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ON THE CONTINUITY OF THE TRACE

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The notion of trace for an element in an algebraic extension of a field can be extended for (not necessary algebraic) elements of same extensions of a complete valued field (see [2]). However there are two different ex- tensions of such notion, namely, the trace as was defined in [7] and continuous trace defined in [2]. According to Example 4.3 in [2] there results that these two kinds of trace seem to be generally different. In this paper, we try to put in light same fundamental relations between there concepts of trace.

1. PRELIMINARIES

1. Let p be a prime number. As usual (see [A]) denote Q_p the field of *p*-adic numbers and by $|\cdot|$ the p-adic module, normalized such that $|p| = 1/p$. Denote $\overline{Q}_p$ a fixed algebraic closure of Q_p and continue to denote by the same symbol $|\cdot|$ the unique extension of *p*-adic module to $\overline{Q}_p$. Furthermore, denote C_p the completion of $\overline{Q}_p$ with respect to module $|\cdot|$ and denote by the same symbol $|\cdot|$ the canonical extension of *p*-adic module to C_p.

2. Denote $G = Gal(\overline{Q}_p / Q_p)$ and consider on G the Krull topology. It is well know (see for example [APZ1]) that G is canonically isomorphic to $Gal_{cant}(C_p / Q_p)$, the group of all continuous Q_p authomorphism of C_p. We shall assume $G = Gal_{cant}(C_p / Q_p)$. One know that G, endowed with Krull topology, is a compact and totaly disconected group, and so it is equiped with a Haar measure χ, normalized such that $\chi(G) = 1$.

Let $x \in C_p$. Denote $O(x) = \{\sigma(x) \mid \sigma \in G\}$ the orbit of x with respect to G and by $H(x) = \{\sigma \in G \mid \sigma(x) = x\}$.

Then $H(x)$ is a closed subgroup of G and the map

$$\Psi_x : G \to O(x), \quad \sigma \to \sigma(x)$$

identifies $G / H(x)$ (endowed with the quotient topology) with $O(x)$ (endowed to the topology induced by C_p) (see [APZ1]. In such a way $O(x)$ is equiped with quotient Haar measure.

The description of this measure, denoted also by χ is given in [2]. Also in [2] is show how one can consider χ as a " p-adic measure". Although this " p-adic measure" is not a measure in usual sense, some functions $f : O(x) \to C_p$ can be integrated with respect of χ. Then in [APZ2] (see also [7]) is defined the *p*-adic integral:

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$$Tr(x) = \int_{O(x)} t d\chi(t) \tag{1}$$

called the "trace of x". However this integral do not exists always. If (1) is defined as usual, one says that the element x has a trace. For example any $x \in \overline{Q}_p$ has a trace, namely one has:

$$Tr(x) = \frac{1}{\deg(x)} tr_{Q_p(x)/Q_p}(x) \tag{2}$$

where $\deg(x) = \lfloor Q_p[x] : Q_p \rfloor$ and $tr_{Q_p(x)/Q_p}(x)$ means the usual trace i.e. the sum of all conjugates of x over Q_p. In [PVZ] are given the general conditions such that the integral (1) there exists, i.e. x has a trace. Also in [APZ2] are indicate some classes of elements x of C_p such that $Tr(x)$ is defined. For an element $x \in C_p$, denote $\widetilde{Q_p[x]}$ the closure of the polynomial ring $Q_p[x]$ in C_p.

In [APZ1] is proved that $\widetilde{Q_p[x]}$ is the smallest closed subfield of C_p which contains x, and $\widetilde{Q_p[x]} \cap \overline{Q}_p = K_x$ is called the algebraic part of $\widetilde{Q_p[x]}$. One has $\tilde{K}_x = \widetilde{Q_p[x]}$, and for any closed subfield L of C_p, there exists at least an element $x \in L$ such that $L = \widetilde{Q_p[x]}$ (see [1]).

