THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 5, Number 2/2004, pp. 000-000

TESTS FOR DISCRIMINATION BETWEEN TWO ALPHA DISTRIBUTIONS

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In this paper hypothesis tests are proposed for discrimination between the populations of two Alpha distributions. This distribution is used for highly skewed data. The tests developed here are uniformly most powerful unbiased and can be used to test various general hypotheses related to this probability distribution which is less known by professional statisticians.

Key words: Alpha distribution, population discrimination, uniformly most powerful unbiased tests.

1. THE ALPHA DISTRIBUTION

Let α and β be positive real numbers. The function $\tilde{\rho}(\cdot;\alpha,\beta):(0,\infty)\to(0,\infty)$ defined by

$$\tilde{\rho}(x;\alpha,\beta)=\frac{\beta}{\Phi(\alpha)\sqrt{2\pi x^2}}\exp\left[-\frac{1}{2}\left(\frac{\beta}{x}-\alpha\right)^2\right] \tag{1}$$

where $\Phi(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha}e^{-\frac{t^2}{2}}dt$ is a probability density function with respect to the Lebesgue measure restricted to $(0,\infty)$.

The probability measure defined over the domain $(0,\infty)$ having the probability density function $\tilde{\rho}(\cdot;\alpha,\beta)$ is called the Alpha distribution. This distribution has two parameters α and β and it is denoted in the following by $A_{\alpha,\beta}$. It was introduced by Drujinin in 1967 [5] and it has been investigated in a series of studies [2], [4], [7], [10] and [14] showing how it can be applied to highly skewed data observed in various industrial processes. The Alpha distribution is less known [8], [9] and as such inferential tools are less developed. The aim of this paper is to fill that gap in the literature and provide uniformly most powerful unbiased tests for discriminating between series of data coming from different populations. The results presented here may have potential applications in reliability modelling and applied statistics for control of industrial processes as already exemplified in [1], [3], and [12].

The paper is organised as follows. The next section briefly introduces the main elements following from the classical exponential family set-up. Section 3 contains the main results of the paper concerning tests of the differences between the parameters of the same type when the other parameters are unknown. The last section summarises the conclusions.

2. STATISTICAL MODELLING

Since $\tilde{\rho}(x;\alpha,\beta)=c(\alpha,\beta)h(x)e^{<Q(\alpha,\beta),S(x)>}$ where $c:(0,\infty)\times(0,\infty)\to(0,\infty)$, $c(\alpha,\beta)=\frac{\beta}{\Phi(\alpha)\sqrt{2\pi}}$, $h:(0,\infty)\to(0,\infty)$, $h(x)=\frac{1}{x^2}$, a $\left(B_{(0,\infty)},B_{(0,\infty)}\right)$- measurable function,

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$Q:(0,\infty)\times(0,\infty)\to R^2$, $Q(\alpha,\beta)=\left(-\frac{\beta^2}{2},\alpha\beta\right)^t$ and $S:(0,\infty)\to R^2$, $S(x)=\left(\frac{1}{x^2},\frac{1}{x}\right)^t$, a $\left(B_{(0,\infty)},B_{R^2}\right)$-measurable function the statistical model

$$\left((0,\infty),B_{(0,\infty)},\{A_{\alpha,\beta}\mid\alpha,\beta>0\}\right) \tag{2}$$

is of exponential type ([6]) and the statistic S is sufficient for inference on the unknown parameters.

Following [13] it is possible to choose a σ-finite measure ν on $\left((0,\infty),B_{(0,\infty)}\right)$ that dominates the statistical model (2). The probability density function of the probability distribution $A_{\alpha,\beta}$ with respect to ν is

$$\rho(x;\alpha,\beta)=c(\alpha,\beta)e^{<Q(\alpha,\beta),S(x)>} \tag{3}$$

for all positive x.
The likelihood function associated to the statistical model represented by (2) is

$$L_n(x^{(n)};\alpha,\beta)=c^n(\alpha,\beta)\exp\left[\alpha\beta\sum_{i=1}^{n}\frac{1}{x_i}-\frac{\beta^2}{2}\sum_{i=1}^{n}\frac{1}{x_i^2}\right] \tag{4}$$

for any $x^{(n)}=(x_1,...,x_n)^t\in(0,\infty)^n$.

