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# APPROXIMATING SOME INFINITE SUMS OF FUNCTIONS WE COME ACROSS IN THE THEORY OF QUANTUM GASES

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Euler and Mac Laurin summation formula is improved, for applying it in more difficult cases, delivered by Statistical Mechanics of Quantum Ideal Gases. The improving refers to a numerical procedure for evaluating high order derivatives, based on a set of parameters and for summing up the series of derivatives so as for reaching convergent and reliable results. The method is worked out in two versions of comparable efficiency. Besides this, an extension of the Robinson series is operated in view of increasing the accuracy and of enlarging the domain of application. Finally, some numerical examples confirm the high precision of the method.

Almost the single numerical tool for summation, put at our disposal by the textbooks of Applied Mathematics, is the well known "Euler & Mac Laurin Summation Formula" [1-4,a].

$$\sum_{j=1}^{j=\infty} f(j) = \int_0^\infty f(x)dx + \sum_{j=1}^{j=N+1}\left\{ f(j) - \int_0^1 f(j-1+x)dx \right\} - \frac{1}{2} f(N+1) - \eta(N+1);$$

$$\eta(x) = \frac{1}{12} f'(x) - \frac{1}{720} f^{III}(x) + \frac{1}{30\ 240} f^{V}(x) - \frac{1}{1\ 209\ 600} f^{VII}(x) + \frac{1}{47\ 900\ 160} f^{IX}(x) - \ldots \quad (1)$$

The expression was adapted by us for infinite summation in the range (1, ∞), the integrals of the formula where assumed to exist and the free parameter N (N = 1,2,3,…), introduced by a trivial generalization of the original formula, has the role to increase the convergence speed of the series of derivatives (The original formula has N = 0 and does hold only if f(x) and all its derivatives are finite for x = 0). Nevertheless, excepting the case when all the integrals and all the derivatives entering the formula (1) may be performed analytically and exactly, the usefulness of the respective formula is drastically reduced. As a rule, the sums of Statistical Mechanics are approximated by integrals, provided that a certain inequality $\left(L >> \lambda_T \sqrt{\pi}\right)$ between the linear size of the enclosure and the thermal wave length is ensured [5], [6].

The purpose of this paper is to circumvent such difficulties, so as for applying a summation procedure, of the Euler & Mc. Laurin type, even in the case when the function f(x) is not exactly integrable and the performing of the successive derivatives is a difficult undertaking. In the first place, we ask for the function $\eta(x)$ an expression of the form

$$\eta(x) = \sum_{s=1}^{s=5} (-1)^{s-1} \cdot \omega_{2s-1}(x, p_s);$$

$$\omega_{2s-1}(x, p_s) = \sum_{l=1}^{l=2s} (-1)^{l-1} C_{2s-1}^{l-1} f\left[ x + \left( s - l + \frac{1}{2} \right) p_s \right] \quad (2)$$

Thereafter, we expand the $\eta$ function in ascending powers of its parameters $p_s$, (s = = 1,2,3,4,5), assuming these parameters confined in the range (0, 1). So, we come to the expression

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$$\eta(x) = p_1 f'(x) - \left(p_2^3 - \frac{1}{24}p_1^3\right) f'''(x) + \left(p_3^5 - \frac{1}{8}p_2^5 + \frac{1}{1920}p_1^5\right)\cdot f^{V}(x) -$$
$$-\left(p_4^7 - \frac{5}{24}p_3^7 + \frac{91}{13\ \ 440}p_2^7 - \frac{1}{322\ \ 560}p_1^7\right) f^{VII}(x) + \tag{3}$$
$$+\left(p_5^9 - \frac{7}{24}p_4^9 + \frac{161}{8\ \ 064}p_3^9 - \frac{41}{193\ \ 536}p_2^9 + \frac{1}{92\ \ 897\ \ 280}p_1^9\right) f^{IX}(x) - \ldots$$

