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POSITIVE SOLUTIONS TO SCHRÖDINGER-TYPE SINGULAR p-LAPLACIAN PROBLEMS WITH INFINITE SEMIPOSITONE BEHAVIOR

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Abstract. This paper establishes the existence of positive solutions to a singular p-Laplacian Schrödinger-type equation exhibiting infinite semipositone structure:

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda \frac{f(u)}{u^\alpha}, \quad x \in \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator ($p > 1$), $\lambda > 0$ is a parameter, $\alpha \in (0,1)$, and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. The nonlinearity $f \in C([0,\infty),\mathbb{R})$ satisfies $f(0) < 0$ and exhibits asymptotic behavior characterized by $\lim_{s\to 0^+} \frac{f(s)}{s^\alpha} = -\infty$, inducing a strong singularity at the origin. The potential $V \in L^\infty(\Omega)$ is allowed to change sign, adding indefinite structure to the problem. By developing a refined sub- and supersolution method adapted to the degenerate nature of the p-Laplacian, we prove the existence of positive solutions $u_\lambda \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ for sufficiently large λ.

***Keywords**:* Singular Schrödinger-type equation, sub- and supersolution, infinite semipositone.
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1. INTRODUCTION AND MAIN RESULT

The study of semipositone problems, where the nonlinear term $f(u)$ is negative at $u = 0$, has played a crucial role in the development of nonlinear PDE theory. These problems have been a cornerstone of the field since their foundational treatment by Lions [1], who highlighted the challenges arising from the negativity of f at zero, a feature that disrupts classical monotonicity arguments and complicates the construction of subsolutions. Early advancements for the Laplacian operator, particularly in bounded domains with Dirichlet boundary conditions, were pioneered by Ko [2], who established existence results for Schrödinger-type equations $-\Delta u + V(x)u = \lambda f(u)/u^\alpha$ using sub- and supersolution methods. Prior to this, significant contributions had been made by Ambrosetti et al. [3] and Castro et al. [4] who expanded the theory to include multiplicity results and mixed boundary conditions, while Rabinowitz [5] developed variational frameworks for nonlinear Schrödinger equations. Despite these strides, the p-Laplacian case—critical in modeling non-Newtonian fluids and quasilinear diffusion—remains underexplored, particularly for infinite semipositone structures where $\lim_{s\to 0^+} f(s)/s^\alpha = -\infty$.

Recent studies have addressed specific aspects of p-Laplacian semipositone problems: Aranda and Godoy [6] proved multiplicity for small λ in convex domains via bifurcation theory; Chu et al. [7] established uniqueness of radial solutions for large λ in balls; and Wang and Suo [8] derived infinitely many solutions for singular-critical growth terms using variational methods. However, the interplay between the p-Laplacian's degeneracy

($p \neq 2$), singular nonlinearities, and sign-changing potentials $V(x)$ remains unresolved. Ko's seminal work [2] on the Laplacian case ($p = 2$) provides a critical foundation, yet the nonlinear growth and weakened regularity of the p-Laplacian demand novel adaptations of classical tools.

We address this gap by analyzing the p-Laplacian infinite semipositone problem:

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda \frac{f(u)}{u^{\alpha}}, \quad x \in \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\lambda > 0$, $V \in L^{\infty}(\Omega)$, and $f \in C([0,\infty),\mathbb{R})$ satisfies $f(0) < 0$ and the following hypotheses:

(H1) $f(s) \leq As^{\gamma}$ for $s \geq 0$, where $\alpha \leq \gamma < p - 1 + \alpha$;

(H2) $f(s) \geq Bs^{\beta}$ for $s \gg 1$, with $0 < \beta < \frac{1}{p}(1-\alpha)(p-1+\alpha)$;

Also the potential function V is required to satisfy:

(H3) There exists a positive constant c_V such that the potential is bounded below by $V(x) \geq -c_V$, with this lower bound strictly greater than $-\|e\|_{\infty}^{1-p}$, where e denotes the unique positive solution to the p-Laplacian problem $-\Delta_p e = 1$ in Ω with homogeneous Dirichlet boundary conditions. Furthermore, the constant c_1 from the Poincaré-type inequality:

$$\|u\|_{L^p(\Omega)}^p \leq c_1 \|\nabla u\|_{L^p(\Omega)}^p \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

must satisfy the coercivity condition $1 - c_V c_1 > 0$.

The central objective of our investigation is to establish the following existence result for the singular p-Laplacian problem.

THEOREM 1. *Under hypotheses (H1)–(H3), problem* (1) *admits a positive solution* $u_\lambda \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ *for* $\lambda \gg 1$.