One say that element $x \in C_p$ has a pseudo-trace, if there exists a sequence $S = \{x_n\}_n$ of $\widetilde{Q_p[x]} \cap \overline{Q}_p$ such that $\lim_n x_n = x$ and that the sequence $\{Tr(x_n)\}_n$ of p-adic numbers is convergent. Then one denote $Tr\,c_s(x) = \lim_n Tr(x_n)$ the pseudo-trace of x with respect to the sequence S. Generally it is possible that x has several pseudo-traces, or it has no any. If all the pseudo-traces of x are coincident, one say that x has a continuous trace and this will be denoted by $Trc(x)$. If the function

$$Tr : \widetilde{Q_p[x]} \cap \overline{Q}_p \to Q_p \tag{3}$$

defined by (2) is continuous, then for any $y \in Q_p[x]$ there exists $Trc(y)$. By above considerations there results the fallowing result:

Proposition 1. Assume that $x \in C_p$ has a pseudo-trace (respectively a continuous trace). Then any element of $Q_p[x]$ has a pseudo-trace (respectively a continuous trace).

Up to now we do not know if any element $x \in C_p$ has a pseudo-trace. By Example 4.3 of [2] there results the existence of elements $x \in C_p$ such that $Tr(x)$ do not exists, but $Trc(x)$ is defined.

In what follow we shall prove that the trace $Tr(x)$ is in fact a pseudo-trace (Theorem 2). Also in Theorem 4 one give a result on the coincidence of $Tr(y)$ and $Tr\,c(y)$ for all elements of $\widetilde{Q_p[x]}$.

2. ANY TRACE IS A PSEUDO-TRACE

Theorem 2 *Let* $x \in C_p$ *be such that* $Tr(x)$ *is defined. Then there exists a sequence* $\{x_n\}_n$ *of elements of* $\widetilde{Q_p[x]} \cap \overline{Q}_p$ *such that* $\lim_n x_n = x$ *and*

$$Tr(x) = \lim_n Tr(x_n) \text{ i.e. } Tr(x) \text{ is a pseudo-trace.}$$

Proof. Let $\{\varepsilon_n\}_n$ be a decreasing sequence of positive real numbers with zero limit. For any $n \geq 1$ denote $H(x,\varepsilon_n) = \{\sigma \in G \mid | x - \sigma(x)| \leq \varepsilon\}$. Then by [7], there results that

$$\lim_n \frac{\varepsilon_n}{|[G : H(x,\varepsilon_n)]|} = 0$$

Now let us fix a natural number *n* and denote

$$K_n = Fix\, H(x,\varepsilon_n) = \{z \in \overline{Q}_p \text{ such that } \sigma(z) = z \text{ for all } \sigma \in H(x,\varepsilon_n)\}$$

Since $H(x) \subseteq H(x,\varepsilon_n)$, then $K_n \subseteq \widetilde{Q_p[x]} \cap \overline{Q}_p$. Let β_n be a sequence in $\widetilde{Q_p[x]} \cap \overline{Q}_p$ such that $\lim_n \beta_n = x$. For m large enough one has: $H(x,\varepsilon_n) = H(\beta_m,\varepsilon_n)$. For such an m, denote $\Delta(\beta_m, K_n) = \max\{|\sigma(\beta_m) - \beta_m| \;\; \sigma \in H(\beta_m,\varepsilon_n)\}$. According to Ax-Sen Theorem (see [5] or [6]), there exists $x_n \in K_n$ such that $|x - x_n| = |\beta_m - x_n| \leq c\Delta(\beta_m, K_m) \leq c\varepsilon_n$, where c is so-called Ax-Sen constant. Moreover one can choice x_n in such way that $K_n = Q_p(x_n)$.By above inequalities and (4) one obtain:

$$\frac{x - x_n}{|[G : H(x,\varepsilon_n)]|} \leq c \frac{\varepsilon_n}{|[G : H(x,\varepsilon_n)]|}$$

Now let

$$A_n = \sum_{i=1}^{r} \alpha_i \, \chi(H(x,\varepsilon_n)/H(x))$$

be a Riemannean sum associated to $Tr(x)$ (see [7]). Here $\chi(H(x,\varepsilon_n)/H(x)) = 1/r$, where $r = [G : H(x,\varepsilon_n)] = \deg(x_n)$. Then one has:

$$|Tr(x_n) - A_n| = \frac{1}{|r|} \left| tr^{(x_n)}_{Q_p(\alpha_n)/Q_p} - \sum_{i=1}^{r} \alpha_i \right| = \frac{1}{|r|} \left| \sum_{i=1}^{r} \sigma_i(x_n) - \sum_{i=1}^{r} \alpha_i \right| \leq \frac{c\varepsilon_n}{|r|} \to 0$$

when $\sigma_1(x_n),\ldots,\sigma_r(x_n)$ are all conjugates of x_n and $\alpha_1,\ldots,\alpha_n$ belongs to correspondings balls $B(\sigma_i(x),\varepsilon_n)$, $1 \leq i \leq r$.

