basis of the Lie algebra $su(4)$ is given by (4.2), (4.3) and by:

$$\tau_9 = E_{14} + E_{41} \quad ; \quad \tau_{10} = \sqrt{-1} \ \ (E_{41} - E_{14}) \quad ; \quad \tau_{11} = E_{24} + E_{42} \quad ; \quad \tau_{12} = \sqrt{-1} \ \ \left(E_{42} - E_{24}\right) \ ;$$
$$\tau_{13} = E_{34} + E_{43} \quad ; \quad \tau_{14} = \sqrt{-1} \ \ \left(E_{43} - E_{34}\right) \quad ; \quad \tau_{15} = \frac{1}{\sqrt{6}}(E_{11} + E_{22} + E_{33} - 3E_{44}) \tag{4.4}$$

If we consider the four-dimensional Hilbert space of two qubits as the tensor product of the two-dimensional Hilbert spaces H which describe the pure states of each qubit then we can take the basis $\{u_1, u_2, u_3, u_4\}$ in $H \otimes H$ in the following way $\{u_1 \otimes u_1, u_1 \otimes u_2, u_2 \otimes u_1, u_2 \otimes u_2\}$. With these conventions and putting the operators in the product space on the left side and the operators in the four-dimensional space on the right side we have the following relations:

$$\tau_1 \otimes \tau_1 = \tau_6 + \tau_9 \quad ; \quad \tau_1 \otimes \tau_2 = -\tau_7 + \tau_{10} \quad ; \quad \tau_1 \otimes \tau_3 = \tau_4 - \tau_{11} \quad ; \quad \tau_2 \otimes \tau_1 = \tau_7 + \tau_{10} \ ;$$
$$\tau_2 \otimes \tau_2 = \tau_6 - \tau_9 \quad ; \quad \tau_2 \otimes \tau_3 = \tau_5 - \tau_{12} \quad ; \quad \tau_3 \otimes \tau_1 = \tau_1 - \tau_{13} \quad ; \quad \tau_3 \otimes \tau_2 = \tau_2 - \tau_{14} \quad ;$$
$$\tau_3 \otimes \tau_3 = \tau_3 + \frac{1}{\sqrt{3}}\tau_8 - \sqrt{\frac{2}{3}}\tau_{15} \tag{4.5}$$

and

$$\tau_1 \otimes I = \tau_4 + \tau_{11} \quad ; \quad \tau_2 \otimes I = \tau_5 + \tau_{12} \quad ; \quad \tau_3 \otimes I = \frac{2}{\sqrt{3}}\tau_8 + \sqrt{\frac{2}{3}}\tau_{15} \quad ; \quad I \otimes \tau_1 = \tau_1 + \tau_{13}$$
$$I \otimes \tau_2 = \tau_2 + \tau_{14} \quad ; \quad I \otimes \tau_3 = \tau_3 - \frac{1}{\sqrt{3}}\tau_8 + \sqrt{\frac{2}{3}}\tau_{15} \quad ; \quad I \otimes I = I. \tag{4.6}$$

Then from the equality:

$$\rho = \frac{1}{4}(I \otimes I) + < x,\tau > \otimes I + I \otimes < y,\tau > + \sum_{k=1}^{3}\sum_{l=1}^{3} K_{kl}(\tau_k \otimes \tau_l) = \frac{1}{4}I + \frac{1}{2} < v,\tau > \tag{4.7}$$

we obtain the following relations between the components of the Bloch vector and the Fano parameters:

$$v_1 = \frac{1}{2}(y_1 + K_{31}) \quad ; \quad v_2 = \frac{1}{2}(y_2 + K_{32}) \quad ; \quad v_3 = \frac{1}{2}(y_3 + K_{33}) \quad ; \quad v_4 = \frac{1}{2}(x_1 + K_{13}) \quad ; \quad v_5 = \frac{1}{2}(x_2 + K_{23})$$
$$v_6 = \frac{1}{2}(K_{11} + K_{22}) \quad ; \quad v_7 = \frac{1}{2}(K_{21} - K_{12}) \quad ; \quad v_8 = \frac{1}{2\sqrt{3}}(2x_3 + y_3 - K_{33}) \quad ; \quad v_9 = \frac{1}{2}(K_{11} - K_{22}). \tag{4.8}$$

and

$$v_{10} = \frac{1}{2}(K_{12} + K_{21}) \quad ; \quad v_{11} = \frac{1}{2}(x_1 - K_{13}) \quad ; \quad v_{12} = \frac{1}{2}(x_2 - K_{23}) \quad ;$$
$$v_{13} = \frac{1}{2}(y_1 - K_{31}) \quad ; \quad v_{14} = \frac{1}{2}(y_2 - K_{32}) \quad ; \quad v_{15} = \frac{1}{\sqrt{6}}(x_3 + y_3 - K_{33}) \tag{4.9}$$