The remainder of the paper is organized as follows: Section 2 develops the necessary technical tools, and Section 3 presents the proof of Theorem 1.

2. FOUNDATIONAL TOOLS AND TECHNICAL FRAMEWORK

This section establishes the essential mathematical infrastructure for analyzing our p-Laplacian problem. We begin by formalizing the critical concepts of sub- and supersolutions, then develop the necessary existence theory through three key lemmas that form the backbone of our subsequent analysis.

Definition 1 (Subsolution/Supersolution). *A function* $\psi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ *is a subsolution of* (1) *if it satisfies the differential inequality* $-\Delta_p \psi + V(x)|\psi|^{p-2}\psi \leq \lambda f(\psi)/\psi^{\alpha}$ *pointwise in* Ω, *while being positive in* Ω *and equal to zero on* $\partial\Omega$. *The supersolution* Z *satisfies the reverse inequality with analogous boundary behavior.*

LEMMA 1 [9–11, Existence via sub- and supersolution Pair]. *Given an ordered pair* $\psi \leq Z$ *of sub- and supersolutions for* (1), *there exists at least one solution* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ *satisfying* $\psi \leq u \leq Z$ *throughout* Ω.

LEMMA 2 (Auxiliary Problem Solvability). *Under hypothesis (H3), the boundary value problem*

$$-\Delta_p \xi + V(x)|\xi|^{p-2}\xi = 1, \quad \xi = 0 \text{ on } \partial\Omega, \tag{2}$$

admits a positive solution $\xi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ *such that* $\xi(x) > 0$ *in* Ω *and with strictly negative normal derivative on* $\partial\Omega$.

Proof. From (H3), it holds that $c_V\|e\|_\infty^{p-1} < 1$, which implies that there exists $K > 0$ such that $1-c_V\|e\|_\infty^{p-1} > \frac{1}{K}$. Let $Z = Ke$. Then we have:

$$\begin{aligned}-\Delta_p Z+V(x)|Z|^{p-2}Z &= K(-\Delta_p e+V(x)|e|^{p-2}e) = K(1+V(x)|e|^{p-2}e)\\ &\geq K(1-c_V\|e\|_\infty^{p-1}) \geq 1,\end{aligned}$$

in Ω and $Z = 0$ on $\partial\Omega$. Hence, Z is a positive supersolution of (2). Note that $\psi \equiv 0$ is a subsolution of (2) but not a solution. By the subsolution-supersolution method, there exists a solution $\xi \in W^{1,p}(\Omega)\cap C(\overline{\Omega})$ such that $0 \leq \xi \leq Z$ in Ω.

Now we claim that $\xi > 0$ in Ω and $\frac{\partial\xi}{\partial\eta} < 0$ on $\partial\Omega$. Suppose that there exists $x_0 \in \Omega$ such that $\xi(x_0) = 0$. From the first equation of (2), we see that:

$$1 = -\Delta_p\xi(x_0)+V(x_0)|\xi(x_0)|^{p-2}\xi(x_0) \leq 0,$$

since V is bounded, which is a contradiction. Therefore, $\xi > 0$ in Ω. By the Hopf boundary lemma, we find that $\frac{\partial\xi}{\partial\eta} < 0$ on $\partial\Omega$. □

LEMMA 3 (Principal Eigenvalue Positivity). *Under hypothesis (H3), the eigenvalue problem* $-\Delta_p\phi+V|\phi|^{p-2}\phi = \lambda\phi$ *with Dirichlet conditions has a smallest eigenvalue* $\lambda_1 > 0$ *with associated positive eigenfunction* ϕ_1.

Proof. The proof proceeds via variational methods and contradiction.

The principal eigenvalue λ_1 is defined as:

$$\lambda_1 = \inf_{\substack{\phi\in W_0^{1,p}(\Omega)\\ \phi\neq 0}} \frac{\int_\Omega|\nabla\phi|^p\,\mathrm{d}x+\int_\Omega V(x)|\phi|^p\,\mathrm{d}x}{\int_\Omega|\phi|^p\,\mathrm{d}x}.$$

The Poincaré inequality and boundedness of V ensure the functional is coercive and weakly lower semicontinuous.