Now, we ask the coincidence of the function $\eta(x)$, defined in (3) with the function $\eta(x)$ defined in (1) for any x. This identifying delivers us a set of 5 equations for the 5 unknown parameters $p_s$ (s = 1,2,3,4,5)

$$p_1 = \frac{1}{12}$$

$$p_2^3 - \frac{1}{24}p_1^3 = \frac{1}{720}$$

$$p_3^5 - \frac{1}{8}p_2^5 + \frac{1}{1\,920}p_1^5 = \frac{1}{30\,240} \tag{4}$$

$$p_4^7 - \frac{5}{24}p_3^7 + \frac{91}{13\,440}p_2^7 - \frac{1}{322\,560}p_1^7 = \frac{1}{1\,209\,600}$$

$$p_5^9 - \frac{7}{24}p_4^9 + \frac{161}{8\,064}p_3^9 - \frac{41}{193\,536}p_2^9 + \frac{1}{92\,897\,280}p_1^9 = \frac{1}{47\,900\,160}$$

The solutions of the algebraic system (4) are given below

| | | |
|---|---|---|
| $p_1$ = 8.333 333 3 (-2) | 8.333 333 333 3 (-2) | |
| $p_2$ = 1. 122 141 3 (-1) | 1.122 141 296 6 (-1) | |
| $p_3$ = 1. 286 862 6 (-1) | 1.286 862 641 4 (-1) | (5) |
| $p_4$ = 1. 378 719 0 (-1) | 1.378 718 959 1 (-1) | |
| $p_5$ = 1. 435 810 3 (-1) | 1.435 810 282 1 (-1) | |

The summation formula acquires the form

$$\sum_{j=1}^{j=\infty} f(j) = \int_0^{\infty} f(x)dx + \sum_{j=1}^{j=N+1}\left\{f(j) - \int_0^1 f(j-1+x)dx\right\} - \frac{1}{2}f(N+1) - \eta(N+1)\ \ ;\quad (N \geq 1) \tag{6}$$

The explicit expression of $\eta$ is

$$\eta = \omega_1(N+1, p_1) - \omega_3(N+1, p_2) + \omega_5(N+1, p_3) - \omega_7(N+1, p_4) + \omega_9(N+1, p_5); \tag{7}$$

The quantities $\omega_j$ are given in Appendix I.

The integrals over the interval (0,1) in formula (6) may be estimated by using pseudo-Tchebysheff type mechanical quadrature[4,a].

$$\int_{-1/2}^{+1/2} h(x)dx = \frac{1}{13}\left\{h(0) + \sum_{s=1}^{s=6}\left[h(-x_s) + h(+x_s)\right]\right\} \tag{8}$$

$$x_1 = 0.082\ 345\ 7 \qquad x_4 = 0.349\ 642\ 7$$
$$x_2 = 0.167\ 435\ 8 \qquad x_5 = 0.351\ 012\ 4$$
$$x_3 = 0.202\ 216\ 1 \qquad x_6 = 0.469\ 574\ 3$$

A more efficient version of the summation formula may be obtained if the derivatives (of various orders) are calculated rather in points $x_N = N + \frac{3}{2}$ $(N \geq 0)$, than $x_N = N + 1$ $(N \geq 1)$. The starting point is the mathematical identity

$$\sum_{j=1}^{j=\infty} f(j) = \int_0^{\infty} f(x)dx - \int_0^{1/2} f(x)dx + \sum_{j=1}^{j=N+1}\int_{-1/2}^{+1/2}\left[f(j) - f(j+x)\right]dx +$$
$$+\theta\left(N+\frac{3}{2}\right),\quad \theta\left(N+\frac{3}{2}\right) = \sum_{j=N+2}^{j=\infty}\int_{-1/2}^{+1/2}\left[f(j) - f(j+x)\right]dx \tag{9a}$$

By using a mathematical procedure similar to that leading to formula (1), one obtains [4,b]

$$\theta\left(N+\frac{3}{2}\right) = \frac{1}{24}f'\left(N+\frac{3}{2}\right) - \frac{7}{5760}f'''\left(N+\frac{3}{2}\right) + \frac{31}{967680}f^{V}\left(N+\frac{3}{2}\right) -$$
$$-\frac{127}{154828800}f^{VII}\left(N+\frac{3}{2}\right) + \frac{73}{3503554560}f^{IX}\left(N+\frac{3}{2}\right) - \ldots \tag{9b}$$

