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OPTIMALITY CONDITIONS FOR NONLINEAR PROGRAMMING WITH GENERALIZED LOCALLY ARCWISE CONNECTED FUNCTIONS

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A nonlinear programming problem with inequality constraints is considered, where the functions involved are ρ-locally arcwise connected, ρ-locally *Q*-connected and ρ-locally *P*-connected and differentiable with respect to an arc. Sufficient optimality conditions are obtained in terms of the right differentials with respect to an arc of the functions.

1. PRELIMINARIES

In this section we introduce the notation and definitions which are used throughout the paper.

Let $\mathbf{R}^n$ be the *n*-dimensional Euclidean space and $\mathbf{R}^n_+$ its nonnegative orthant $\{x \in \mathbf{R}^n, x_j \geqq 0, j = 1,\dots, n\}$. Throughout the paper, the following conventions for vectors in $\mathbf{R}^n$ will be followed:

$x > y$ if and only if $x_i > y_i$, $i = 1, \dots, n$,

$x \geqq y$ if and only if $x_i \geqq y_i$, $i = 1,\dots, n$,

$x \geq y$ if and only if $x_i \geqq y_i$, $i = 1,\dots, n$, but $x \neq y$.

Throughout the paper, all definitions and theorems are numbered consecutively in a single numeration system in each section.

Let $X^0 \subseteq \mathbf{R}^n$ be a nonempty and compact subset of $\mathbf{R}^n$.

Definition 1.1. Let $\bar{x}$, $x \in X^0$. A continuous mapping $H_{\bar{x},x}: [0,1] \to \mathbf{R}^n$ with

$$H_{\bar{x},x}(0) = \bar{x},\ H_{\bar{x},x}(1) = x$$

is called an arc from $\bar{x}$ to x.

Definition 1.2. [4] We say that the set $X^0 \subseteq \mathbf{R}^n$ is a locally arcwise connected set at $\bar{x}$ ($\bar{x} \in X^0$) (X^0 is LAC($\bar{x}$), for short) if for any $x \in X^0$ there exist a positive number $a(x,\bar{x})$, with $0 < a(x,\bar{x}) \leqq 1$, and a continuous arc $H_{\bar{x},x}$ such that $H_{\bar{x},x}(\lambda) \in X^0$ for any $\lambda \in (0, a(x,\bar{x}))$.

We say that the set X^0 is locally arcwise connected if X^0 is locally arcwise connected at any $x \in X^0$.

If we choose the function $H_{\bar{x},x}$ of the form $H_{\bar{x},x}(\lambda) = (1-\lambda)\bar{x} + \lambda x$, we retrieve the definition of locally starshaped set as given by Ewing [2].

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Definition 1.3. [7] Let $f : X^0 \to \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R}^n$ is a locally arcwise connected set at $\bar{x} \in X^0$ with the corresponding function $H_{\bar{x},x}(\lambda)$ and a maximum positive number $a(x,\bar{x})$ satisfying the required conditions. Also let $\rho \in \mathbf{R}$ and $\mathrm{d}(\cdot,\cdot) : X^0 \times X^0 \to \mathbf{R}_+$ such that $\mathrm{d}(x,\bar{x}) \neq 0$ for $x \neq \bar{x}$. We say that f is:

(i_1) ρ*-locally arcwise connected* at $\bar{x}$ (f is $\rho - LCN(\bar{x})$, for short) if for any $x \in X^0$ there exist a positive number $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$ and an arc $H_{\bar{x},x}$ in X^0 on $[0, \mathrm{d}(x,\bar{x})]$ such that

$$f(H_{\bar{x},x}(\lambda)) \leqq \lambda f(x) + (1-\lambda) f(\bar{x}) - \rho\lambda\, \mathrm{d}(x,\bar{x}),\ 0 \leq \lambda \leq \mathrm{d}(x,\bar{x}). \tag{1.1}$$

(i_2) ρ*-locally Q-connected* at $\bar{x}$ ($\rho - LQCN(\bar{x})$) if for any $x \in X^0$ there exist a positive number $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$) and an arc $H_{\bar{x},x}$ in X^0 on $[0, \mathrm{d}(x,\bar{x})]$ such that

$$\left.\begin{array}{l} f(x) \leqq f(\bar{x}) \\ 0 \leq \lambda \leq \mathrm{d}(x,\bar{x}) \end{array}\right\} \Rightarrow f(H_{\bar{x},x}(\lambda)) - f(\bar{x}) \leqq -\rho\lambda \mathrm{d}(x,\bar{x}).$$

