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AERODYNAMIC ANALYSIS OF AN AIRFOIL IN DARRIEUS MOTION

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Two methods for computing the aerodynamic loads on an airfoil in Darrieus motion are presented. One method is based on Blasius theorem for unsteady flow and the other uses the vortex-impulse concept. For both methods the wake is modeled by using point vortices. A comparison with exact solutions for pitching and plunging motions gives good agreement. Results are shown for a one-bladed Darrieus rotor at a tip speed ratio of three and two chord sizes.

Key words: Aerodynamics, Wind turbines, Free-vortex methods, Unsteady flow, Rotor vortex wake.

1. INTRODUCTION

The blades of a Darrieus rotor follow a circular path with significant cyclic variations in relative velocity and angle of attack. Currently, the determination of loads and performance of Darrieus rotors is accomplished using quasi-steady aerodynamics [1-2]. In these approaches the aerodynamic forces on the airfoil are obtained by means of some lift and drag coefficient taken from static airfoil data. The use of quasi-steady aerodynamics for the determination of forces and moments is questioned. The main aim of the present paper is to calculate unsteady aerodynamic loads on an airfoil in Darrieus motion (Fig. 1) and to compare these loads predictions with the loads obtained from a quasi-steady no wake approach.

The calculation of airfoil forces and moments in general unsteady motion can be performed by analytical methods based on a point-vortex model of the wake. The best procedure would consist of the integration of the pressure equation [3]. However, it can not describe the generation of a vortex wake due to the temporal variation of the airfoil circulation as dictated by Kelvin's theorem. Therefore, the purpose of this paper is to describe a simple analytical model with which accurate values of the normal force, moment and leading edge suction can be obtained for unsteady motions of airfoils. This model is based on the marching-vortex concept, where motion begins from an impulsive start with the subsequent generation of a vortex wake, modeled be a sequence of discrete vortices shed at equal time intervals. Thus, for steady-state motion, the force and moment responses are asymptotically achieved. An application of such a method was given in a previous paper for two-dimensional airfoils undergoing general periodic motions, using both planar and deformable discrete-vortex wakes [4]. However, the bound airfoil vorticity was continuous in order to provide accurate calculations for leading-edge suction. This had the drawback of requiring fairly time-consuming chordwise integration to find the normal force and pitching moment at each time step. Also, it offered no simple extension to finite-span wings.

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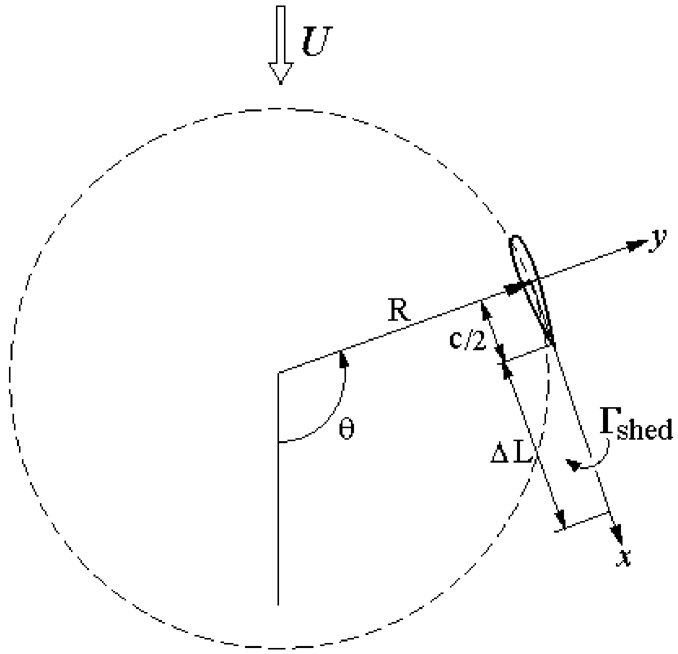


Fig. 1 – Darrieus rotor coordinate system.