Let $\{\phi_n\}$ be a minimizing sequence normalized by $\|\phi_n\|_{L^p} = 1$. By reflexivity of $W_0^{1,p}(\Omega)$, a subsequence converges weakly to $\phi_1 \in W_0^{1,p}(\Omega)$. Weak lower semicontinuity yields:

$$\int_\Omega|\nabla\phi_1|^p\,\mathrm{d}x+\int_\Omega V(x)|\phi_1|^p\,\mathrm{d}x \leq \lambda_1. \tag{3}$$

Thus, ϕ_1 minimizes the Rayleigh quotient and satisfies the Euler-Lagrange equation:

$$\int_\Omega|\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla\psi\,\mathrm{d}x+\int_\Omega V(x)|\phi_1|^{p-2}\phi_1\psi\,\mathrm{d}x = \lambda_1\int_\Omega|\phi_1|^{p-2}\phi_1\psi\,\mathrm{d}x,$$

for all $\psi \in W_0^{1,p}(\Omega)$.

By the strong maximum principle for the p-Laplacian, non-negative solutions are either identically zero or strictly positive. Since ϕ_1 minimizes $\mathscr{R}(\phi)$, it cannot vanish identically. Hence, $\phi_1 > 0$ in Ω.

We assume, for contradiction, that $\lambda_1 \leq 0$. Then, by (3):

$$\int_\Omega|\nabla\phi_1|^p\,\mathrm{d}x+\int_\Omega V(x)|\phi_1|^p\,\mathrm{d}x \leq 0.$$

Since $1-c_1c_V > 0$ and $\int_\Omega V(x)|\phi_1|^p\,\mathrm{d}x \geq -c_V\int_\Omega|\phi_1|^p\,\mathrm{d}x$, it follows that:

$$c_1\int_\Omega|\nabla\phi_1|^p\,\mathrm{d}x < \int_\Omega|\phi_1|^p\,\mathrm{d}x,$$

this leads to a contradiction. Therefore, $\lambda_1 > 0$. □

LEMMA 4 (Singular Problem Solvability). *Under hypothesis (H3), the singular problem*

$$\begin{cases} -\Delta_p \zeta + V(x)|\zeta|^{p-2}\zeta = \frac{1}{\zeta^\alpha}, & x \in \Omega, \\ \zeta = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

admits a positive solution $\zeta \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ *with negative normal derivative on* $\partial\Omega$*.*

Proof. Define the energy functional $E : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with (4):

$$E(\zeta) = \frac{1}{p}\int_\Omega |\nabla\zeta|^p \,\mathrm{d}x - \frac{\lambda}{1-\alpha}\int_\Omega (\zeta^+)^{1-\alpha}\,\mathrm{d}x + \frac{1}{p}\int_\Omega V(x)|\zeta|^p \,\mathrm{d}x,$$

where $\alpha \in (0,1)$ and $\zeta^+ = \max(\zeta, 0)$. Using the Poincaré inequality $\|\zeta\|_{L^p} \leq c_1\|\nabla\zeta\|_{L^p}$ and the boundedness of $V(x)$ from $(H3)$, we derive:

$$\begin{aligned} E(\zeta) &\geq \frac{1}{p}\|\nabla\zeta\|_{L^p}^p - \frac{\lambda}{1-\alpha}\|\zeta\|_{L^{1-\alpha}}^{1-\alpha} - \frac{c_V}{p}\|\zeta\|_{L^p}^p \\ &\geq \frac{1-c_V c_1}{p}\|\nabla\zeta\|_{L^p}^p - \frac{\lambda}{1-\alpha}\|\zeta\|_{L^{1-\alpha}}^{1-\alpha} \end{aligned}$$

and using the Poincaré inequality $\|\zeta\|_{L^p} \leq c_1\|\nabla\zeta\|_{L^p}$, we deduce:

$$E(\zeta) \geq \frac{1}{p}(1-c_V c_1)\|\nabla\zeta\|_{L^p}^p - \frac{\lambda}{1-\alpha}\|\zeta\|_{L^{1-\alpha}}^{1-\alpha}.$$

Since $1 - c_V c_1 > 0$, we conclude that $E(\zeta) \geq \frac{1}{p}(1-c_V c_1)\|\nabla\zeta\|_{L^p}^p \geq 0$, and therefore $E(\zeta)$ is coercive and weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. By the Direct Method in the Calculus of Variations, E attains its global minimum at some $\zeta \in W_0^{1,p}(\Omega)$.