3. TESTS FOR DISCRIMINATING BETWEEN TWO ALPHA POPULATIONS

The novelty of this paper consists in testing the discrepancy between two distributions from the Alpha family. Consider the statistical model

$$\otimes_{i=1}^{2}\left((0,\infty),B_{(0,\infty)};\{A_{\alpha_i,\beta_i}\mid\alpha_i,\beta_i>0\}\right)^{n_i} \tag{5}$$

that is dominated by the measure $\nu^{n_1}\otimes\nu^{n_2}$, where $n_1,n_2\in N^*$ are given.

The probability density of the probability distribution $(A_{\alpha_1,\beta_1})^{n_1}\otimes(A_{\alpha_2,\beta_2})^{n_2}$ with respect to $\nu^{n_1}\otimes\nu^{n_2}$ is

$$\begin{aligned}&L_{n_1,n_2}(x^{(n_1)},y^{(n_2)};\alpha_1,\beta_1,\alpha_2,\beta_2)=\\&=\kappa(\alpha_1,\beta_1,\alpha_2,\beta_2)\exp\left[\alpha_1\beta_1\sum_{i=1}^{n_1}\frac{1}{x_i}+\alpha_2\beta_2\sum_{i=1}^{n_2}\frac{1}{y_i}-\frac{\beta_1^2}{2}\sum_{i=1}^{n_1}\frac{1}{x_i^2}-\frac{\beta_2^2}{2}\sum_{i=1}^{n_2}\frac{1}{y_i^2}\right]\end{aligned} \tag{6}$$

for all positive x's and y's, with $\kappa(\alpha_1,\beta_1,\alpha_2,\beta_2)=c^{n_1}(\alpha_1,\beta_1)c^{n_2}(\alpha_2,\beta_2)$.

Here we consider first the tests for the comparison of parameters β_1,β_2 when the other parameters α_1,α_2 are *unknown*. It is worth pointing out that the research cited above on the statistical inference for the Alpha distribution assumes that $\alpha>3$, motivated mainly by the values used in empirical studies. While this greatly simplifies the inferential process due to the fact that $\Phi(\alpha)\cong 1$ it is an unnecessary restriction.

The function in (6) can be rewritten equivalently as

$$\begin{aligned}&\tilde{L}_{n_1,n_2}(x^{(n_1)},y^{(n_2)};\theta,w_1,w_2,w_3)=\\&=c^*(\theta,w_1,w_2,w_3)\exp[\theta U(x^{(n_1)},y^{(n_2)})+w_1T_1(x^{(n_1)},y^{(n_2)})+w_2T_2(x^{(n_1)},y^{(n_2)})+w_3T_3(x^{(n_1)},y^{(n_2)})]\end{aligned} \tag{7}$$

where

$$\theta=-\frac{\beta_1^2}{2}+\lambda_0\frac{\beta_2^2}{2},\ w_1=-\frac{\beta_2^2}{2},\ w_2=n_1\alpha_1\beta_1,\ w_3=n_2\alpha_2\beta_2$$

$$U(x^{(n_1)}, y^{(n_2)}) = \sum_{i=1}^{n_1} \frac{1}{x_i^2}, \quad T_1(x^{(n_1)}, y^{(n_2)}) = \sum_{i=1}^{n_2} \frac{1}{y_i^2} + \lambda_0 \sum_{i=1}^{n_1} \frac{1}{x_i^2}$$

$$T_2(x^{(n_1)}, y^{(n_2)}) = \frac{1}{n_1} \sum_{i=1}^{n_1} \frac{1}{x_i}, T_3(x^{(n_1)}, y^{(n_2)}) = \frac{1}{n_2} \sum_{i=1}^{n_2} \frac{1}{y_i}$$

and where $\lambda_0 > 0$ is known.

Hence, the statistic $(U, T_1, T_2, T_3)^t$ is sufficient for the set of parameters $(\theta, w_1, w_2, w_3)^t$. In the following $s(m, n; \varepsilon)$ denotes the quantile of order ε for the Fisher-Snedecor distribution $S_{m,n}$ with *m* and *n* degrees of freedom and $F_{m,n}$ denotes the cumulative distribution and $\rho_{m,n}$ denotes the probability distribution function of the same distribution $S_{m,n}$.