The converse relations are given by [11]:

$$
\begin{aligned}
&x_1 = v_4 + v_{11} \quad ; \quad x_2 = v_5 + v_{12} \quad ; \quad x_3 = \frac{2}{\sqrt{3}} v_8 + \sqrt{\frac{2}{3}} v_{15} \quad ; \\
&y_1 = v_1 + v_{13} \quad ; \quad y_2 = v_2 + v_{14} \quad ; \quad y_3 = v_3 - \frac{1}{\sqrt{3}} v_8 + \sqrt{\frac{2}{3}} v_{15} \quad ; \\
&K_{11} = v_6 + v_9 \quad ; \quad K_{12} = -v_7 + v_{10} \quad ; \quad K_{13} = v_4 - v_{11}
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
&K_{21} = v_7 + v_{10} \quad ; \quad K_{22} = v_6 - v_9 \quad ; \quad K_{23} = v_5 - v_{12} \quad ; \\
&K_{31} = v_1 - v_{13} \quad ; \quad K_{32} = v_2 - v_{14} \quad ; \quad K_{33} = v_3 + \frac{1}{\sqrt{3}} v_8 - \sqrt{\frac{2}{3}} v_{15}
\end{aligned}
\tag{4.11}
$$

5. THE FANO PARAMETERS FOR PSEUDO-PURE STATES

In order to obtain the solutions of the equations (2.12) we must have the concrete expressions for the components of the vector $v\bigcup v$. These components are given using the values of the components of the symmetric tensor d_{ijk} for the Lie algebra $su(4)$ taken from [8]. We have:

$$
\begin{aligned}
&(v\bigcup v)_1 = \frac{2}{\sqrt{3}} v_1 v_8 + v_4 v_6 + v_5 v_7 + v_9 v_{11} + v_{10} v_{12} + \sqrt{\frac{2}{3}} v_1 v_{15} \\
&(v\bigcup v)_2 = \frac{2}{\sqrt{3}} v_2 v_8 - v_4 v_7 + v_5 v_6 + v_{10} v_{11} - v_9 v_{12} + \sqrt{\frac{2}{3}} v_2 v_{15} \\
&(v\bigcup v)_3 = \frac{2}{\sqrt{3}} v_3 v_8 + \frac{1}{2}(v_4{}^2 + v_5{}^2 - v_6{}^2 - v_7{}^2 + v_9{}^2 + v_{10}{}^2 - v_{11}{}^2 - v_{12}{}^2) + \sqrt{\frac{2}{3}} v_3 v_{15}
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
&(v\bigcup v)_4 = v_1 v_6 - v_2 v_7 + v_3 v_4 - \frac{1}{\sqrt{3}} v_4 v_8 + v_9 v_{13} + v_{10} v_{14} + \sqrt{\frac{2}{3}} v_4 v_{15} \\
&(v\bigcup v)_5 = v_1 v_7 + v_2 v_6 + v_3 v_5 - \frac{1}{\sqrt{3}} v_5 v_8 + v_{10} v_{13} - v_9 v_{14} + \sqrt{\frac{2}{3}} v_5 v_{15} \\
&(v\bigcup v)_6 = v_1 v_4 + v_2 v_5 - v_3 v_6 - \frac{1}{\sqrt{3}} v_6 v_8 + v_{11} v_{13} + v_{12} v_{14} + \sqrt{\frac{2}{3}} v_6 v_{15} \\
&(v\bigcup v)_7 = v_1 v_5 - v_2 v_4 - v_3 v_7 - \frac{1}{\sqrt{3}} v_7 v_8 + v_{12} v_{13} - v_{11} v_{14} + \sqrt{\frac{2}{3}} v_7 v_{15}
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
&(v\bigcup v)_8 = \frac{1}{2\sqrt{3}}(v_9{}^2 + v_{10}{}^2 + v_{11}{}^2 + v_{12}{}^2 - v_4{}^2 - v_5{}^2 - v_6{}^2 - v_7{}^2) + \\
&\frac{1}{\sqrt{3}}(v_1{}^2 + v_2{}^2 + v_3{}^2 - v_8{}^2 - v_{13}{}^2 - v_{14}{}^2) + \sqrt{\frac{2}{3}} v_8 v_{15}
\end{aligned}
\tag{5.3}
$$

