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# SUMMATION FORMULAE FOR GENOCCHI, BERNOULLI, AND EULER NUMBERS AND POLYNOMIALS

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**Abstract**. By means of two known combinatorial identities involving polynomials, recurrence relations, and the telescoping technique, we obtain new explicit expressions for Genocchi, Bernoulli and Euler numbers/polynomials, along with some other interesting transformation formulas and explicit double sum combinatorial identities.

***Keywords:*** Genocchi numbers and polynomials, Bernoulli numbers and polynomials, Euler numbers and polynomials.

*Mathematics Subject Classification (MSC2020):* Primary 11B68, Secondary 11B83

## 1. INTRODUCTION AND MOTIVATION

The classical Bernoulli polynomials and numbers $B_n(x)$ and $B_n$, Euler polynomials and numbers $E_n(x)$ and $E_n$ as well as Genocchi polynomials/numbers $G_n(x)$ and $G_n$ are defined, respectively, by the following exponential generating functions [1,4,7]

$$\sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!} = \frac{z}{e^z-1}e^{xz}, \quad \sum_{k=0}^{\infty} B_k\frac{z^k}{k!} = \frac{z}{e^z-1}, \quad \text{with } |z|<2\pi; \tag{1}$$

$$\sum_{k=0}^{\infty} E_k(x)\frac{z^k}{k!} = \frac{2}{e^z+1}e^{xz}, \quad \sum_{k=0}^{\infty} E_k\frac{z^k}{k!} = \frac{2e^z}{e^{2z}+1}, \quad \text{with } |z|<\pi; \tag{2}$$

$$\sum_{k=0}^{\infty} G_k(x)\frac{z^k}{k!} = \frac{2z}{e^z+1}e^{xz}, \quad \sum_{k=0}^{\infty} G_k\frac{z^k}{k!} = \frac{2z}{e^z+1}, \quad \text{with } |z|<\pi. \tag{3}$$

They are not only have a wide range of applications in mathematics but also in other fields, such as quantum physics (quantum groups) [1]. It is obviously that $G_k \equiv G_k(0)$, $B_k \equiv B_k(0)$ and $E_k \equiv 2^k E_k(\frac{1}{2})$. Based on the generating functions of the three polynomials, we have the binomial relations below:

$$\mathscr{W}_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{W}_k(x)y^{n-k}, \text{ with } \mathscr{W}_k(x) \in \{B_k(x), E_k(x), G_k(x)\}$$

whose particular cases yield

$$G_n(x)=\sum_{k=0}^{n}G_k\binom{n}{k}x^{n-k},\ B_n(x)=\sum_{k=0}^{n}B_k\binom{n}{k}x^{n-k},\ \text{and}\ \ E_n(x)=\sum_{k=0}^{n}\frac{E_k}{2^k}\binom{n}{k}\big(x-\frac{1}{2}\big)^{n-k}.$$

Differentiating of the generating functions with respect to $x$, we have

$$G_n'(x)=nG_{n-1}(x),\ B_n'(x)=nB_{n-1}(x)\ \text{and}\ E_n'(x)=nE_{n-1}(x). \tag{4}$$

In view of $G_0(x)=0$ and $B_0(x)=E_0(x)=1$, we have

$$\deg G_n(x)=n-1,\ \deg B_n(x)=\deg E_n(x)=n.$$

Throughout the paper, the symbol $\mathbb{N}_0$ denotes the set of natural numbers, i.e., $\mathbb{N}_0=\{0,1,2,\cdots\}$. Meanwhile, we define $\mathbb{N}$ as the set of positive integers, i.e., $\mathbb{N}=\mathbb{N}_0\setminus\{0\}=\{1,2,\cdots\}$.

The Bernoulli and Euler polynomials can be expressed by the following double sum

$$B_n(x)=\sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(x+j)^n,\quad n\in\mathbb{N}_0,$$
$$E_n(x)=\frac{1}{2^n}\sum_{k=1}^{n+1}\sum_{j=0}^{k-1}(-1)^j\binom{n+1}{k}(x+j)^n,\quad n\in\mathbb{N}_0,$$

respectively. Bernoulli and Euler numbers have the explicit formulas [5]

$$B_n=\sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n,\quad n\in\mathbb{N},$$
$$E_n=\frac{1}{2^n}\sum_{k=1}^{n+1}\binom{n+1}{k}\sum_{j=0}^{k-1}(-1)^j(2j+1)^n,\quad n\in\mathbb{N}_0.$$

Also by the relations $G_n(x)=nE_{n-1}(x)$ and $G_{2n}=2(1-2^{2n}B_{2n})$, we can obtain the explicit formulas of Genocchi polynomials and numbers.