(i_3) ρ*-locally P-connected* at $\bar{x}$ ($\rho - LPCN(\bar{x})$) if for any $x \in X^0$ there exist a positive number $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$, an arc $H_{\bar{x},x}$ in X^0 on $[0, \mathrm{d}(x,\bar{x})]$, and a positive number $\gamma_{\bar{x},x}$ such that

$$\left.\begin{array}{l} f(x) < f(\bar{x}) \\ 0 \leq \lambda \leq \mathrm{d}(x,\bar{x}) \end{array}\right\} \Rightarrow f(H_{\bar{x},x}(\lambda)) \leqq f(\bar{x}) - \lambda\gamma_{\bar{x},x} - \rho\lambda \mathrm{d}(x,\bar{x}).$$

(i_4) ρ*-locally strictly P-connected* at $\bar{x}$ ($\rho - LSTPCN(\bar{x})$) if for any $x \in X^0$ there exist a positive number $\mathrm{d}(x,\bar{x}) \leqq a(x,\bar{x})$, an arc $H_{\bar{x},x}$ in X^0 on $[0, \mathrm{d}(x,\bar{x})]$, and a positive number $\gamma_{\bar{x},x}$ such that

$$\left.\begin{array}{l} x \neq \bar{x}, f(x) < f(\bar{x}) \\ 0 \leq \lambda \leq \mathrm{d}(x,\bar{x}) \end{array}\right\} \Rightarrow f(H_{\bar{x},x}(\lambda)) < f(\bar{x}) - \lambda\gamma_{\bar{x},x} - \rho\lambda \mathrm{d}(x,\bar{x}).$$

The function f is said to be ρ-locally strictly arcwise connected at $\bar{x} \in X^0 (\rho - LSCN(\bar{x}))$ if for each $x \in X^0$, $x \neq x^0$, the inequality (1.1) is strict.

If f is $\rho - LCN(\bar{x})$ ($\rho - LSCN(\bar{x})$) at each $\bar{x} \in X^0$, then f is said to be $\rho - LCN$ ($\rho - LSCN$) on X^0.

If f is $\rho - LQCN$ at each $\bar{x} \in X^0$, then f is said to be $\rho - LQCN$ on X^0.

If f is $\rho - LPCN$ at each $\bar{x} \in X^0$, then f is said to be $\rho - LPCN$ on X^0.

Definition 1.4. [3] Let $f : X^0 \to \mathbf{R}$ be a function, where $X^0 \subseteq \mathbf{R}^n$ is a locally arcwise connected set at $\bar{x} \in X^0$, with the corresponding function $H_{\bar{x},x}(\lambda)$ and a maximum positive number $a(x,\bar{x})$ satisfying the required conditions. The right differential of f at $\bar{x}$ with respect to the arc $H_{\bar{x},x}(\lambda)$ is defined as

$$(\mathrm{d}f)^+(\bar{x}, H_{\bar{x},x}(0^+)) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}[f(H_{\bar{x},x}(\lambda)) - f(\bar{x})]$$

provided the limit exists.

If f is differentiable at any $\bar{x} \in X^0$, then f is said to be differentiable on X^0.

2. SUFFICIENT OPTIMALITY CRITERIA

Consider the nonlinear programming problem

$$\text{(P)} \qquad \begin{cases} \text{Minimize } f(x) \\ \text{subject to}: g(x) \leqq 0, x \in X^0, \end{cases}$$

where

i) $X^0 \subseteq \mathbf{R}^n$ is a nonempty open locally arcwise connected set;

ii) $f : X^0 \to \mathbf{R}$;

iii) $g = (g_i)_{1 \le i \le m} : X^0 \to \mathbf{R}^m$;

iv) the right differentials of f and g_j, $j = 1,...,m$ at $\bar{x}$ exist with respect to the same arc $H_{\bar{x},x}(\lambda)$.

Let $X = \{x \in X^0 \mid g(x) \leqq 0\}$ be the set of all feasible solutions to (P).