2. AERODYNAMIC ANALYSIS

2.1 The unsteady aerodynamics of a pitching airfoil

Consider a right cylinder as shown above in Fig. 1. The coordinate x, y are locked in the cylinder which is rotating with angular velocity $\mathbf{\Omega}$ and translating with velocities U_x and U_y. By transforming the cylinder (actually a flat plate) into a circle, the complex potential may be written using the boundary function plus techniques from the complex theory. Letting the complex velocity $Q = U_x - iU_y$, the boundary function $B(z)$ is given by

$$B(z) = Qz - \overline{Q}\overline{z} - i\Omega z\overline{z}\ . \tag{1}$$

Transforming to the circle plane, $z = f(\varsigma)$ and we obtain

$$B(\varsigma) = Qf(\varsigma) - \overline{Q}\bar{f}(\varsigma) - i\Omega f(\varsigma)\bar{f}(\varsigma). \tag{2}$$

With the transformation $z = \varsigma + \dfrac{r^2}{\varsigma}$, where $r = c/4$, the airfoil of chord c is transformed into a circle of radius $c/4$. Using the above transform we have

$$B(\varsigma) = Q\left(\varsigma + \frac{r^2}{\varsigma}\right) - \overline{Q}\left(\overline{\varsigma} + \frac{r^2}{\overline{\varsigma}}\right) - i\Omega\left(\varsigma + \frac{r^2}{\varsigma}\right)\left(\overline{\varsigma} + \frac{r^2}{\overline{\varsigma}}\right)$$

On the circle $\varsigma\overline{\varsigma} = r^2$ so that the boundary function becomes

$$B(\varsigma) = \left(Q - \overline{Q}\right)\left(\varsigma + \frac{r^2}{\varsigma}\right) - i\Omega\left(\varsigma + \frac{r^2}{\varsigma}\right)^2, \tag{3}$$

which indeed is a function of ς alone. Now the boundary function $B_1(\varsigma)$ contains only the negative powers of ς and is equal to the complex potential of the flow generated by the moving body. Thus

$$B_1(\varsigma)=W(\varsigma)=\left(Q-\overline{Q}\right)\frac{r^2}{\varsigma}-i\Omega\frac{r^4}{\varsigma^2}\,. \tag{4}$$

If there are circulation and wake, we may add a vortex at the origin (clockwise is positive) and a sequence of discrete vortices shed at equal time intervals giving

$$W(\varsigma)=\left(Q-\overline{Q}\right)\frac{r^2}{\varsigma}+\frac{i\Gamma}{2\pi}\ln\varsigma-i\Omega\frac{r^4}{\varsigma^2}+\sum_{n=1}^{N}\frac{i\Gamma_n}{2\pi}\ln(\varsigma-\varsigma_n)-\sum_{n=1}^{N}\frac{i\Gamma_n}{2\pi}\ln\left(\varsigma-\frac{r^2}{\varsigma_n}\right) \tag{5}$$

Here Γ_n and ς_n are the strength and position of the nth shed vortex.

The complex velocity in the circle plane is

$$q\equiv\frac{\mathrm{d}W}{\mathrm{d}\varsigma}$$

and in the cylinder (airfoil) plane is

$$q^*=\frac{dW}{dz}=\frac{dW}{d\varsigma}\frac{1}{dz/d\varsigma}=\frac{dW}{d\varsigma}\frac{1}{1-\dfrac{r^2}{\varsigma^2}}=\frac{q}{1-r^2/\varsigma^2}$$

Thus, noting $Q-\overline{Q}=-2iU_y$,

$$q^*=\left[2iU_y\frac{r^2}{\varsigma^2}+\frac{i\Gamma}{2\pi\varsigma}+2i\Omega\frac{r^4}{\varsigma^3}+\sum_{n=1}^{N}\frac{i\Gamma_n}{2\pi}\left(\frac{1}{\varsigma-\varsigma_n}-\frac{1}{\varsigma-\dfrac{r^2}{\varsigma_n}}\right)\right]\left(1-\frac{r^2}{\varsigma^2}\right)^{-1} \tag{6}$$