Consider now the statistic

$$V(x^{(n_1)}, y^{(n_2)}) = \frac{\lambda_0(n_2 - 1)}{n_1 - 1} \frac{\sum_{i=1}^{n_1} \left(\frac{1}{x_i} - \frac{1}{n_1} \sum_{j=1}^{n_1} \frac{1}{x_j} \right)^2}{\sum_{i=1}^{n_2} \left(\frac{1}{y_i} - \frac{1}{n_2} \sum_{j=1}^{n_2} \frac{1}{y_j} \right)^2}$$

This statistic can be linked to the maximum likelihood estimator developed in [14] for the parameter α. This statistic has an important property. If the null hypothesis cannot be rejected then the statistic *V* has an *F* distribution which does not depend on the parameters of the Alpha distribution.

Theorem 1. Let $\lambda_0 > 0$ and $\varepsilon \in (0,1)$ be given. Then

i. The test $\varphi_1 = 1_{(V > s(n_1 - 1, n_2 - 1; 1 - \varepsilon)}$, is uniformly most powerful unbiased at the level of significance ε for testing the null hypothesis $H_0^{(1)} : \beta_1^2 \le \lambda_0 \beta_2^2$ versus the alternative $H_1^{(1)} : \beta_1^2 > \lambda_0 \beta_2^2$.

ii. The test $\varphi_2 = 1_{(V < s(n_1 - 1, n_2 - 1; 1 - \varepsilon)}$, is uniformly most powerful unbiased at the level of significance ε for testing the null hypothesis $H_0^{(2)} : \beta_1^2 \ge \lambda_0 \beta_2^2$ versus the alternative $H_1^{(2)} : \beta_1^2 < \lambda_0 \beta_2^2$.

iii. The test $\varphi_3 = 1_{(V^* < c_1) \cup (V^* > c_2)}$, where

$$V^*(x^{(n_1)}, y^{(n_2)}) = \frac{\lambda_0 \sum_{i=1}^{n_1} \left(\frac{1}{x_i} - \frac{1}{n_1} \sum_{j=1}^{n_1} \frac{1}{x_j} \right)^2}{\lambda_0 \sum_{i=1}^{n_1} \left(\frac{1}{x_i} - \frac{1}{n_1} \sum_{j=1}^{n_1} \frac{1}{x_j} \right)^2 + \sum_{i=1}^{n_2} \left(\frac{1}{y_i} - \frac{1}{n_2} \sum_{j=1}^{n_2} \frac{1}{y_j} \right)^2}, \tag{8}$$

is uniformly most powerful unbiased at the level of significance ε for testing the null hypothesis $H_0^{(3)} : \beta_1^2 = \lambda_0 \beta_2^2$ versus the alternative $H_1^{(3)} : \beta_1^2 \ne \lambda_0 \beta_2^2$.. The scalars c_1, c_2 satisfy the conditions

$$\int_{c_1}^{c_2} \rho_{\frac{n_1 - 1}{2}, \frac{n_2 - 1}{2}}(x)dx = 1 - \varepsilon = \int_{c_1}^{c_2} \rho_{\frac{n_1 + 1}{2}, \frac{n_2 - 1}{2}}(x)dx \tag{9}$$

Proof: i. Since $V = \frac{\lambda_0(n_2 - 1)}{n_1 - 1} \frac{U - T_2^2}{T_1 - \lambda_0 U - n_2 T_3^2}$ we conclude that this statistic is a non-decreasing function of *U*. In addition, when $\beta_1^2 = \lambda_0 \beta_2^2$, the statistic *V* can be rewritten as

$$V(x^{(n_1)}, y^{(n_2)}) = \frac{\dfrac{\beta_1^2}{n_1 - 1}\displaystyle\sum_{i=1}^{n_1}\left(\frac{1}{x_i} - \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{x_j}\right)^2}{\dfrac{\beta_2^2}{n_2 - 1}\displaystyle\sum_{i=1}^{n_2}\left(\frac{1}{y_i} - \frac{1}{n_2}\sum_{j=1}^{n_2}\frac{1}{y_j}\right)^2}$$

The variable at the numerator has a $\chi^2(n_1 - 1)$ distribution while the variable at the denominator has a $\chi^2(n_2 - 1)$ distribution and moreover, the two variables are independent statistics with respect to $(A_{\alpha_1,\beta_1})^{n_1} \otimes (A_{\alpha_2,\beta_2})^{n_2}$. Thus, when $\beta_1^2 = \lambda_0\beta_2^2$, it is true that $V \sim S_{n_1-1,n_2-1}$, that is

$$[(A_{\alpha_1,\beta_1})^{n_1} \otimes (A_{\alpha_2,\beta_2})^{n_2}] \circ V^{-1} = S_{n_1-1,n_2-1} \tag{10}$$