Let

$$N_\varepsilon(\bar{x}) = \{x \in \mathbf{R}^n \mid \|x - \bar{x}\| < \varepsilon\}.$$

Definition 2.1. a) $\bar{x}$ is said to be a local minimum solution to problem (P) if $\bar{x} \in X$ and there exists $\varepsilon > 0$ such that $x \in N_\varepsilon(\bar{x}) \cap X \Rightarrow f(\bar{x}) \leqq f(x)$.

b) $\bar{x}$ is said to be the minimum solution to problem (P) if $\bar{x} \in X$ and $f(\bar{x}) = \min_{x \in X} f(x)$.

For $\bar{x} \in X$ we denote by $I = I(\bar{x}) = \{i \mid g_i(\bar{x}) = 0\}$ the set of indices of active constraints at $\bar{x}$, by $J = J(\bar{x}) = \{i \mid g_i(\bar{x}) < 0\}$ the set of indices of nonactive constraints at $\bar{x}$, and set $g_I = (g_i)_{i \in I}$. Obviously $I \cup J = \{1,2,...,m\}$.

Let $u \in \mathbf{R}^m$ be such that $u \geqq 0$ and $u^T g(\bar{x}) = 0$. Obviously, $u_I \geqq 0$ and $u_J = 0$ where u_I and u_J denotes the subvectors of u corresponding to the index sets I and J, respectively.

Let $K = \{i \in I : u_i > 0\}$ and $L = \{i \in I : u_i = 0\}; K \cup L = I$.

Let g_K and g_L be the subvectors of g_I corresponding to the index sets K and L, respectively.

In this section we give sufficient optimality theorems for problem (P).

First, we give a sufficient optimality theorem of the Kuhn-Tucker type. The functions f and g are not differentiable but are directional differentiable with respect to the same arc $H_{\bar{x},x}(\lambda)$ at $\lambda = 0$.

Let $\{K_1, K_2, K_3\}$ be a partition of the index set K; thus $K_i \subset K$ for each $i = 1,2,3$, $K_r \cap K_s = \text{Ø}$ for each $r, s \in \{1,2,3\}$ with $r \neq s$, and $\bigcup_{i=1}^{3} K_i = K$.

Theorem 4.3. given by Kaul and Lyall [3] is special case of the following result.

Theorem 2.2 *Let $\bar{x} \in X^0 \subseteq \boldsymbol{R}^n$, where X^0 is a locally arcwise connected set and let $\bar{u} \in \boldsymbol{R}^m$. We assume that there exist the right differentials at $\bar{x}$ with respect to the same arc $H_{\bar{x},x}$ of f and g and $(\bar{x}, \bar{u})$ satisfies the following conditions:*

$$(\mathrm{d}f)^+(\bar{x}, H_{\bar{x},x}(0^+)) + \bar{u}^T (\mathrm{d}g)^+(\bar{x}, H_{\bar{x},x}(0^+)) \geqq 0, \ \forall \ x \in X, \tag{2.1}$$

$$\bar{u}^T g(\bar{x}) = 0, \tag{2.2}$$

$$g(\bar{x}) \leqq 0, \tag{2.3}$$

$$\bar{u} \geqq 0,\ \bar{u} \neq 0 \tag{2.4}$$

Assume furthermore that

$i_1)$ $$g_i, i \in K_1, \text{ is } \alpha_i - LQCN(\bar{x}), \tag{2.5}$$

$i_2)$ $$u_{K_2}^T g_{K_2} \text{ is } \beta - LQCN(\bar{x}) \tag{2.6}$$

$i_3)$ $$f + u_{K_3}^T g_{K_3} \text{ is } \gamma - LPCN(\bar{x}) \tag{2.7}$$

$i_4)$ $$\sum_{i \in K_1} \alpha_i u_i + \beta + \gamma \geq 0. \tag{2.8}$$

Then $\bar{x}$ *is a minimum solution to Problem (P).*

The following result is a special case of Theorem 2.2., where the conditions are special cases of (2.5) through (2.8).

Theorem 2.3. *Let* $\bar{x} \in X^0 \subseteq \mathbf{R}^n$, *where* X^0 *is a locally arcwise connected set and let* $\bar{u} \in \mathbf{R}^m$. *We assume that there exist the right differentials at* $\bar{x}$ *with respect to the same arc* $H_{\bar{x},x}$ *of* f *and* g *and* $(\bar{x},\bar{u})$ *satisfies conditions* (2.1) - (2.4).

Assume furthermore that any one of the following hypotheses is satisfied.