Formulas (2) still apply now, for $\theta\left(N+\frac{3}{2}\right)$ instead of $\eta(N+1)$. Accordingly, the equations (4) are kept unchanged, excepting the free terms, which should be replaced by the coefficients of the expansion (9b). So, new parameters $p_s$(s = 1,2,3,4,5,), slightly different form (5), are obtained

$$\begin{aligned} p_1 &= 4.\ 166\ 666\ 667(-2)\\ p_2 &= 1.\ 068\ 030\ 809(-1)\\ p_3 &= 1.\ 275\ 594\ 018(-1)\\ p_4 &= 1.\ 375\ 939\ 001(-1)\\ p_5 &= 1.\ 435\ 065\ 681(-1)\end{aligned} \tag{10}$$

The explicit expression of $\theta$ is

$$\theta = \omega_1\left(N+\frac{3}{2},\ p_1\right) - \omega_3\left(N+\frac{3}{2},\ p_2\right) + \omega_5\left(N+\frac{3}{2},\ p_3\right) - \omega_7\left(N+\frac{3}{2},\ p_4\right) + \omega_9\left(N+\frac{3}{2},\ p_5\right) \tag{11}$$

The quantities $\omega_j$ are given in Appendix II.

The expression $\omega_1 - \omega_3 + \omega_5 - \omega_7 + \omega_9$ standing either for $\zeta$ in (6) or for $\theta$ in (9), is actually a truncation of an infinite alternative series. For completing the missing terms (in the case of slowly convergent series) we recommend to apply the Padé-approximant procedure: [7], [8].

$$\omega_1 - \omega_3 + \omega_5 - \omega_7 + \omega_9 - \ldots = \omega_1 - \lambda \quad ; \quad \lambda = \frac{A_1}{1+}\ \frac{+A_2}{1+}\ \frac{+A_3}{1+}\ \frac{+A_4}{1+\ldots}$$

$$A_1 = \omega_3, \quad A_2 = \frac{\omega_5}{\omega_3}, \quad A_3 = \frac{\omega_7}{\omega_5} - \frac{\omega_5}{\omega_3} \quad A_4 = \frac{\omega_9 \cdot \omega_5 - \omega_7^2}{\omega_7 \cdot \omega_3 - \omega_5^2} \cdot \frac{\omega_3}{\omega_5}. \tag{12}$$

Sometimes convenient way to evaluate the integrals over the interval $(0, \infty)$, in (6) and (9) is to resort to Hermite type numerical integration. For instance, the integrals we come across in the theory of ideal quantum gases, when we want to calibrate the particle spectrum of fermions [9], [10].

$$I_+ = \int_0^\infty \frac{2\pi\sqrt{x}dx}{e^{ax+\alpha}+1}, \qquad (\alpha \geq 0) \tag{13}$$

may, through the variable change $x = \frac{1}{a}u^2$, be brought to the form

$$I_+ = \left(\frac{\pi}{a}\right)^{3/2} \cdot G_+(\alpha), \quad G_+(\alpha) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{u^2 e^{-u^2} du}{e^\alpha + e^{-u^2}} \tag{14}$$

enabling us to apply the Hermite type integration formula (with weight $e^{-u^2}$ and interval $-\infty < u < +\infty$) [11].

$$G_+(\alpha) = \sum_{l=1}^{l=n} \frac{C_l}{e^\alpha + a_l} \tag{15}$$

The constants $(C_l,\ a_l)$ are connected to the original ones by

$$\sum_{l=1}^{l=n} C_l = 1, \quad C_l = \frac{4}{\sqrt{\pi}} A_l x_l^2, \quad a_l = e^{-x_l^2} \tag{16}$$

The calibration relation (16) is fulfilled by taking the first 11 weights $A_l$ and roots $x_l$ of the 32-root formula. The constants of the formula (15) are given below.

| l | $C_l$ | $a_l$ |
|---|---|---|
| 1 | 3.214 784 1 (-2) | 0.962 748 7 |
| 2 | 2.142 706 7 (-1) | 0.710 206 0 |
| 3 | 3.255 232 3 (-1) | 0.385 369 3 |
| 4 | 2.562 238 3 (-1) | 0.152 906 1 |
| 5 | 1.237 774 7 (-1) | 0.439 544 7 (-1) |
| 6 | 3.882 204 3 (-2) | 0.903 430 1 (-2) |
| 7 | 8.038 595 4 (-3) | 0.130 413 5 (-2) |
| 8 | 1.094 652 3 (-3) | 0.129 089 9 (-3) |
| 9 | 9.620 110 7 (-5) | 0.848 650 2 (-5) |
| 10 | 5.275 995 2 (-6) | 0.354 899 4 (-6) |
| 11 | 1.714 700 0 (-7) | 0.889 530 0 (-8) |