This means that the statistic V is a free parameter statistic over the domain $\Omega_0 = \{(\alpha_1,\beta_1,\alpha_2,\beta_2) \mid \alpha_1,\beta_1,\alpha_2,\beta_2 > 0, \beta_1^2 = \lambda_0\beta_2^2\}$. Then, using theorem 1, chapter 5, from [11] it follows that the test given by the critical region $C_1 = \{(x_1,...,x_{n_1}, y_1,..., y_{n_2}) \mid V(x_1,...,x_{n_1}, y_1,..., y_{n_2}) > c\}$ where c is determined, under $\beta_1^2 = \lambda_0\beta_2^2$, from the condition

$$A_{\alpha_1,\beta_1}^{n_1} \otimes A_{\alpha_2,\beta_2}^{n_2}(C_1) = \varepsilon \tag{11}$$

is uniformly most powerful unbiased test for testing the null hypothesis $\widetilde{H}_0^{(1)} : \theta \le 0$ (which is equivalent to the null hypothesis $H_0^{(1)} : \beta_1^2 \le \lambda_0\beta_2^2$) versus the alternative $\widetilde{H}_1^{(1)} : \theta > 0$ at the level of significance ε. The proof of i. is finalised if we show that $c = s(n_1 - 1, n_2 - 1; 1 - \varepsilon)$. From (11) we get

$$\varepsilon = [A_{\alpha_1,\beta_1}^{n_1} \otimes A_{\alpha_2,\beta_2}^{n_2}(C_1) = 1 - F_{(n_1-1),(n_2-1)}(c)$$

and thus $c = s(n_1 - 1, n_2 - 1; 1 - \varepsilon)$. The proof for ii. is similar with i.

iii. It is easy to see that the statistic W can be rewritten as $W = \dfrac{\lambda_0 U - n_1\lambda_0 T_2^2}{T_1 - n_1\lambda_0 T_2^2 - n_2 T_3^2}$ and since the numerator is positive it follows then that W as a function of U is linear and increasing.

When $\beta_1^2 = \lambda_0\beta_2^2$ we get that

$$W(x^{(n_1)}, y^{(n_2)}) = \frac{\beta_1^2\displaystyle\sum_{i=1}^{n_1}\left(\frac{1}{x_i} - \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{x_j}\right)^2}{\beta_1^2\displaystyle\sum_{i=1}^{n_1}\left(\frac{1}{x_i} - \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{x_j}\right)^2 + \beta_2^2\sum_{i=1}^{n_2}\left(\frac{1}{y_i} - \frac{1}{n_2}\sum_{j=1}^{n_2}\frac{1}{y_j}\right)^2}.$$

It can be noticed that the statistics W_1, W_2 defined by $W_1(x^{(n_1)}, y^{(n_2)}) = \beta_1^2\displaystyle\sum_{i=1}^{n_1}\left(\frac{1}{x_i} - \frac{1}{n_1}\sum_{j=1}^{n_1}\frac{1}{x_j}\right)^2$ and $W_2(x^{(n_1)}, y^{(n_2)}) = \beta_2^2\displaystyle\sum_{i=1}^{n_2}\left(\frac{1}{y_i} - \frac{1}{n_2}\sum_{j=1}^{n_2}\frac{1}{y_j}\right)^2$ are independent and have the probability distributions $\chi^2(n_1 - 1), \chi^2(n_2 - 1)$. Thus, when $\beta_1^2 = \lambda_0\beta_2^2$, the statistic W has with respect to the probability measure $(A_{\alpha_1,\beta_1})^{n_1} \otimes (A_{\alpha_2,\beta_2})^{n_2}$, the distribution Beta $b_{\frac{n_1-1}{2},\frac{n_2-1}{2}}$.