and

$$(v\bigcup v)_9 = v_1v_{11} - v_2v_{12} + v_3v_9 + v_4v_{13} - v_5v_{14} + \frac{1}{\sqrt{3}}v_9v_8 - \sqrt{\frac{2}{3}}v_9v_{15}$$
$$(v\bigcup v)_{10} = v_1v_{12} + v_2v_{11} + v_3v_{10} + v_4v_{14} + v_5v_{13} + \frac{1}{\sqrt{3}}v_{10}v_8 - \sqrt{\frac{2}{3}}v_{10}v_{15}$$
$$(v\bigcup v)_{11} = v_1v_9 + v_2v_{10} - v_3v_{11} + v_6v_{13} - v_7v_{14} + \frac{1}{\sqrt{3}}v_{11}v_8 - \sqrt{\frac{2}{3}}v_{11}v_{15} \qquad (5.4)$$
$$(v\bigcup v)_{12} = v_1v_{10} - v_2v_9 - v_3v_{12} + v_6v_{14} + v_7v_{13} + \frac{1}{\sqrt{3}}v_{12}v_8 - \sqrt{\frac{2}{3}}v_{12}v_{15}$$

and

$$(v\bigcup v)_{13} = -\frac{2}{\sqrt{3}}v_{13}v_8 + v_4v_9 + v_5v_{10} + v_6v_{11} + v_7v_{12} - \sqrt{\frac{2}{3}}v_{13}v_{15}$$
$$(v\bigcup v)_{14} = -\frac{2}{\sqrt{3}}v_{14}v_8 + v_4v_{10} - v_5v_9 + v_6v_{12} - v_7v_{11} - \sqrt{\frac{2}{3}}v_{14}v_{15} \qquad (5.5)$$
$$(v\bigcup v)_{15} = \frac{1}{\sqrt{6}}(v_1{}^2 + v_2{}^2 + v_3{}^2 + v_4{}^2 + v_4{}^2 + v_5{}^2 + v_6{}^2 + v_7{}^2 + v_8{}^2) -$$
$$\frac{1}{\sqrt{6}}(v_9{}^2 + v_{10}{}^2 + v_{11}{}^2 + v_{12}{}^2 + v_{13}{}^2 + v_{14}{}^2) - \sqrt{\frac{2}{3}}v_{15}{}^2$$

When we put these expressions in the equations (2.12) we obtain an apparently intractable system of equations. The remarkable fact discovered by Kummer in [12] and [13] is the very simple and tractable form of these equations when they are written for the Fano parameters. These equation were already been obtained by us (see the equations (3.6), (3.7) and (3.8)] as a particularization of the general equations (3.2)-(3.5), taken from [17]). We shall find these equations directly from the equations for the Bloch vector using the relations between the Bloch and Fano parametrizations given above. We shall define the Fano parameters (denoted by the same symbols with a hat) associated with the vector $v\bigcup v$ as functions of the Fano parameters of the vector v:

$$\hat{x}_1 = (v\bigcup v)_4 + (v\bigcup v)_{11} = K_{11}y_1 + K_{12}y_2 + K_{13}y_3$$
$$\hat{x}_2 = (v\bigcup v)_5 + (v\bigcup v)_{12} = K_{21}y_1 + K_{22}y_2 + K_{23}y_3$$
$$\hat{x}_3 = \frac{2}{\sqrt{3}}(v\bigcup v)_8 + \sqrt{\frac{2}{3}}(v\bigcup v)_{15} = K_{31}y_1 + K_{32}y_2 + K_{33}y_3$$
$$\hat{y}_1 = (v\bigcup v)_1 + (v\bigcup v)_{13} = K_{11}x_1 + K_{21}x_2 + K_{31}x_3$$
$$\hat{y}_2 = (v\bigcup v)_2 + (v\bigcup v)_{14} = K_{12}x_1 + K_{22}x_2 + K_{32}x_3 \qquad (5.6)$$
$$\hat{y}_3 = (v\bigcup v)_3 - \frac{2}{\sqrt{3}}(v\bigcup v)_8 + \sqrt{\frac{2}{3}}(v\bigcup v)_{15} = K_{13}x_1 + K_{23}x_2 + K_{33}x_3$$
$$\hat{K}_{11} = (v\bigcup v)_6 + (v\bigcup v)_9 = x_1y_1 - (K_{22}K_{33} - K_{23}K_{32})$$
$$\hat{K}_{12} = -(v\bigcup v)_7 + (v\bigcup v)_{10} = x_1y_2 + (K_{21}K_{33} - K_{23}K_{31})$$
$$\hat{K}_{13} = (v\bigcup v)_4 - (v\bigcup v)_{11} = x_1y_3 - (K_{21}K_{32} - K_{22}K_{31})$$