(17)

A specific mono-parametric class of mathematical functions is to be found in the theory of bosonic gases, namely [12].

$$G\left(n+\frac{1}{2},\ \alpha\right) = \frac{1}{\left(n-\frac{1}{2}\right)!} \int_0^\infty \frac{x^{n-\frac{1}{2}}dx}{e^{x+\alpha}-1}, \quad (n = -1,0,1,2,3,\ldots) \tag{18}$$

A direct physical meaning may be assigned to the cases n = 1 (particle spectrum) and n = 2 (energy spectrum). The functions G may be expressed as infinite series by

$$G\left(n+\frac{1}{2},\alpha\right)=\sum_{j=1}^{j=\infty}\frac{e^{-\alpha j}}{j^{n+\frac{1}{2}}} \qquad 193)$$

Among the various elements of this class, a certain mathematical connection, through the intermediary of the Riemann's $\zeta$ function, is established, in such a way that the entire class G may be known if a single element of G is known

$$G\left(n-\frac{1}{2},\ \alpha\right)=-\frac{\partial}{\partial\alpha}G\left(n+\frac{1}{2},\alpha\right)$$

$$G\left(n+\frac{1}{2},\ \alpha\right)=\zeta\left(n+\frac{1}{2}\right)-\int_0^{\alpha}G\left(n-\frac{1}{2},\ \alpha\right)d\alpha \qquad (20)$$

$$\zeta(3/2)=2.612\ 375\ 4,\quad \zeta=(5/2)=1.341\ 487\ 2,\quad \zeta=(7/2)=1.126\ 733\ 9$$

We choose $G\left(\frac{1}{2},\ \alpha\right)$ as generating element for the whole class G. After separating the singularity, the expression of $G\left(\frac{1}{2},\ \alpha\right)$ becomes

$$G\left(\frac{1}{2},\ \alpha\right)=\sqrt{\frac{\pi}{\alpha}}+\frac{2}{\sqrt{\pi}}\int_0^{\infty}\left\{\frac{1}{e^{u^2+\alpha}-1}-\frac{1}{u^2+\alpha}\right\}du \qquad (21)$$

Starting with (21) and performing a series expansion in the ascending powers of $\alpha$ under the integral sign, one obtains few terms of the series of G:

$$G\left(\frac{1}{2},\ \alpha\right)=\sqrt{\frac{\pi}{\alpha}}-\frac{2}{\sqrt{\pi}}\int_0^{\infty}\left\{\frac{1}{u^2}-\frac{1}{e^{u^2}-1}\right\}du+\alpha\frac{2}{\sqrt{\pi}}\int_0^{\infty}\left\{\frac{1}{u^4}-\frac{e^{u^2}}{\left(e^{u^2}-1\right)^2}\right\}du \qquad (22)$$

F. London presumably used this procedure, based on successive extractions, in 1954, in his trial to evaluate $G\left(\frac{3}{2},\ \alpha\right)$ and $G\left(\frac{5}{2},\ \alpha\right)$ for very small $\alpha$ [13].

The coefficients of the expansion of G in terms of $\alpha$ in (22) are expressed as rather intricate integrals, and this is a difficulty preventing us from the extending of the approximation toward greater values of $\alpha$. However, such an extension is possible, provided that a non-conventional procedure, we expound in the sequel, is used. Our starting point is the identity:

$$\sum_{j=1}^{j=\infty}f(j)=\int_0^{\infty}f(x)dx-\int_0^{1/2}f(x)dx-\sum_{j=1}^{j=\infty}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\left[f(j+x)-f(j)\right]dx \qquad (23)$$

used in the case $f(x)=\frac{e^{-\alpha x}}{\sqrt{x}}$. We first calculate the two integrals entering the expression (23)

$$\int_0^\infty e^{-\alpha x}\frac{dx}{\sqrt{x}}=\sqrt{\frac{\pi}{\alpha}} \quad ; \quad \int_0^{1/2} e^{-\alpha x}\frac{dx}{\sqrt{x}}=\sum_{s=0}^{\infty}(-1)^s\frac{\alpha^s}{s!}\frac{1}{\left(s+\frac{1}{2}\right)2^{s+\frac{1}{2}}} \tag{24}$$