$i_1)$ a) $f + u_K^T g_K$ is $\gamma - LPCN(\bar{x})$, where $\gamma \geqq 0$;

$i_2)$ a) $g_i, i \in K$, is $\alpha_i - LQCN(\bar{x})$,

b) f is $\gamma - LPCN(\bar{x})$,

c) $\sum_{i \in K} \alpha_i u_i + \gamma \geq 0$;

$i_3)$ a) $u_K^T g_K$ is $\beta - LQCN(\bar{x})$,

b) f is $\gamma - LPCN(\bar{x})$,

c) $\beta + \gamma \geq 0$;

$i_4)$ a) $u_{K_2}^T g_{K_2}$ is $\beta - LQCN(\bar{x})$,

b) $f + u_{K_3}^T g_{K_3}$ is $\gamma - LPCN(\bar{x})$, where $\{K_2, K_3\}$ is a partition of K,

c) $\beta + \gamma \geq 0$;

$i_5)$ a) g_i, $i \in K_1$, is $\alpha_i - LQCN(\bar{x})$,

b) $f + u_{K_3}^T g_{K_3}$ is $\gamma - LPCN(\bar{x})$, where $\{K_1, K_3\}$ is a partition of K,

c) $\sum_{i \in K_1} \alpha_i u_i + \gamma \geq 0$;

$i_6)$ a) $g_i, i \in K_1$, is $\alpha_i - LQCN(\bar{x})$,

b) $u_{K_2}^T g_{K_2}$ is $\beta - LQCN(\bar{x})$,

c) f is $\gamma - LPCN(\bar{x})$,

d) $\sum_{i \in K_1} \alpha_i u_i + \beta + \gamma \geq 0$, where $\{K_1, K_2\}$ is a partition of K.

Then $\bar{x}$ *is a minimum solution to problem (P).*

In what follows we consider sufficient optimality conditions of the Fritz John type.

Let $(\bar{x}, v_0, v)$ be a Fritz John point, where $\bar{x} \in X^0$ (a locally arcwise connected set), $v_0 \in \mathbf{R}$, and $v \in \mathbf{R}^m$. Assume that $(\bar{x}, v_0, v)$ satisfies the following conditions:

$$v_0 (\mathrm{d}f)^+ (\bar{x}, H_{\bar{x},x}(0^+)) + v^T (\mathrm{d}g)^+ (\bar{x}, H_{\bar{x},x}(0^+)) \geqq 0, \ \forall \ x \in X \tag{2.9}$$

$$v^T g(\bar{x}) = 0 \tag{2.10}$$

$$(v_0, v) \geqq 0 \tag{2.11}$$

If $v_0 = 0$, then conditions (2.9)-(2.11) become

$$v^T (\mathrm{d}g)^+ (\bar{x}, H_{\bar{x},x}(0^+)) \geqq 0, \ \forall \ x \in X \tag{2.12}$$

$$v^T g(\bar{x}) = 0 \tag{2.13}$$

$$v \geqq 0 \tag{2.14}$$

Let I and J be the sets defined at the beginning of this section. Let $M = \{i \in I : v_i > 0\}$ and $N = \{i \in I : v_i = 0\}$. Obviously, $M \cup N = I$. Let g_M and g_N be the subvectors of g_I corresponding to the index sets M and N, respectively.

Theorem 2.4. *Let $\bar{x} \in X^0 \subseteq \boldsymbol{R}^n$, where X^0 is a locally arcwise connected set. We assume that there exist the right differentials at $\bar{x}$ with respect to the same arc $H_{\bar{x},x}$ of f and g. Let $(\bar{x}, v_0, v)$ be a Fritz John point which satisfy conditions* (2.9)-(2.11).

i) *If $v_0 > 0$, let the assumptions of Theorem* 2.2 *hold with*

$$\bar{u} = v_0^{-1} v$$

ii) *If $v_0 = 0$, let $(\bar{x}, 0, v)$ satisfy* (2.12)-(2.14) *and the following hypotheses are satisfied*

a) $g_i, i \in M_1$, *is* $\alpha_i - LQCN(\bar{x})$,

b) $v_{M_2}^T g_{M_2}$ *is* $\beta - LQCN(\bar{x})$, *where* $\{M_1, M_2\}$ *is a partition of* M,

c) $\sum_{i \in M_1} \alpha_i v_i + \beta > 0$.

Then $\bar{x}$ is a global minimum solution to Problem (P).

The proofs will appear in [10].

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