In the next section, we shall establish, to the best of my knowledge, new double-sum expressions for these polynomials and numbers. In Section 3, we shall derive several interesting summation formulae concerning these polynomials and numbers.

## 2. NEW DOUBLE SUM EXPRESSIONS FOR GENOCCHI, BERNOULLI AND EULER NUMBERS AND POLYNOMIALS

In this section, we shall use the following Lemma 1 [6, §7] (also see [3, P 82]) and Lemma 2 [3, Z.7] to derive new explicit expressions of Genocchi, Bernoulli and Euler numbers and polynomials.

*Lemma* 1. For any polynomial $f(x)$ in $x$ of degree $\leq n$, there holds the identity

$$f(x+y)=y\binom{y+n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k}. \tag{5}$$

*Lemma* 2. For any polynomial $f(x)$ in $x$ of degree $\leq n$, there holds the identity

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{f(x-k)}{k}=f(x)H_n-f'(x), \tag{6}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the Harmonic numbers.

### 2.1. New double sum expression of Genocchi polynomials and numbers

Regarding Genocchi polynomials, their new double-sum expressions are presented as follows.

*Theorem* 3. For $m \le n \in \mathbb{N}$, the following relation holds

$$G_m(x) = \frac{m}{2^n}\Big\{x^{m-1} - \sum_{k=1}^{n}\binom{n+1}{k+1}\sum_{i=1}^{k}(-1)^i(x-i)^{m-1}\Big\}. \tag{7}$$

*Proof.* By means of the recurrence relation $G_k(x) + G_k(x-1) = 2k(x-1)^{k-1}$, we can obtain that

$$(-1)^k G_m(x-k) = 2m\sum_{i=1}^{k}(-1)^i(x-i)^{m-1} + G_m(x). \tag{8}$$

For $m \le n$, by using Lemma 1, we have

$$G_m(x+1) = \sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}G_m(x-k) = (2^{n+1}-1)G_m(x) + 2m\sum_{k=0}^{n}\binom{n+1}{k+1}\sum_{i=1}^{k}(-1)^i(x-i)^{m-1}.$$

Substituting $G_m(x+1) = 2mx^{m-1} - G_m(x)$ into the formula above and performing an uncomplicated calculation enables us to obtain the desired result. □

For particular case $n = m$, we have the following corollary.

*Corollary* 1. For $m \in \mathbb{N}$, the following relation holds

$$G_m(x) = \frac{m}{2^m}\Big\{x^{m-1} - \sum_{k=1}^{m}\binom{m+1}{k+1}\sum_{i=1}^{k}(-1)^i(x-i)^{m-1}\Big\}. \tag{9}$$

Obviously, letting $x = 0$ in this theorem, we obtain the following double sum representation of Genocchi numbers.

*Corollary* 2. For $1 < m \le n \in \mathbb{N}$, the following relation holds

$$G_m = (-1)^m\frac{m}{2^n}\sum_{k=1}^{n}\binom{n+1}{k+1}\sum_{i=1}^{k}(-1)^i i^{m-1}; \tag{10}$$

$$G_m = (-1)^m\frac{m}{2^m}\sum_{k=1}^{m}\binom{m+1}{k+1}\sum_{i=1}^{k}(-1)^i i^{m-1}. \tag{11}$$

### 2.2. New double sum expression of Bernoulli polynomials and numbers

From the relation between Genocchi and Bernoulli polynomials [2]

$$G_m(x) = 2\big\{B_m(x) - 2^m B_m(\frac{x}{2})\big\}$$

and Theorem 3, we can obtain the corolary below.

*Corollary* 3. *(Double sum on Bernoulli polynomials)*

$$B_m(x) = 2^m B_m(\frac{x}{2}) + \frac{m}{2^{m+1}}\Big\{x^{m-1} - \sum_{k=1}^{m}\binom{m+1}{k+1}\sum_{i=1}^{k}(-1)^i(x-i)^{m-1}\Big\}. \tag{12}$$

Besides. in view of the well-known result $G_m = 2(1-2^m)B_m$ and Corollary 2, we immediately have the double sum expression of Bernoulli numbers below.