The force on the airfoil may be determined from the extended Blasius theorem. Using the formulation developed in [3] the conjugate force is given by

$$\overline{F}\equiv X-iY=\frac{i\rho}{2}\int_C\left(\frac{dW}{dz}\right)^2dz-\rho\Omega\int_C\overline{z}\frac{d\overline{W}}{d\overline{z}}d\overline{z}+i\rho\frac{\partial}{\partial t}\int_C\overline{W}d\overline{z}+i\rho Q\Gamma \tag{7}$$

and the moment about the origin (anticlock wise is positive) is given by

$$M=\mathrm{Re}\left[-\frac{\rho}{2}\int_C z\left(\frac{dW}{dz}\right)^2dz+\rho Q\int_C z\frac{dW}{dz}dz-\rho\frac{\partial}{\partial t}\int_C zWdz\right] \tag{8}$$

where the integrals are evaluated along the airfoil contour. Since the airfoil contour in the ζ plane is a circle, the above integrals are evaluated by means of the residual theorem. Thus the complex force and the moment may be written in terms of circle coordinates as

$$X - iY = \frac{i\rho}{2}\left(2\sum \Gamma_n q_n^* - 2\pi i \sum \frac{\Gamma_n^2 c^2}{8\varsigma_n^3\left(1-\dfrac{c^2}{16\varsigma_n^2}\right)^2}\right) - \Omega\rho\left[-\frac{\pi c^2}{4}U_y + \sum \Gamma_n\left(\bar{\varsigma}_n^* + \frac{c^2}{16\bar{\varsigma}_n}\right)\right] + i\rho\Gamma Q + \\ + i\rho\Omega\frac{\partial}{\partial t}\left[\frac{\pi c^2}{4}U_y - \Gamma\frac{c}{2} - \sum \Gamma_n\left(\bar{\varsigma}_n^* + \frac{c^2}{16\bar{\varsigma}_n}\right)\right] \tag{9}$$

$$M = \mathrm{Re}\left\{-\frac{\rho}{2}\left(2\sum \Gamma_n q_n^* - \frac{i}{2\pi}\sum \frac{\Gamma_n c^2}{8\varsigma_n^3\left(1-\dfrac{c^2}{16\varsigma_n^2}\right)^2}\right)\left(\varsigma_n + \frac{c^2}{16\varsigma_n}\right) + \rho Q\left[-\frac{\pi c^2}{4}U_y + \sum \Gamma_n\left(\varsigma_n^* + \frac{c^2}{16\varsigma_n}\right)\right] + \right. \\ \left. + \rho\frac{\partial}{\partial t}\left[\frac{\pi\Omega c^4}{128} + \frac{\Gamma c^2}{16} + \sum \frac{\Gamma_n}{2}\left(\varsigma_n^{*2} + \left(\frac{c}{4}\right)^4\frac{1}{\varsigma_n^2}\right)\right]\right\} \tag{10}$$

where q_n^* is the complex velocity induced at the nth vortex by all causes in the airfoil plane and ς_n^* is the position of the nth image vortex in the circle plane. The above relations can be restated in dimensionless from as

$$C_F \equiv \frac{X - iY}{\rho U^2 R}, \quad C_M \equiv \frac{M}{\rho U^2 R^2}. \tag{11}$$

where R and U are the proper length and velocity scales respectively. Using the following definitions

- q_{0_n} = the complex velocity at the origin in the airfoil plane induced by the nth vortex;
- $q_{L.E._n}$ = the complex velocity at the leading edge induced by the nth vortex;
- $q_{T.E._n}$ = the complex velocity at the trailing edge induced by the nth vortex,

the equations (9,10) can be written in dimensionless form as

$$C_F = i\sum \frac{\Gamma_n}{U^2 R}\left(q_n^* - q_{0_n} + \frac{q_{T.E._n} + q_{L.E._n}}{2}\right) + \frac{\pi c^2\Omega}{4UR}\frac{U_y}{U} - \frac{\Omega R}{U}\sum \frac{\Gamma_n}{UR^2}\left(\bar{\varsigma}_n^* + \frac{c^2}{16\bar{\varsigma}_n}\right) + i\frac{\Gamma Q}{U^2 R} - \\ - i\frac{\partial}{\partial t}\left[-\frac{\pi c^2}{4UR}\frac{U_y}{U} + \frac{\Gamma c}{2U^2 R} + \frac{1}{U}\sum \frac{\Gamma_n}{UR}\left(\bar{\varsigma}_n^* + \frac{c^2}{16\bar{\varsigma}_n}\right)\right] \tag{12}$$