and

$$\begin{aligned}
\widehat{K}_{21} &= (v\bigcup v)_7 + (v\bigcup v)_{10} = x_2 y_1 + (K_{12}K_{33} - K_{13}K_{32}) \\
\widehat{K}_{22} &= (v\bigcup v)_6 + (v\bigcup v)_9 = x_2 y_2 - (K_{11}K_{33} - K_{13}K_{31}) \\
\widehat{K}_{23} &= (v\bigcup v)_5 - (v\bigcup v)_{12} = x_2 y_3 + (K_{11}K_{32} - K_{12}K_{31}) \\
\widehat{K}_{31} &= (v\bigcup v)_1 - (v\bigcup v)_{13} = x_3 y_1 - (K_{12}K_{23} - K_{13}K_{22}) \\
\widehat{K}_{32} &= (v\bigcup v)_2 - (v\bigcup v)_{14} = x_3 y_2 + (K_{11}K_{23} - K_{13}K_{21}) \\
\widehat{K}_{33} &= (v\bigcup v)_3 + \frac{1}{\sqrt{3}}(v\bigcup v)_8 - \sqrt{\frac{2}{3}}(v\bigcup v)_{15} = x_3 y_3 - (K_{11}K_{22} - K_{12}K_{21})
\end{aligned} \tag{5.7}$$

Then if we define the operator K on the 3-dimensional Euclidean space R^3 in which the Bloch vectors x and y live we obtain:

$$\begin{aligned}
\widehat{x} &= Ky \\
\widehat{y} &= K^T x \\
\widehat{K} &= xy^T - (adjK)^T
\end{aligned} \tag{5.8}$$

from which the analogue for the Kummer equations in the case of pseudo-pure states follows immediately:

$$\begin{aligned}
Ky &= \mu x \\
K^T x &= \mu y \\
xy^T - (adjK)^T &= \mu K
\end{aligned} \tag{5.9}$$

In the above given equations we have used the following notations: K^T denotes the transpose of K and $adjK = K^{-1}\det K$. In order to solve the equations (5.10) we shall use an idea of T. Constantinescu and V. Ramakrishna [14-15]. We shall multiply the last equation from (5.10) on the right side with K^T and also we shall multiply it transpose with K. Then we obtain:

$$\begin{aligned}
(\det K)I &= \mu(xx^T - KK^T) \\
(\det K)I &= \mu(yy^T - K^T K)
\end{aligned} \tag{5.10}$$

If we multiply these equations on the right hand side with x and y respectively then we obtain:

$$\begin{aligned}
\det K &= \mu(<x,x> - \mu^2) \\
\det K &= \mu(<y,y> - \mu^2)
\end{aligned} \tag{5.11}$$

Hence for any pseudo-pure state we have the important restriction:

$$<x,x> = <y,y> \tag{5.12}$$

Replacing the values of $\det K$ given by the equations (5.12) in the equations (5.11) respectively we obtain the important equations:

$$\begin{aligned}
KK^T &= xx^T + (-<x,x> + \mu^2)I \\
K^T K &= yy^T + (-<y,y> + \mu^2)I
\end{aligned} \tag{5.13}$$

Let us consider that the Bloch vector x of the first qubit is given. Then the Bloch vector y of the second qubit and the correlation matrix K are the solutions of the following equations:

$$KK^T = xx^T + (-<x,x> + \mu^2)I$$
$$\det K = \mu(<x,x> - \mu^2) \tag{5.14}$$
$$y = \frac{1}{\mu}K^T x$$