Thereafter, we calculate

$$\int_{-1/2}^{+1/2}\left[f(j+x)-f(j)\right]dx=\sum_{s=0}^{s=\infty}(-1)^s\frac{\alpha^s}{s!}\psi(j,s) \quad ; \quad \psi(j,s)\equiv\frac{\left(j+\frac{1}{2}\right)^{s+\frac{1}{2}}-\left(j-\frac{1}{2}\right)^{s+\frac{1}{2}}}{s+\frac{1}{2}}-j^{s-\frac{1}{2}} \tag{25}$$

So, the function $G\left(\frac{1}{2},\alpha\right)$ acquires the form

$$G\left(\frac{1}{2},\alpha\right)=\sqrt{\frac{\pi}{\alpha}}-\sum_{s=0}^{s=\infty}(-1)^s\frac{\alpha^s}{s!}\left\{\frac{1}{\left(s+\frac{1}{2}\right)^{s+\frac{1}{2}}}+C_s\right\} \quad ; \quad C_s=F.P.\sum_{j=1}^{j=\infty}\psi(j,s) \tag{26}$$

The simple summation in the expression of $C_s$ in (26) is divergent. For this reason, a special caution is to be paid to extract the finite part (F.P.). We accomplish this by resorting to the modified version of the Euler & Mac Laurin summation formula (in which the derivatives are determined in points $x=N+\frac{1}{2}$).

$$C_s=\lim_{N\to\infty}\left\{\sum_{j=1}^{j=N}\psi(j,s)+R_N\right\}$$

$$\begin{aligned}
R_N=&\left\{\frac{1}{s+\frac{1}{2}}\left(N+\frac{1}{2}\right)^{s+\frac{1}{2}}-\frac{1}{\left(s+\frac{1}{2}\right)\left(s+\frac{3}{2}\right)}\left[(N+1)^{s+\frac{3}{2}}-N^{s+\frac{3}{2}}\right]\right\}+\\
&+\frac{1}{24}\left\{(N+1)^{s-\frac{1}{2}}-N^{s-\frac{1}{2}}-\left(s-\frac{1}{2}\right)\left(s+\frac{1}{2}\right)^{s-\frac{3}{2}}\right\}-\\
&-\frac{7}{5\,760}\left(s-\frac{1}{2}\right)\left(s-\frac{3}{2}\right)\left\{(N+1)^{s-\frac{5}{2}}-N^{s-\frac{5}{2}}-\left(s-\frac{5}{2}\right)\left(N+\frac{1}{2}\right)^{s-\frac{7}{2}}\right\}+\\
&+\frac{31\left(s-\frac{1}{2}\right)\left(s-\frac{3}{2}\right)\left(s-\frac{5}{2}\right)\left(s-\frac{7}{2}\right)}{967\,680}\left\{(N+1)^{s-\frac{9}{2}}-N^{s-\frac{9}{2}}-\left(s-\frac{9}{2}\right)\left(N+\frac{1}{2}\right)^{s-\frac{11}{2}}\right\}-
\end{aligned} \tag{27}$$

$$-\frac{127\left(s-\frac{1}{2}\right)\left(s-\frac{3}{2}\right)\left(s-\frac{5}{2}\right)\left(s-\frac{7}{2}\right)\left(s-\frac{9}{2}\right)\left(s-\frac{11}{2}\right)}{154\ \ 828\ \ 800}\left\{(N+1)^{s-\frac{13}{2}}-N^{s-\frac{13}{2}}-\left(s-\frac{13}{2}\right)\left(N+\frac{1}{2}\right)^{s-\frac{15}{2}}\right\}+\ldots$$

In this way, one obtains for the first coefficients $C_s$ the table:

| s | $C_s$ | s | $C_s$ | |
|---|---|---|---|---|
| 0 | +0.046 141 0 | 4 | -0.014 262 00 | |
| 1 | -0.027 816 1 | 5 | -0.000 926 23 | (28) |
| 2 | -0.045 225 6 | 6 | +0.000 972 10 | |
| 3 | -0.033 770 9 | 7 | -0.003 484 20 | |