Using again theorem 1, chapter 5, from [11], the test characterized by the critical region $C_3 = (W < c_1) \cup (W > c_2)$ where c_1, c_2 are determined from the conditions

$$E_\mu(\varphi_3) = \varepsilon,\ E_\mu(\varphi_3 W) = \varepsilon E_\mu(W) \tag{12}$$

where $\mu = A_{\alpha_1,\beta_1} \otimes A_{\alpha_2,\beta_2}$, is uniformly most powerful unbiased at the level of significance ε for testing the null hypothesis $\widetilde{H}_0^{(3)} : \theta = 0$ (equivalent to the hypothesis $H_0^{(3)}$) versus the alternative $\widetilde{H}_0^{(3)} : \theta \neq 0$ (equivalent to the hypothesis $H_1^{(3)}$). The proof is finished by showing that the condition (12) is equivalent with (9). This is true because

$$E_\mu(W) = \frac{(n_1 - 1)}{n_1 + n_2 - 2}$$

and

$$x\rho_{\frac{n_1-1}{2},\frac{n_2-1}{2}}(x) = \frac{n_1 - 1}{n_1 + n_2 - 1}\rho_{\frac{n_1+1}{2},\frac{n_2-1}{2}}(x)$$

for any 0< x < 1.

Now we consider tests for the parameters α_1, α_2 when the parameters β_1, β_2 are unknown but equal to β.In this case the likelihood function given above in (6) is

$$L_{n_1,n_2}{}^*(x^{(n_1)}, y^{(n_2)}; \alpha_1, \alpha_2, \beta) = \kappa^*(\alpha_1,\alpha_2,\beta)\exp\left[\alpha_1\beta\sum_{i=1}^{n_1}\frac{1}{x_i} + \alpha_2\beta\sum_{i=1}^{n_2}\frac{1}{y_i} - \frac{\beta^2}{2}\left(\sum_{i=1}^{n_1}\frac{1}{x_i^2} + \sum_{i=1}^{n_2}\frac{1}{y_i^2}\right)\right]$$

or in equivalent form as

$$\begin{aligned}&\tilde{L}^*_{n_1,n_2}(x^{(n_1)}, y^{(n_2)}; \theta, w_1, w_2) = \\ &= \tilde{k}^*(\theta, w_1, w_2)\exp[\theta^* U^*(x^{(n_1)}, y^{(n_2)}) + w_1^* T_1^*(x^{(n_1)}, y^{(n_2)}) + w_2^* T_2^*(x^{(n_1)}, y^{(n_2)})] + w_3 T_3(x^{(n_1)}, y^{(n_2)})]\end{aligned}$$

where

$$\theta^* = \frac{(\alpha_1 - \alpha_2)\beta}{\frac{1}{n_1} + \frac{1}{n_2}},\ w_1^* = \frac{n_1\alpha_1 + n_2\alpha_2}{n_1 + n_2}\beta,\ w_2^* = -\frac{\beta^2}{2},$$

$$U^*(x^{(n_1)}, y^{(n_2)}) = \frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{x_i} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{1}{y_i},\ T_1^*(x^{(n_1)}, y^{(n_2)}) = \sum_{i=1}^{n_1}\frac{1}{x_i} + \sum_{i=1}^{n_2}\frac{1}{y_i}$$

$$T_2^*(x^{(n_1)}, y^{(n_2)}) = \sum_{i=1}^{n_1}\frac{1}{x_i^2} + \sum_{i=1}^{n_2}\frac{1}{y_i^2}$$

Hence, the statistic $S^* = (U^*, T_1^*, T_2^*)^t$ is sufficient for the parameter $(\theta^*, w_1^*, w_2^*)^t$. Lets consider the statistic $W^* : (0,\infty)^{n_1} \times (0,\infty)^{n_2} \to R$ defined as

$$W(x^{(n_1)}, y^{(n_2)}) = \sqrt{\frac{n_1 + n_2 - 2}{\frac{1}{n_1} + \frac{1}{n_2}}}\frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{x_i} - \frac{1}{n_2}\sum_{i=1}^{n}\frac{1}{y_i}}{\sqrt{\sum_{i=1}^{n_1}\left(\frac{1}{x_i} - \frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{x_i}\right)^2 + \sum_{i=1}^{n_2}\left(\frac{1}{y_i} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{1}{y_i}\right)^2}} \tag{13}$$