$$C_M = \mathrm{Re}\left\{-\sum \frac{\Gamma_n}{U^2 R}\left(q_n^* - q_{0_n} + \frac{q_{T.E._n} + q_{L.E._n}}{2}\right)\frac{z_n}{R} + \frac{Q}{U}\left[\frac{\pi}{4}\left(\frac{c}{R}\right)^2\frac{U_y}{U} + \sum \frac{\Gamma_n}{UR^2}\left(\varsigma_n^* + \frac{c^2}{16\varsigma_n}\right)\right] + \right. \\ \left. + \frac{\partial}{\partial t}\left[\frac{\pi}{128}\left(\frac{c}{R}\right)^2\frac{c^2\Omega}{U^2} + \frac{c^2\Gamma}{16U^2R^2} + \sum \frac{\Gamma_n}{2U^2R^2}\left(\varsigma_n^{*2} + \left(\frac{c}{4}\right)^4\frac{1}{\varsigma_n^2}\right)\right]\right\} \tag{13}$$

The time variable in the Bernoulli equation has not been scaled. For the plunging airfoil we take $R = c$ and use $\tau = Ut/c$. For the Darrieus case, Ω is constant and we use $\theta = \Omega t$ as the independent variable.

2.2 Vortex-impulse method

The approach given above was developed by integration of the pressure on the airfoil. An alternate method to calculate the forces is the vortex-impulse method described by Milne-Thomson [3]. The fluid momentum of a body in an infinite fluid can be written in complex form as

$$I = -i\int_C W dz \tag{14}$$

Using equation (5) for the complex potential and performing the integration around a path that is deformed into an infinite radius circle we obtain

$$I = 2\pi i\left[-2r^2 U_y - \sum \frac{\Gamma_n}{2\pi}\left(\varsigma_n - \varsigma_n^*\right)\right] \tag{15}$$

Since the time derivative of the impulse is the force, successive evaluation of I at different positions of the airfoil can be used via numerical differentiation to determine the force on the airfoil. Equation (15) involves only the velocity normal to the airfoil and shed vortices. The force evaluated in this manner can be compared with the pressure integration on the airfoil surface

2.3 Marching-vortex model

The vortices are advanced with each time step Δt and new vortex positions are determined from

$$x_n(t+\Delta t) = x_n(t) + u_n(t)\Delta t \tag{16}$$

where u_n is the x- direction velocity of the nth vortex. A similar expression was used for $y_n(t)$. Each external vortex carries the same circulation through its path in the wake, established at the time it is shed. Circulation is conserved and as a result of the system starting from rest (with zero circulation), the total circulation in the system at any time t remains zero. The total circulation is given by

$$\underbrace{\sum_{n=1}^{N}\Gamma_n}_{\text{free vort.}} - \underbrace{\sum_{n=1}^{N}\Gamma_n^m}_{\text{image vort.}} + \underbrace{\sum_{n=1}^{N}\Gamma_n^c}_{\text{induced center vort.}} + \underbrace{\Gamma_{Q.S.}}_{\text{quasi-st. circ.}} + \underbrace{\Gamma_p}_{\text{pitching circ.}} = 0$$

It may be noted that at any time t

$$\sum_{n=1}^{N}\Gamma_n^c + \Gamma_{Q.S.} + \Gamma_p = 0 \tag{17}$$

since the external and image circulations cancel each other. Consider the system at time $t+\Delta t$ after the vortices and the airfoil have been moved to other new positions in the airfoil plane. The coordinates of each free vortex may be transformed to the circle plane where the induced center circulation corresponding to a new shed vortex may be calculated from

$$\Gamma_{N+1}^c = -\Gamma_{Q.S.} - \Gamma_p - \sum_{n=1}^{N}\Gamma_n^c \tag{18}$$

where

$$\Gamma_{Q.S.} = -\pi c U_y, \Gamma_p = \frac{-\pi\Omega c^2}{4}, \Gamma_n^c = \Gamma_n \frac{\rho_n^2 - 1}{\rho_n^2 - 2\rho_n\cos\phi_n + 1}$$

and $\varsigma_n = \rho_n e^{i\phi_n}$ is the position of the nth vortex in the circle plane.