The series expansion of $G\left(\frac{1}{2},\alpha\right)$ turns out to be

$$G\left(\frac{1}{2},\alpha\right)=\sqrt{\pi}\alpha^{-\frac{1}{2}}-\left(C_0+\sqrt{2}\right)+\frac{1}{1!}\left(C_1+\frac{\sqrt{2}}{6}\right)\alpha-\frac{1}{2!}\left(C_2+\frac{\sqrt{2}}{20}\right)\alpha^2+$$
$$+\frac{1}{3!}\left(C_3+\frac{\sqrt{2}}{56}\right)\alpha^3-\frac{1}{4!}\left(C_4+\frac{\sqrt{2}}{144}\right)\alpha^4+\frac{1}{5!}\left(C_5+\frac{\sqrt{2}}{352}\right)\alpha^5-\frac{1}{6!}\left(C_6+\frac{\sqrt{2}}{832}\right)\alpha^6+\frac{1}{7!}\left(C_7+\frac{\sqrt{2}}{1920}\right)\alpha^7-\ldots \tag{29}$$

In completely explicit form the G-functions are given in Appendix III.

An alternative procedure, to work out the calibration of the particle – and energy – spectra of ideal quantum gases (either bosons or fermions), is the resorting to the Born-type series as they stand, but asymptotically evaluating the remainders [14], [15].

$$I_P^{\pm}(q)=\frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{\sqrt{x}dx}{qe^x\pm1}=\frac{1}{q}\mp\frac{1}{2^{3/2}}\frac{1}{q^2}+\frac{1}{3^{3/2}}\frac{1}{q^3}\mp\frac{1}{4^{3/2}}\frac{1}{q^4}+\ldots$$
$$I_E^{\pm}(q)=\frac{4}{3\sqrt{\pi}}\int_0^{\infty}\frac{x^{3/2}dx}{qe^x\pm1}=\frac{1}{q}\mp\frac{1}{2^{5/2}}\frac{1}{q^2}+\frac{1}{3^{5/2}}\frac{1}{q^3}\mp\frac{1}{4^{5/2}}\frac{1}{q^4}+\ldots \tag{30}$$

We can write for $I_P^-(q)$

$$I_P^-(q)=S_N(q)+R_N(q); \tag{31}$$

$$S_N(q)=\sum_{j=1}^{j=N}j^{-3/2}\cdot\frac{1}{q^j}, \qquad R_N(q)=\sum_{j=1}^{j=N}(N+j)^{-3/2}\cdot q^{-(N+j)} \tag{32}$$

Resorting to the integral representation of the general term in the remainder series

$$(N+j)^{-3/2}\cdot q^{-(N+j)}=q^{-(N+j)}\cdot\frac{2}{\sqrt{\pi}}\int_0^{\infty}\sqrt{x}e^{-(N+j)x}dx \tag{32}$$

the expression of the remainder may be cast in the compact form

$$R_N=\frac{1}{(N+j)^{3/2}\cdot q^{(N+1)}}\cdot\frac{2}{\sqrt{\pi}}\cdot\int_0^{\infty}\frac{\sqrt{x}e^{-x}dx}{\left(1-\frac{1}{q}e^{-\frac{x}{N+1}}\right)} \tag{33}$$

Likewise, for $I_P^+(q)$ on obtains

$$I_P^+(q) = \sum_{j=1}^{j=N} (-1)^{j-1} \cdot j^{-3/2} \cdot \frac{1}{q^j} + R_N \quad ; \quad R_N = \frac{(-1)^N}{(N+1)^{3/2} \cdot q^{(N+1)}} \cdot \frac{2}{\sqrt{\pi}} \cdot \int_0^\infty \frac{\sqrt{x} e^{-x} dx}{\left(1 + \frac{1}{q} e^{-\frac{x}{N+1}}\right)} \tag{34}$$

The integrals in (34) may be evaluated resorting to the mechanical quadrature formula

$$\int_0^\infty \sqrt{x} e^{-x} f(x) dx = \sum A_k f(x_k), \qquad \sum A_k = \frac{\sqrt{\pi}}{2} \tag{35}$$