This shows that

$$\lim_n Tr(x_n) = Tr(x), \text{ i.e. } x \text{ has a pseudo-trace.}$$

Corollary 3. Assume that $x \in C_p$ has a trace and the function (3) is continuous. Then $Tr\, c(x)$ is defined and one has:

$$Tr(x) = Tr\, c(x)$$

3. MAIN RESULT

Let $x \in C_p$. According to [APZ1] there exists a sequence $\{M_n(x)\}_{n \geq 0}$ of polynomials of $Q_p[x]$ such that:

i) $\deg M_n(x) = n$, and $|M_n(x)| \leq 1$, $n \geq 0$.

ii) For any $y\in \widetilde{Q_p[x]}$, there exists an unique sequence $\{a_n\}_{n\geq 0}$ of p-adic numbers, such that $\lim_n a_n = 0$, and that:

$$y=\sum_{n\geq 0} a_n M_n(x) \tag{4}$$

Moreover one has: $|y| = \sup_n |a_n M_n(x)|$.

Now let us assume that $Tr(x)$ is defined. Then $Tr(M_n(x))$ is also defined for all $n\geq 0$, and so for $y\in Q_p[x]$ given by (4) one can consider the series:

$$S(y)=\sum_{n\geq 0} a_n Tr(M_n(x)) \tag{5}$$

Generaly one do not say anything on the convergence of such series. However one has the following results:

Theorem 4. Let $x\in C_p$ be such that $Tr(x)$ is defined. The following assertions on equivalent:

i) For any $y\in \widetilde{Q_p[x]}$ the series (5) is convergent.

ii) The function

$$Tr:\widetilde{Q_p[x]}\cap \bar{Q}_p \to Q_p,\quad y\mapsto Tr(y)$$

is continuous.

Then for any $y\in \widetilde{Q_p[x]}$ are defined both $Tr(y)$ and $Tr\,c(y)$ and one has:

$$Tr(y)=Tr\,c(y)=S(y)$$

Proof. i) $\Rightarrow$ ii) Since the series (5) is convergent for all $y\in Q_p[x]$, there results that for a suitable real number $M>0$ one has:

$$|Tr(M_n(x))|\leq M$$

for all $n\geq 0$.

Then the function

$$S:\widetilde{Q_p[x]}\to Q_p,\ \ y\mapsto S(y)$$

is continuous and lineary. Now one assert that for any $y\in \widetilde{Q_p[x]}\cap \bar{Q}_p$ one has: $S(y)=Tr(y)$. For that let us consider the equality (4), and denote $y_m=\sum_{i=0}^{m} a_i M_i(x)$, for all $m\geq 0$. It is clear that $\lim_m y_m = y$. Furthermore, denote by $\sigma_1=e,\ldots,\sigma_r$ a system of representatives for the right casets of G with respect to $H(y)=\{\sigma\in G \mid \sigma(y)=y\}$. Sence $y\in \widetilde{Q_p[x]}$, one has $H(x)\subseteq H(y)$, and let us denote $D_i=\sigma_i(H(y)/H(x))$, $1\leq i\leq m$. It is clear that $\{D_i\}_{1\leq i\leq m}$ give an open (and closed) covering of $O(x)=G/H(x)$ by mutual disjoint subsets. The element y can be viewed as a local constant function $y: O(x)\to C_p$ defined by $y(\sigma(x))=\sigma(y)$. Then for any $\sigma(x)\in D_i$, one has: $y(\sigma(x))=\sigma_i(y)$.