The relation between the strength of the new shed vortex and the center induced circulation is determined from Kutta-Joukowski condition. If this vortex is assumed to be located along the x-axis, the relation can be reduced to a simple expression in the airfoil plane.

$$\Gamma_{n+1}^{c}=\Gamma_{n+1}\sqrt{\frac{2x_{n+1}+c}{2x_{n+1}-c}} \tag{19}$$

In reality the vorticity is springing continuously in the form of a sheet. A first approximation to the problem is that the sheet is of constant strength ($=\Gamma_{n+1}/\Delta L$). Thus, one obtains for a sheet of length ΔL

$$\Gamma_{n+1}^{c}=\frac{\Gamma_{n+1}}{\Delta L}\int_{c/2}^{c/2+\Delta L}\sqrt{\frac{2x+c}{2x-c}}dx\,, \tag{20}$$

or

$$\frac{\Gamma_{n+1}}{\Gamma_{n+1}^{c}}=\frac{\Delta L/c}{\sqrt{\frac{\Delta L}{c}\left(1+\frac{\Delta L}{c}\right)}+\ln\left(\sqrt{\frac{\Delta L}{c}}+\sqrt{1+\frac{\Delta L}{c}}\right)} \tag{21}$$

Now, if we know the value of Γ_{n+1}^{c} (eq. 18)) and the length of the sheet ΔL, we can determine the strength of the shed vortex, Γ_{n+1}. The length of the sheet ΔL is determined from equation (16) using the trailing edge complex velocity at time t in the airfoil plane

$$q^{*}_{T.Q.}=U_x-i\frac{c}{2}\Omega-\frac{2}{\pi c}\sum_{n=1}^{N}\Gamma_n\frac{\rho_n\left(\rho_n^2-1\right)\sin\phi_n}{\left(\rho_n^2-2\rho_n\cos\phi_n+1\right)^2} \tag{22}$$

and then the equation (19) (the Kutta-Joukowski condition for a discrete vortex) yields the location of the last shed vortex

$$\frac{x_{n+1}}{c/2}=\frac{\left(\Gamma_{n+1}^{c}\right)^2+\left(\Gamma_{n+1}\right)^2}{\left(\Gamma_{n+1}^{c}\right)^2-\left(\Gamma_{n+1}\right)^2}\,. \tag{23}$$

This process is then repeated for the next motion step.

3. Numerical Results

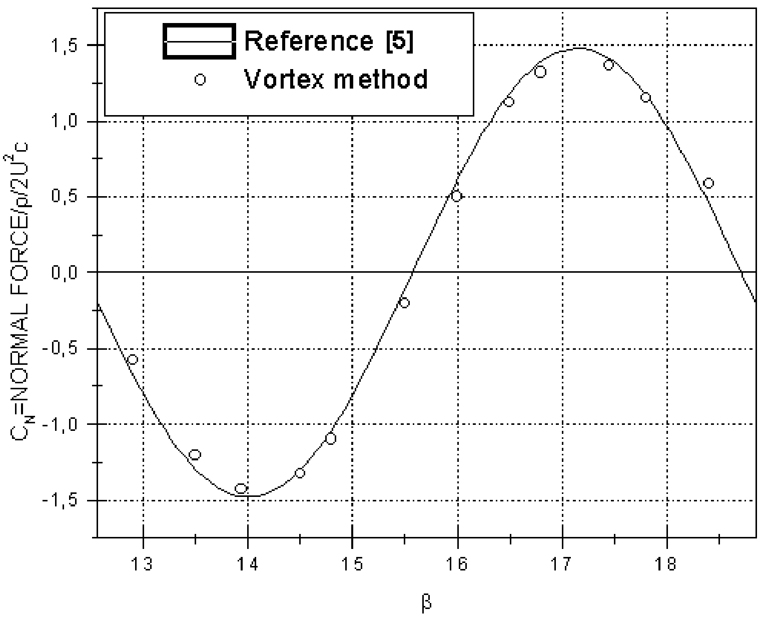
In order to check the methods, the forces on an airfoil undergoing oscillatory midchord plunging and pitching was calculated and compared with the solutions from Garrick [5]. Fig 2 shows the results for $k\left(=\omega c/2U\right)=0.5$ and one cycle of the motion with pitching $\theta=0.1\sin(\omega t)$ and plunging $h=0.5c\sin(\omega t+\pi/2)$. Agreement between the numerical approach and the Garrick solution was considered to be good.