So, the remainder formula becomes

$$R_N = \frac{(\pm 1)^N}{(N+1)^{3/2} \cdot q^{(N+1)}} \cdot \sum_{k=1}^{k=16} \frac{B_k}{\left(1 \mp \frac{1}{q} e^{-\frac{x}{N+1}}\right)} \tag{36}$$

Here, $B_k = \frac{2}{\sqrt{\pi}} A_k$ ; $(A_k, x_k)$- are the first 16 weights and zeros of the 32 – root formula

| k | $B_k$ | $x_k$ |
|---|---|---|
| 1 | 4.330 037 6 (-2) | 7.535 274 3 (-2) |
| 2 | 1.383 330 8 (-1) | 3.015 846 3 (-1) |
| 3 | 2.138 702 3 (-1) | 6.792 188 1 (-1) |
| 4 | 2.245 744 9 (-1) | 1.209 134 7 |
| 5 | 1.779 385 0 (-1) | 1.892 579 2 |
| 6 | 1.113 731 3 (-1) | 2.731 183 3 |
| 7 | 5.636 486 2 (-2) | 3.726 984 2 |
| 8 | 2.335 905 5 (-2) | 4.882 453 3 |
| 9 | 7.981 651 4 (-3) | 6.200 532 2 |
| 10 | 2.255 957 6 (-3) | 7.684 677 0 |
| 11 | 5.278 473 7 (-4) | 9.338 912 8 |
| 12 | 1.021 298 7 (-4) | 1.116 790 1(+1) |
| 13 | 1.629 742 6 (-5) | 1.317 702 3 (+1) |
| 14 | 2.136 260 9 (-6) | 1.537 248 0 (+1) |
| 15 | 2.287 784 4 (-7) | 1.776 142 5 (+1) |
| 16 | 1.988 143 6 (-8) | 2.035 211 7 (+1) |

(37)

The convergence of the sum $S_N + R_N$, with constants in the table (37), is very rapid for fermions and still satisfactory for bosons. For bosons, with $q^{-1} = 0.99$, one obtains the following results

| $N$ | $S_N$ | $R_N$ | $S_N + R_N$ |
|---|---|---|---|
| 9 | 1.917 385 7 | 0.353 773 2 | 2.271 158 9 |
| 19 | 2.088 193 0 | 0.183 442 4 | 2.271 635 4 |
| 29 | 2.154 740 3 | 0.116 917 3 | 2.271 657 6 |
| 39 | 2.190 233 0 | 0.081 426 8 | 2.271 659 8 |

(38)

Summing up directly 1024 terms and taking into account the remainder $R_{1024} \sim 1 \times 10^{-7}$, one obtains for $S_\infty + R_\infty$ the value 2.271 660 7. The result may be compared to that delivered by the Robinson function method $G(3/2, \alpha) = G(3/2, -ln\, 0.99) = 2.271\ 660\ 1$.

**Some applications**. For checking the efficiency of the summation formulas, improved in this paper, we consider the sum

$$S=\sum_{j=1}^{j=\infty}\frac{2\pi\sqrt{j}}{e^{aj+\alpha}-1}\quad;\quad \alpha=-\ln 0.99=1.005\ 033\ 6\cdot 10^{-2}\quad;\quad a=0.5;\qquad a=10^{-3} \tag{39}$$

For this purpose, we resort to formula (9) and use equally the expansion

$$f(x)=\frac{2\pi\sqrt{x}}{e^{ax+\alpha}-1}=\frac{2\pi\sqrt{x}}{ax+\alpha}\left\{1-\frac{1}{2}(ax+\alpha)+\frac{1}{12}(ax+\alpha)^2-\right.$$
$$\left.-\frac{1}{720}(ax+\alpha)^4+\frac{1}{30\ \ 240}(ax+\alpha)^6-\frac{1}{1\ \ 209\ \ 600}(ax+\alpha)^8+...\right\}\ \text{for}\quad 0<(ax+\alpha)<1. \tag{40}$$