Let $\phi_1 > 0$ denote the normalized first eigenfunction of the p-Laplacian with $\|\phi_1\|_{L^p} = 1$. For small $\varepsilon > 0$, evaluate $E(\varepsilon\phi_1)$:

$$E(\varepsilon\phi_1) = \frac{\varepsilon^p}{p}\int_\Omega |\nabla\phi_1|^p \,\mathrm{d}x - \frac{\lambda\varepsilon^{1-\alpha}}{1-\alpha}\int_\Omega \phi_1^{1-\alpha}\,\mathrm{d}x + \frac{\varepsilon^p}{p}\int_\Omega V(x)|\phi_1|^p \,\mathrm{d}x.$$

Since

$$\int_\Omega V(x)|\phi_1|^p \,\mathrm{d}x = \lambda_1 \underbrace{\int_\Omega |\phi_1|^p \,\mathrm{d}x}_{=1} - \int_\Omega |\nabla\phi_1|^p \,\mathrm{d}x,$$

we conclude:

$$E(\varepsilon\phi_1) = \frac{\varepsilon^p \lambda_1}{p} - \frac{\lambda\varepsilon^{1-\alpha}}{1-\alpha}\int_\Omega \phi_1^{1-\alpha}\,\mathrm{d}x.$$

Given $1 - \alpha < p$, the singular term $-\frac{\lambda\varepsilon^{1-\alpha}}{1-\alpha}\int_\Omega \phi_1^{1-\alpha}\,\mathrm{d}x$ dominates as $\varepsilon \to 0^+$, forcing $E(\varepsilon\phi_1) < 0$.

Since $E(0) = 0 > E(\varepsilon\phi_1)$, it follows that $\zeta = 0$ cannot be the global minimizer of $E(\zeta)$. Therefore, $\zeta \neq 0$ in Ω.

Decompose $\zeta = \zeta^+ - \zeta^-$. Since $E(|\zeta|) \leq E(\zeta)$ and equality holds only if $\zeta^- = 0$ a.e., the minimizer must satisfy $\zeta \geq 0$ a.e. in Ω. If $\zeta^+ = 0$, then $E(\zeta) \geq 0$, contradicting $E(\varepsilon\phi_1) < 0$. Hence, $\zeta > 0$ in Ω.

Define $\tilde{\zeta} = \zeta - \xi\|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}}$, where ξ solves equation (2). Suppose $\tilde{\Omega} = \{x \in \Omega : \tilde{\zeta} < 0\} \neq \emptyset$.

Evaluating the p-Laplacian differential operator at $\tilde{\zeta}$ gives:

$$-\Delta_p\tilde{\zeta} = -\Delta_p\zeta + \|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}}(-\Delta_p\xi).$$

Using the original equations for ζ and ξ, compute:

$$-\Delta_p\tilde{\zeta} = \left(\frac{1}{\zeta^\alpha} - V(x)|\zeta|^{p-2}\zeta\right) + \|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}}\left(1 - \|V\|_\infty|\xi|^{p-2}\xi\right).$$

Thus, for any $x \in \tilde{\Omega}$:

$$\begin{aligned}
-\Delta_p\tilde{\zeta} &= \left(\|V\|_\infty - V(x)\right)|\zeta|^{p-2}\zeta + \|V\|_\infty\left(\|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}}|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta\right) \\
&\quad + \frac{1}{\zeta^\alpha} - \|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}} \\
&\geq \frac{1}{\zeta^\alpha} - \|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}} \\
&\geq \xi^{-\alpha}\|\xi\|_\infty^{\frac{\alpha^2}{1+\alpha}} - \|\xi\|_\infty^{-\frac{\alpha^2}{1+\alpha}} \\
&\geq \|\xi\|_\infty^{-\alpha}\|\xi\|_\infty^{\frac{\alpha}{1+\alpha}} - \|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}} = 0.
\end{aligned}$$

Now, applying the Maximum Principle for $\tilde{\zeta}$, since $-\Delta_p\tilde{\zeta} \geq 0$ in $\tilde{\Omega}$ and $\tilde{\zeta}(x) < 0$ in $\tilde{\Omega}$, the Maximum Principle implies that $\tilde{\zeta}$ cannot achieve a negative minimum inside $\tilde{\Omega}$ unless $\tilde{\zeta}(x) = 0$ on the boundary of $\tilde{\Omega}$.

This leads to a contradiction, as we assumed that $\tilde{\zeta}(x) < 0$ in $\tilde{\Omega}$, and the only way this could be consistent with the Maximum Principle is if $\tilde{\zeta}(x) = 0$ on the boundary of $\tilde{\Omega}$, which contradicts the assumption that $\zeta(x) < \xi(x)\|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}}$.

Thus, $\tilde{\Omega} = \emptyset$, and therefore:

$$\zeta(x) \geq \xi(x)\|\xi\|_\infty^{-\frac{\alpha}{1+\alpha}} \quad \text{in } \Omega.$$

Since $\xi > 0$ in Ω and $\frac{\partial\xi}{\partial\eta} < 0$ on $\partial\Omega$, it follows that $\zeta(x) > 0$ for all $x \in \Omega$, and $\frac{\partial\zeta}{\partial\eta} < 0$ on $\partial\Omega$. ☐

3. PROOF OF THE MAIN THEOREM

We present a comprehensive proof of Theorem 1 through meticulous construction of ordered sub- and supersolutions. The proof contains three principal components: (I) precise subsolution construction, (II) supersolution development with compatibility analysis, and (III) application of the sub- and supersolution principle with regularity considerations.