Next, a one-bladed Darrieus rotor was analyzed in order to examine the forces and moments on blade. Figures 3 and 4 show the normal and tangential force coefficients for the five revolution of a single blade at a tip speed ratio $\left(R\Omega/U\right)$ of three. Two values of the chord to radius ratio were used: 0.200 and 0.063. Three curves are shown for each case. The solid line shows the contribution of all terms in equation (12), the dot and dash curve shows the value of the two term approximated coefficient

$$C_{F,2}=\underbrace{i\frac{\Gamma_C}{UR}\left(\frac{Q_{rel}}{U}\right)}_{\substack{\text{Kutta-Joukowski}\\ \text{lift term}}}-\underbrace{\frac{\pi}{4}\left(\frac{c}{R}\right)^2\frac{R\Omega}{U}e^{-i\theta}}_{\substack{\text{pitching circulation}\\ \text{and apparent mass term}}}\,, \tag{24}$$

where $\Gamma_c=-\sum\Gamma_n$ and $Q_{rel}=Q+\sum q_n^{*c}$ is the local complex velocity relative to the blade origin, and the dashed line shows the quasi-steady approximation (that would arise if there were no wake)

$$C_{F,Q.S.} = i\frac{\Gamma_{Q.S.}}{UR}\left(\frac{Q}{U}\right) - \frac{\pi}{4}\left(\frac{c}{R}\right)^2 \frac{R\Omega}{U} e^{-i\theta} . \tag{25}$$