It is easy to see that

$$W^{*} = \sqrt{\frac{n_1+n_2-2}{\frac{1}{n_1}+\frac{1}{n_2}}}\frac{U^{*}}{\sqrt{T_2^{*2}-\frac{1}{n_1+n_2}T_1^{*2}-\frac{n_1 n_2}{n_1+n_2}U^{*2}}}$$

and therefore W^{*} is an increasing function of the statistic U^{*}. In addition

$$W^{*}(x^{(n_1)}, y^{(n_2)}) = \frac{\dfrac{\beta\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{x_i}-\frac{1}{n_2}\sum_{i=1}^{n}\frac{1}{y_i}\right)}{\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}}{\sqrt{\left[\sum_{i=1}^{n_1}(\frac{1}{x_i}-\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{x_i})^2+\sum_{i=1}^{n_2}(\frac{1}{y_i}-\frac{1}{n_2}\sum_{i=1}^{n_2}\frac{1}{y_i})^2\frac{\beta^2}{n_1+n_2-2}\right]}} \tag{14}$$

Since the statistics defined as $\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{x_i}$ and $\frac{1}{n_2}\sum_{i=1}^{n_2}\frac{1}{y_i}$ are independent and have, with respect with the probability measure $(A_{\alpha_1,\beta_1})^{n_1}\otimes(A_{\alpha_1,\beta_2})^{n_2}$, the normal distributions $N\left(\frac{\alpha_1}{\beta},\frac{1}{n_1\beta^2}\right)$, $N\left(\frac{\alpha_2}{\beta},\frac{1}{n_2\beta^2}\right)$ it follows from (14) that the numerator of that expression is distributed $N(0,1)$. Similarly, the statistic from the denominator is distributed $\chi^2(n_1+n_2-1)$ and consequently W^{*} is Student distributed with n_1+n_2-1 degrees of freedom. Hence, the statistic W^{*} is free over the domain $\Omega^{*} = \{(\alpha_1,\beta;\alpha_1,\beta) \mid \alpha_1,\beta>0\}$.

Lets denote by $s_{m;q}, F_m, \rho_m$ the quantile of order *q*, the cumulative distribution function and the density probability function respectively, of the Student distribution with m degrees of freedom. The following theorem is true now.

Theorem 2. Let $\varepsilon\in(0,1)$ be given. Then

i. The test φ_1 given by the critical region $C_1 = \left(W^{*} > s_{n_1+n_2-2;1-\varepsilon}\right)$, is uniformly most powerful unbiased at the level of significance ε for testing the null hypothesis $H_0^{(1)}: \alpha_1 \le \alpha_2$ versus the alternative $H_1^{(1)}: \alpha_1 > \alpha_2$.

ii. The test φ_2 given by the critical region $C_2 = \left(W^{*} < s_{n_1+n_2-2;\varepsilon}\right)$, is uniformly most powerful unbiased at the level of significance ε for testing the null hypothesis $H_0^{(2)}: \alpha_1 \ge \alpha_2$ versus the alternative $H_1^{(2)}: \alpha_1 < \alpha_2$.

iii. The test φ_3 given by the critical region $C_3 = \left(|W|^{*} > s_{n_1+n_2-2;1-\frac{\varepsilon}{2}}\right)$, is uniformly most powerful unbiased at the level of significance ε for testing the null hypothesis $H_0^{(3)}: \alpha_1 = \alpha_2$ versus the alternative $H_1^{(3)}: \alpha_1 \ne \alpha_2$.

Proof: i. Taking into account the properties of the statistic W^{*} and using theorem 1, chapter 5 from [11] it follows that the test $\varphi_1 = 1_{(W^{*}>c)}$ where *c* is determined from the condition

$$E_{\mu^{*}}(\varphi_1) = \varepsilon \tag{15}$$

with $\mu^* = A^{n_1}_{\alpha_1,\beta} \otimes A^{n_2}_{\alpha_1,\beta}$, is uniformly most powerful unbiased test for testing the hypothesis $\widetilde{H}^{(1)}_0 : \theta^* \leq 0$, equivalent to the null hypothesis $H^{(1)}_0 : \alpha_1 \leq \alpha_2$, versus the alternative $\widetilde{H}^{(1)}_1 : \theta^* > 0$.
Moreover, from (15) it follows that

$$\varepsilon = E_{\mu^*}(\varphi_1) = (A^{n_1}_{\alpha_1,\beta} \otimes A^{n_2}_{\alpha_1,\beta})(C_1) = S_{n_1+n_2-2}(c,\infty) = 1 - F_{n_1+n_2-2}(c)$$