One has

$$Tr(y)=\frac{1}{r}\sum_{i=1}^{r}\sigma_i(y)=\int_{O(x)} y(\sigma(x))d\chi(\sigma(x))=\sum_{i=1}^{r}\int_{D_i} y(\sigma(x))d\chi(\sigma(x))$$

Furthermore, for any $\sigma(x)\in D_i$ one has:

$$\lim_m y_m(\sigma(x))=\sigma_i(y),$$

and this limit is uniform with respect to $\sigma(x)$. Now one show that

$$\lim_m \int_{D_i} y_m(\sigma(x))d\chi(\sigma(x))=\frac{1}{r}\sigma_i(y).$$

For that let $\delta>0$ be a real number, and let $m(\delta)$ be a natural number such that $|y_m(\sigma(x))-\sigma_i(y)|<\delta$, for all $\sigma(x)\in D_i$; whereas $m\geq m(\delta)$. Now let $\{B[\sigma_{ij}(x),\delta]\}_{1\leq j\leq d(m,\delta)}$ be all closed balls of radius δ, any two disjoint which covers $O(x)$. Then by the definition of trace (see [2] or [7]), one has:

$$\left|\frac{1}{rd(m,\delta)}\sum_{j=1}^{d(m,\delta)} y_m(\sigma_{ij}(x))-\frac{1}{r}\sigma_i(x)\right|=\left|\frac{1}{rd(m,\delta)}\left(\sum_{j=1}^{d(m,\delta)}\sigma_{ij}(y_m)-d(m,\delta)\sigma_i(y)\right)\right|\leq$$

$$\leq\frac{\delta}{|rd(m,\delta)|}=\frac{\delta}{|I(x,\delta)|}.$$

where $I(x,\delta)=rd(m,\delta)$ is just the number of closed balls of radius δ, any two disjoint which covers $O(x)$.

Since $Tr(x)$ is defined, then (see[PVZ]) one has:

$$\lim_\delta \frac{\delta}{|I(x,\delta)|}=0\,.$$

Hence if $\varepsilon>0$ is a real number, there exists another real number $\delta(\varepsilon)>0$ such that

$$\frac{\delta}{|I(x,\delta)|}<\varepsilon$$

whereas $\delta\leq\delta(\varepsilon)$.

Then for all $m\geq m(\delta(\varepsilon))$, and all $\delta\leq\delta(\varepsilon)$, one has:

$$\left|\frac{1}{rd(m,\delta)}\sum_{i=1}^{d(m,\delta)} y_m(\sigma_{ij}(x))-\frac{1}{r}\sigma_i(x)\right|<\varepsilon$$

This shows that

$$\lim_m \int_{D_i} y_m(\sigma(x))d\chi(\sigma(x))=\frac{1}{r}\sigma_i(y).$$

But

$$S(y)=\lim_m S(y_m)=\lim_m\left(\sum_{i=1}^{r}\int_{D_i} y_m(\sigma(x)d\chi(\sigma(x)))\right)=\sum_{i=1}^{r}\frac{1}{r}\sigma_i(y)=Tr(y),$$

As claimed.

In conclusion, for all $y\in \widetilde{Q_p[x]}\cap\overline{Q}_p$ one has:

$Tr(y) = S(y) = Tr\,c(y)$ and so by Theorem 2 one has Tr: $\widetilde{Q_p[x]} \cap \bar{Q}_p \to Q_p$ is continuous.

ii)⇒i) By Theorem 2 one has:

$Tr(x) = Tr\,c(x)$, and $Tr(M_n(x)) = Tr\,c(M_n(x))$ for all $n \geq 0$. Also if $y \in \widetilde{Q_p[x]}$ is given by (4), consider

$$y_m = \sum_{i=0}^{m} a_i M_i(x).$$

Then by Theorem 2 one has: $Tr(y_m) = Tr\,c(y_m)$ for all $m \geq 0$. Furthermore, since $y = \lim_m y_m$, then $Tr\,c(y) = \lim_m Tr\,c(y_m)$.

But $Tr\,c(y_m) = \sum_{n=0}^{m} a_n Tr(M_n(x))$, and so $Tr\,c(y) = S(y)$ i.e. the series (5) is convergent.

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