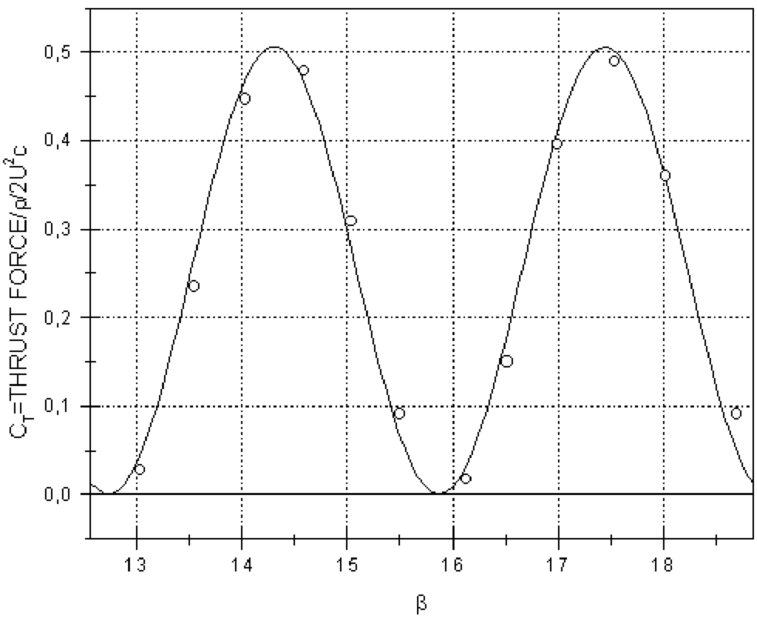


Fig. 2 – Airfoil lift and thrust for pitching and plunging motion, $k = 0.5$

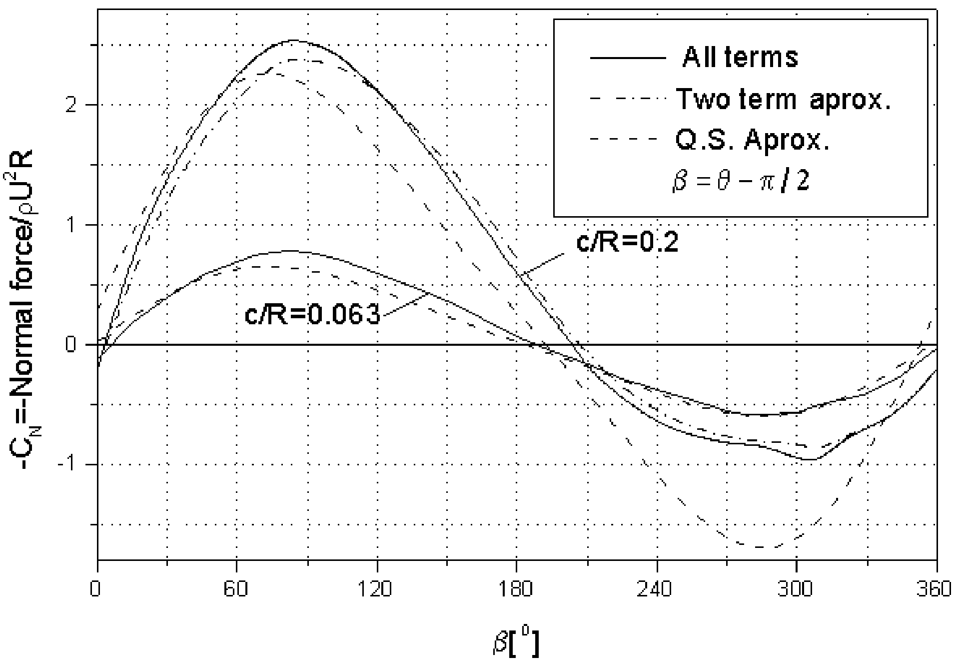


Fig. 3 – Normal force variation at tip speed ratio of 3.

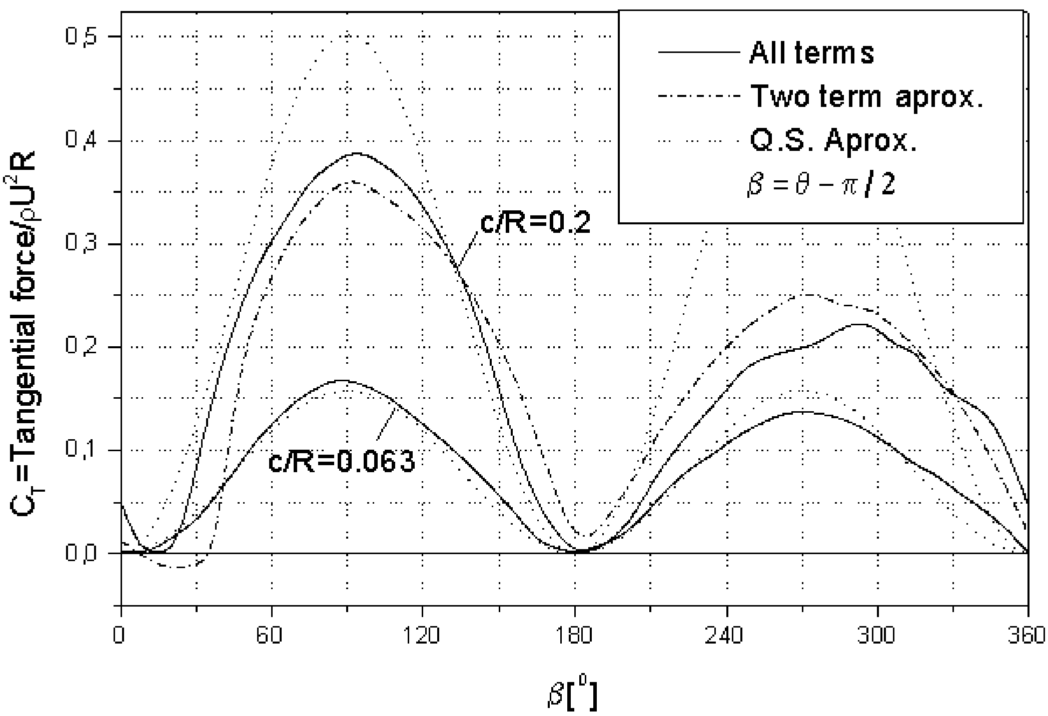


Fig. 4 – Tangential force variation at tip speed ratio of 3.