and so $c = s_{n_1+n_2-2;1-\varepsilon}$ and thus $\varphi_1 = 1_{(W^* > s_{n_1+n_2-2;1-\varepsilon})}$.
ii. Similar with i.
iii Lets consider the statistic $Z : (0,\infty)^{n_1} \times (0,\infty)^{n_2} \to R$ defined by

$$Z(x^{(n_1)}, y^{(n_2)}) = \frac{\frac{1}{n_1}\sum_{i=1}^{n_1}\frac{1}{x_i} - \frac{1}{n_2}\sum_{i=1}^{n_2}\frac{1}{y_i}}{\sqrt{\sum_{i=1}^{n_1}\frac{1}{x_i^2} + \sum_{i=1}^{n_2}\frac{1}{y_i^2} - \frac{1}{n_1+n_2}\left(\sum_{i=1}^{n_1}\frac{1}{x_i} + \sum_{i=1}^{n_2}\frac{1}{y_i}\right)^2}} \tag{16}$$

This statistic can be rewritten as $Z = \frac{U^*}{\sqrt{T_2^* - \frac{1}{n_1+n_2}T_1^{*2}}}$ which shows that Z is a linear function of U^* and it is also increasing. In addition, denoting $\widetilde{W} = \sqrt{\frac{\frac{1}{n_1}+\frac{1}{n_2}}{n_1+n_2-2}}W^*$, we get that $Z^2 = \frac{\widetilde{W}^2}{1+\frac{n_1 n_2}{n_1+n_2}\widetilde{W}^2}$. Over the domain $\Omega_{00} = \{(\alpha_1,\beta;\alpha_1,\beta) \mid \alpha_1,\beta > 0\}$ the statistic W^* is free and therefore independent of T_1^* and T_2^*. Consequently the statistic Z is also independent of T_1^* and T_2^*, and thus, free over the domain Ω_{00}.

The distribution of the statistic Z, with respect to the probability measure $(A_{\alpha_1,\beta_1})^{n_1} \otimes (A_{\alpha_1,\beta_2})^{n_2}$ is symmetric from the origin. Then there is a pure test φ_3 such that it is the most powerful unbiased test at the level of significance ε for testing the null hypothesis $\widetilde{H}^{(3)}_0 : \theta^* = 0$, which is equivalent to $H^{(3)}_0 : \alpha_1 = \alpha_2$, versus the alternative $\widetilde{H}^{(3)}_0 : \theta^* \neq 0$. This test is characterised by the critical region

$$(| Z |> c) \tag{17}$$

Since |Z| is an increasing function of the argument $\widetilde{W}$ and thus of the statistic W^*, we can replace the critical region given in (17) by the critical region

$$C_3 = (| W^* |> k)$$

where k is calculated from the condition $(A^{n_1}_{\alpha_1,\beta} \otimes A^{n_2}_{\alpha_1,\beta})(C_3) = \varepsilon$. In conclusion we can say that

$$\varepsilon = .(A^{n_1}_{\alpha_1,\beta} \otimes A^{n_2}_{\alpha_1,\beta})(C_3) = S_{n_1+n_2-2}(\{x \in R \| x \mid > k\}) = F_{n_1+n_2-2}(-k) + 1 - F_{n_1+n_2-2}(k) = 2 - 2F_{n_1+n_2-2}(k)$$

which means that $k = s_{n_1+n_2-2;1-\frac{\varepsilon}{2}}$ and therefore $C_3 = \left(| W |^* > s_{n_1+n_2-2;1-\frac{\varepsilon}{2}}\right)$.

4. CONCLUSION

The Alpha distribution is a two-parameter class of distributions that is less known in the literature in spite of proving to be very useful in modelling skewed data. Statistical models based on this relatively unknown distribution may provide a general platform for inference related to many areas of statistics, probability and engineering.

In this paper some uniformly most powerful unbiased tests were proposed to discriminate between two Alpha populations. The tests are developed for each type of parameter while keeping the other unknown. The hypothesis being tested covers the whole spectrum of possibilities.

ACKNOWLEDGEMENTS

We are pleased to thank Acad. Prof. Marius Iosifescu and Prof. Viorel Gh. Voda for helpful comments that improved the clarity of the paper.

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Received February 23, 2004