*Corollary* 4. For $m > 1$, the following relation holds

$$B_m = \frac{(-1)^m m}{2^{m+1}(1-2^m)} \sum_{k=1}^{m} \binom{m+1}{k+1} \sum_{i=1}^{k} (-1)^i i^{m-1}. \tag{13}$$

*Theorem* 4. For $m < n \in \mathbb{N}$, the following relation holds

$$B_m(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \sum_{i=1}^{k} (x-i)^m. \tag{14}$$

*Proof.* According to the recurrence relation of Bernoulli polynomials [7]

$$B_m(x+1) = B_m(x) + mx^{m-1}, \tag{15}$$

and by means of telescoping method, we have

$$B_m(x-k) = B_m(x) - m\sum_{i=1}^{k} (x-i)^{m-1}. \tag{16}$$

Replacing $f(x-k)$ by $B_m(x-k)$ in Lemma 2 and in view of (16) and (4), we get

$$B_m(x) \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} - m \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \sum_{i=1}^{k} (x-i)^{m-1} = B_m(x)H_n - mB_{m-1}(x).$$

Based on the representation of Harmonic numbers in terms of alternating binomial sum

$$H_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k}, \tag{17}$$

the above identity can be rewritten as

$$B_{m-1}(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \sum_{i=1}^{k} (x-i)^{m-1},$$

which is equivalent to the desired result. □

Particularly, by taking the replacement $m \to n$ and $n \to n+1$ in Theorem 4, we have the corollary below.

*Corollary* 5. For $n \in \mathbb{N}_0$, the following relation holds

$$B_n(x) = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} \sum_{i=1}^{k} (x-i)^n. \tag{18}$$

Letting $x = 0$ in Theorem 4 and Corollary 5, we have the following corollary about the double sum expression of Bernoulli numbers.

*Corollary* 6. For $m < n \in \mathbb{N}$, the following relation holds

$$B_m = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \sum_{i=1}^{k} (-i)^m; \tag{19}$$

$$B_n = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} \sum_{i=1}^{k} (-i)^n. \tag{20}$$

### 2.3. New double sum expression of Euler polynomials and numbers

Regarding Euler polynomials, we present the following novel double-sum expressions.

*Theorem* 5. For $m \le n \in \mathbb{N}_0$, the following relation holds

$$E_m(x) = \frac{1}{2^n}\Big\{x^m - \sum_{k=1}^{n} \binom{n+1}{k+1} \sum_{i=1}^{k} (-1)^i (x-i)^m\Big\}. \tag{21}$$

*Proof.* Based on the recurrence ration $E_m(x+1)+E_m(x) = 2x^m$ and telescoping approach, we have

$$(-1)^k E_m(x-k) = E_m(x) + 2\sum_{i=1}^{k} (-1)^i (x-i)^m. \tag{22}$$

According to Lemma 1, we obtain

$$\begin{aligned} E_m(x+1) &= \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} E_m(x-k) = \sum_{k=0}^{n} \binom{n+1}{k+1} \Big\{E_m(x) + 2\sum_{i=1}^{k} (-1)^i (x-i)^m\Big\} \\ &= (2^{n+1}-1)E_m(x) + 2\sum_{k=0}^{n} \binom{n+1}{k+1} \sum_{i=1}^{k} (-1)^i (x-i)^m, \end{aligned}$$

which completes the proof by using $E_m(x+1)+E_m(x) = 2x^m$. □

In view of $E_m = 2^m E_m(\frac{1}{2})$, we have the following corollary about the double sum expression for Euler numbers.

*Corollary* 7. For $m \le n \in \mathbb{N}_0$, the following relation holds

$$E_m = \frac{1}{2^n}\Big\{1 - \sum_{k=1}^{n} \binom{n+1}{k+1} \sum_{i=1}^{k} (-1)^i (1-2i)^m\Big\}.$$

## 3. SEVERAL SUMMATION TRANSFORMATION FORMULAE

In this section, we shall establish, according to Lemma 1 and Lemma 2, several summation formulae concerning Genocchi, Bernoulli and Euler numbers and polynomials.

### 3.1. Summation transformation formulae involving Genocchi polynomials and numbers

Based on Genocchi polynomials and Lemma 2, we have the following results.