Similarly if the Bloch vector y of the second qubit is given then the Bloch vector x of the first qubit and the correlation matrix K are the solutions of the following equations:

$$K^T K = yy^T + (-<y,y> + \mu^2)I$$
$$\det K = \mu(<y,y> - \mu^2) \tag{5.15}$$
$$x = \frac{1}{\mu}Ky$$

We remark that the symmetric, positive definite, 3X3 matrices $K^T K$ and KK^T have the same eigenvalues $\zeta_1, \zeta_2, \zeta_3$ which are the roots of the Cayley-Hamilton equations:

$$\zeta^3 - Tr(K^T K)\zeta^2 + \frac{1}{2}[(TrK^T K) - Tr(K^T K)^2]\zeta - (\det K)^2 = 0$$
$$\zeta^3 - Tr(KK^T)\zeta^2 + \frac{1}{2}[(TrKK^T) - Tr(KK^T)^2]\zeta - (\det K)^2 = 0 \tag{5.16}$$

respectively. Indeed we have:

$$Tr(K^T K) = -2<y,y> + 3\mu^2$$
$$Tr(KK^T) = -2<x,x> + 3\mu^2$$
$$\det K = \mu(<y,y> - \mu^2) \tag{5.17}$$
$$Tr(K^T K)^2 = 2<y,y>^2 - 4\mu^2<y,y> + 3\mu^4$$
$$Tr(KK^T)^2 = 2<x,x>^2 - 4\mu^2<x,x> + 3\mu^4$$

i.e., the matrices $K^T K$ and KK^T have identical Cayley-Hamilton equations given by:

$$\zeta^3 - (3\mu^2 - 2<x,x>)\zeta^2 + (3\mu^4 - 4\mu^2<x,x> + <x,x>^2)\zeta - \mu^2(<x,x> - \mu^2)^2 = 0 \tag{5.18}$$

This equation factorizes in the following way:

$$(\zeta - \mu^2)(\zeta - (\mu^2 - <x,x>))^2 = 0 \tag{5.19}$$

Hence the common eigenvalues of the positive definite matrices $K^T K$ and KK^T are $c_1 = \mu^2$ and $c_2 = c_3 = \mu^2 - <x,x>$.(We remark that the restriction $0 \le \mu^2 - <x,x>$ is a consequence of the fact that KK^T and KK^T are positive operators.) From this result and from the equations (5.14) we obtain the spectral decompositions of the operators $K^T K$ and KK^T :

$$KK^T = \mu^2 \frac{xx^T}{<x,x>} + (-<x,x> + \mu^2)(I - \frac{xx^T}{<x,x>})$$
$$K^T K = \mu^2 \frac{yy^T}{<y,y>} + (-<y,y> + \mu^2)(I - \frac{yy^T}{<y,y>}) \tag{5.20}$$

respectively. Let us define $|K|_x = \sqrt{KK^T}$ and $|K|_y = \sqrt{K^T K}$. Then we have:

$$
\begin{aligned}
|K|_x &= \mu \frac{xx^T}{<x,x>} + \sqrt{\mu^2 - <x,x>}(I - \frac{xx^T}{<x,x>}) \\
|K|_y &= \mu \frac{yy^T}{<y,y>} + \sqrt{\mu^2 - <y,y>}(I - \frac{yy^T}{<y,y>})
\end{aligned}
\tag{5.21}
$$

and

$$
\begin{aligned}
K &= \pm |K|_x \, O \\
K &= \pm O |K|_y
\end{aligned}
\tag{5.22}
$$

(where O_x and O_y are rotation matrices in the 3-dimensional Euclidean space of the corresponding Bloch vectors of qubits) and respectively

$$
\begin{aligned}
y &= \pm \frac{1}{\mu} O^T \, |K|_X \;\; x = \pm O^T x \\
x &= \pm \frac{1}{\mu} O |K|_y \;\; y = \pm O y
\end{aligned}
\tag{5.23}
$$

Hence, for the two qubits pseudo-pure states the Bloch vectors of the corresponding qubits are related by a rotation. There we have two singular cases: a) when $\mu^2 = <x,x> = <y,y>$ and b) when $<x,x> = <y,y> = 0$. In the first case we have $K = \frac{xx^T}{\mu}$ and $y = x$. In the second case we have $K = \mu O$ and $x = y = 0$.

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