The second term in the approximate relations (24) and (25) does not net work, but does contribute to the loads. The contribution of the moment to the power due mainly to the term

$$C_M \cong \frac{\pi}{4}\left(\frac{c}{R}\right)^2\left(\frac{R\Omega}{U} + \sin\theta\right)\cos\theta\,, \tag{26}$$

is negligible mall, even for the larger chord.

The unsteady coefficients of tangential force and power, divided by the corresponding quasisteady quantities are plotted versus averaged reduced frequency $k = c/2R$ in fig. 5. According to this, the influence of unsteadiness is surprisingly large, especially on the predicted average power.

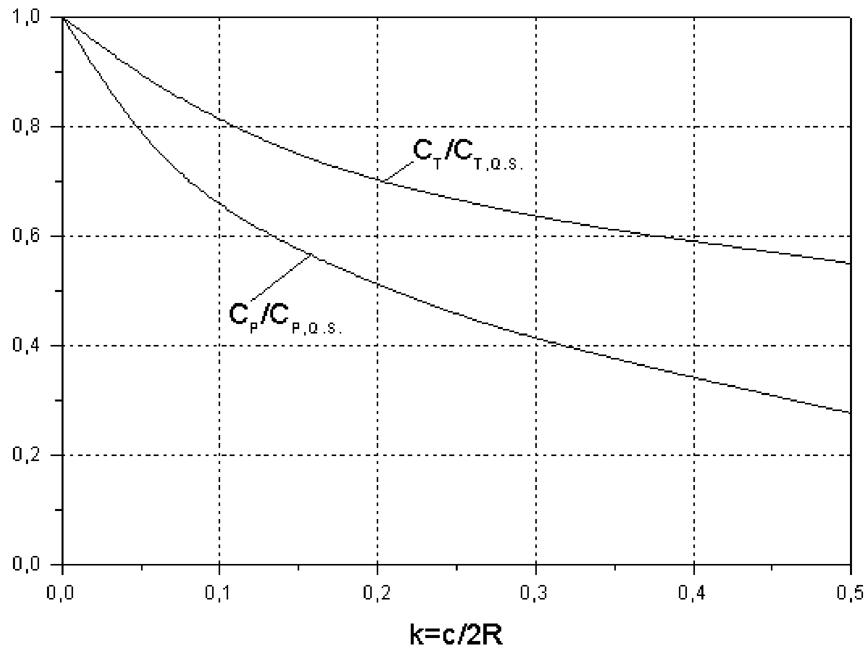


Fig. 5 – Calculated unsteady coefficients of thrust and power, divided by the corresponding quantities, versus averaged reduced frequency *k*.

CONCLUSIONS

A free-vortex analysis of an idealized Darrieus rotor blade has been developed. The wake model is based on the marching-vortex concept, where motion begins from an impulsive start with the subsequent generation of a vortex wake, modeled by a sequence of discrete vortices shed at equal time intervals. At each time step a new shed vortex appears and its strength and location are rigorously determined. The force on the airfoil has been determined by two methods, integration of the pressure over the plate and from the impulse of the wake vortices. Both methods have yielded the same numerical results. For airfoils undergoing oscillatory midchord plunging and pitching, comparisons have been done with the solutions from Garrick [5]. Perhaps the most interesting result is the very substantial loss in rotor power predicted by aerodynamic theory in the case of high chord to radius ratios (fig. 5). However, the numerical results indicate that the forces and moment on a Drrieus rotor blade may be adequately approximated by quasy-steady type relationships if accurate determination of local velocity and circulation is used.

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