*Theorem* 6. For $n \in \mathbb{N}$, the following relation holds

$$\sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} \sum_{i=1}^{k} (-1)^{i-1} (x-i)^{n-1} = \frac{G_n(x)}{n} \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{2k-1} \binom{n}{2k-1} - \frac{G_{n-1}(x)}{2}. \tag{23}$$

*Proof.* Taking replacement $f(x-k) \to G_n(x-k)$ in Lemma 2 and in view of (8) and (4), we have

$$2n\sum_{k=1}^{n}\frac{1}{k}\binom{n}{k}\sum_{i=1}^{k}(-1)^{i-1}(x-i)^{n-1} = G_n(x)\sum_{k=1}^{n}\frac{1}{k}\binom{n}{k} + H_nG_n(x) - nG_{n-1}(x).$$

Taking into account (17) and the following relation

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\binom{n}{k} + \sum_{k=1}^{n}\frac{1}{k}\binom{n}{k} = 2\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\frac{1}{2k-1}\binom{n}{2k-1}, \tag{24}$$

we can complete the proof. □

Letting $x=0$ in this theorem, we get the following corollary concerning Genocchi numbers.

*Corollary* 8. For $n \in \mathbb{N}$, the following relation holds

$$\sum_{k=1}^{n}\frac{1}{k}\binom{n}{k}\sum_{i=1}^{k}(-1)^{n-i}i^{n-1} = \frac{G_n}{n}\sum_{k=1}^{\lceil\frac{n}{2}\rceil}\frac{1}{2k-1}\binom{n}{2k-1} - \frac{G_{n-1}}{2}. \tag{25}$$

### 3.2. Summation transformation formulae via Bernoulli polynomials

According to Bernoulli polynomials and Lemma 1, we can establish the following identities.

*Theorem* 7. For $m,n \in \mathbb{N}$, the following relations hold

$$\sum_{k=1}^{n}(-1)^k\binom{n+1}{k+1}\sum_{i=1}^{k}(x-i)^{m-1} = -x^{m-1}; \tag{26}$$

$$\sum_{k=1}^{m}(-1)^k\binom{m+1}{k+1}\sum_{i=1}^{k}(x-i)^{m-1} = -x^{m-1}. \tag{27}$$

*Proof.* For $m \le n$, in view of Lemma 1 and by means of (16), we can compute

$$\begin{aligned} B_m(x+1) &= \sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}B_m(x-k) \\ &= \sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\Big\{B_m(x) - m\sum_{i=1}^{k}(x-i)^{m-1}\Big\} \\ &= B_m(x) - m\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}\sum_{i=1}^{k}(x-i)^{m-1}. \end{aligned}$$

Combining the recurrence relation (15), we obtain the double summation formula (26). Then the second one (27) just a special case of (26) when $n=m$. □

By letting $x=0$ in Theorem 7, we derive the following further identity, which equals zero except the case $m=1$.

*Corollary* 9.

$$\sum_{k=1}^{m}(-1)^k\binom{m+1}{k+1}\sum_{i=1}^{k}i^{m-1} = -\chi(m=1),$$

where $\chi$ stands for the logical function with $\chi(true)=1$ and $\chi(false)=0$.

### 3.3. Summation transformation formulae involving Euler polynomials and numbers

Bases on Euler polynomials and Lemma 2, we can derive the following theorem and corollary.

*Theorem* 8. For $n \in \mathbb{N}$, the following relation holds

$$\sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} \sum_{i=1}^{k} (-1)^i (x-i)^n = \frac{n}{2} E_{n-1}(x) - E_n(x) \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{2k-1} \binom{n}{2k-1}. \tag{28}$$

*Proof.* Making the replacement $f(x-k) \to E_n(x-k)$ in Lemma 2 and using (22) and (4) yields

$$\sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} \left\{ E_n(x) + 2 \sum_{i=1}^{k} (-1)^i (x-i)^n \right\} = nE_{n-1}(x) - E_n(x) H_n.$$

In view of (17) and (24), we complete the proof. □

Letting $x = \frac{1}{2}$ in Theorem 8, we obtain the corollary below.

*Corollary* 10. For $n \in \mathbb{N}$, the following relation holds

$$\sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} \sum_{i=1}^{k} (-1)^i (1-2i)^n = nE_{n-1} - E_n \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{2k-1} \binom{n}{2k-1}. \tag{29}$$

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