A direct integration in (40) delivers the result

$$I_\lambda(a,\alpha)\equiv\int_0^\lambda f(x)dx=\frac{4\pi}{a}\sqrt{\lambda}\left(1-\sqrt{\frac{\alpha}{a\lambda}}\tan^{-1}\sqrt{\frac{a\lambda}{\alpha}}\right)-\frac{3}{2}\pi\lambda^{3/2}+$$
$$+\frac{\pi}{3}\lambda^{3/2}\left(\frac{1}{5}a\lambda+\frac{1}{3}\alpha\right)-\frac{\pi}{180}\lambda^{3/2}\left(\frac{1}{9}a^3\lambda^3+\frac{3}{7}\alpha a^2\lambda^2+\frac{3}{5}\alpha^2a\lambda+\frac{1}{3}\alpha^3\right)+...\qquad 0<\lambda\le 2 \tag{41}$$

and the integral

$$I=\int_0^\infty f(x)dx=\left(\frac{\pi}{a}\right)^{3/2}\cdot G(3/2,\alpha),\ \text{id est} \tag{42a}$$

$$I\left(a=0.5;\ \alpha=1.005\ 033\ 6\cdot 10^{-2}\right)=35.777\ 770$$
$$I\left(a=10^{-3};\ \alpha=1.005\ 033\ 6\cdot 10^{-2}\right)=400\ 007.6 \tag{42b}$$

The result is given (for N = 0) under the form

$$S=I-I_{\frac{1}{2}}+\left\{f(1)-\left(I_{\frac{3}{2}}-I_{\frac{1}{2}}\right)\right\}+\theta(3/2) \tag{43}$$

with the following elements of calculation:

| *a* | *f(1)* | $I_{1/2}$ | $I_{3/2}$ |
|---|---|---|---|
| 1.(-3) | 5.654 609 6 (+2) | 1.423 670 (+2) | 6.998 217 6 (+2) |
| 5.(-1) | 9.443 073 9 | 1.215 876 (+1) | 2.204 053 2 (+1) |

(44)

| a | $I_{3/2}$- $I_{1/2}$ | $\theta_{3/2}$ | S |
|---|---|---|---|
| 1.(-3) | 5.574 547 6 (+2) | +6.673 300 | 3.998 799 1 (+5) |
| 5.(-1) | 0.988 177 2 (+1) | -164 628 2 (-1) | 2.301 568 4 (+1) |

[The derivation of the statistical factor $\left(e^{ax+}\pm 1\right)^{-1}$ is essentially based on the rough approximation $n!\sim n^n e^{-n}$. When a more realistic approximation, due to Stirling, is used, namely $n!\approx n^n\cdot e^{-n}\sqrt{2\pi n}$, the obtaining of the mentioned factor is no longer possible. As the physical phenomenon called "bosonic condensation" is a direct consequence of the behavior of the statistical factor in the proximity of x = 0, a certain doubt, upon this phenomenon too, cannot be avoided.]

Summing up directly the first 35 terms of the sum S in (39) with $a = 0.5$ and $\alpha = 1.005\ 033\ 6 \times 10^{-2}$ one obtains the value S = 2.301 568 3 (+1) is in coincidence, within an error of about $\leq 1\times10^{-7}$ with the calculated value.

It remained to us the task of proving that the accuracy of the two alternative summation formulas (6) and (9) (with the specified mechanism for summing up the derivative series) is comparable.

In this purpose, we adopt as trial function $f(x) = (1+x)^{-2}$, leading to the exact results $S = \frac{\pi^2}{6} - 1$. Applying now the mentioned formulas (for N = 1 and N = 0) one obtains

$$\frac{\pi^2}{6} = \frac{59}{36} - \eta(2) \quad = 1.644\ 934\ 3 \quad (N=1) \quad ; \quad \frac{\pi^2}{6} = \frac{33}{20} + \theta(3/2) = 1.644\ 933\ 9 \quad (N=0) \qquad (45)$$

The results are indeed comparable to the exact value $\frac{\pi^2}{6} = 1.644\ 934\ 1$, within an error of about $\pm 2 \cdot 10^{-7}$.

**Concluding remarks**. This paper put at the disposal of the theorists, working in the field of Statistical Mechanics (and equally in other domains of research) a powerful mathematical tool for calculating infinite sums. The restrictions concerning the smallness of the finite volume effects are no longer necessary. Nor the intricate aspect of the functions to be summed up is an impediment. We appreciate the results as a noteworthy contribution for processing the experimental data.

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