Proof. Let $\zeta > 0$ be the unique positive solution to the singular boundary value problem (4), whose existence is guaranteed by Lemma 4. Define the normalized function:

$$w(x) := \frac{\zeta(x)}{\|\zeta\|_{L^\infty(\Omega)}} \quad \text{satisfying} \quad \begin{cases} \|w\|_{L^\infty(\Omega)} = 1 \\ -\Delta_p w + V|w|^{p-2}w = \frac{1}{\|\zeta\|_{L^\infty(\Omega)}^{p-1+\alpha}} w^{-\alpha} \end{cases}$$

Given the positivity of w in Ω and the boundary conditions, there exist $\delta, m > 0$ such that in $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \delta\}$, the following inequality holds:

$$\left(\frac{p}{p-1+\alpha}\right)^{p-1}\left[\gamma|\nabla w|^p - \left(c_v\delta_0 + \|\zeta\|_{L^\infty}^{-(p-1+\alpha)}\right)w^{1-\alpha}\right] \geq m, \tag{5}$$

where $\gamma := \frac{(1-\alpha)(p-1)}{p-1+\alpha}$ and $\delta_0 := 1 - \left(\frac{p-1+\alpha}{p}\right)^{p-1}$. Moreover, there exists $\mu \in (0,1)$ such that:

$$\mu \leq w(x) \leq 1 \quad \text{in } \Omega \setminus \Omega_\delta.$$

Now, define the subsolution candidate through the ansatz:

$$\psi(x) := \lambda^{\frac{r}{p-1}} w(x)^{\frac{p}{p-1+\alpha}}, \quad r \in \left(\frac{p-1}{p-1+\alpha}, \frac{p-1}{p-1+\alpha-\beta} \right).$$

The gradient structure reveals:

$$|\nabla \psi|^{p-2} \nabla \psi = \lambda^r \left(\frac{p}{p-1+\alpha} \right)^{p-1} w^{\gamma} |\nabla w|^{p-2} \nabla w.$$

Through detailed computation using the p-Laplacian's nonlinear structure:

$$-\Delta_p \psi = \lambda^r \left(\frac{p}{p-1+\alpha} \right)^{p-1} \left[w^{\gamma} \Delta_p w + \gamma w^{\gamma-1} |\nabla w|^p \right].$$

Substituting w's governing equation and hypothesis (H3), we obtain:

$$\begin{aligned}
-\Delta_p \psi + V|\psi|^{p-2}\psi &= -\lambda^r \left(\frac{p}{p-1+\alpha} \right)^{p-1} \\
&\quad \times \left[w^{\gamma} \left(V|w|^{p-2} w - \frac{1}{\|\zeta\|_{\infty}^{p-1+\alpha}} \frac{1}{w^{\alpha}} \right) + \gamma w^{\gamma-1} |\nabla w|^p \right] \\
&= -\lambda^r \left(\frac{p}{p-1+\alpha} \right)^{p-1} w^{\gamma-1} \\
&\quad \times \left[\gamma |\nabla w|^p - \frac{1}{\|\zeta\|_{\infty}^{p-1+\alpha}} w^{1-\alpha} + V \left(1 - \left(\frac{p-1+\alpha}{p} \right)^{p-1} \right) w^p \right] \\
&\leq -\lambda^r \left(\frac{p}{p-1+\alpha} \right)^{p-1} w^{\gamma-1} \\
&\quad \times \left[\gamma |\nabla w|^p - \frac{1}{\|\zeta\|_{\infty}^{p-1+\alpha}} w^{1-\alpha} - c_v \delta_0 w^p \right].
\end{aligned}$$

Since $w \leq 1$, we have $w^p \leq w^{1-\alpha}$. Thus:

$$\begin{aligned}
-\Delta_p \psi + V|\psi|^{p-2}\psi &\leq -\lambda^r \left(\frac{p}{p-1+\alpha} \right)^{p-1} w^{\gamma-1} \\
&\quad \times \left[\gamma |\nabla w|^p - \left(c_v \delta_0 + \|\zeta\|_{L^{\infty}}^{-(p-1+\alpha)} \right) w^{1-\alpha} \right]
\end{aligned} \tag{6}$$

Given that $r > \frac{p-1}{p-1+\alpha}$, it immediately follows that $1 - r - \frac{r\alpha}{p-1} < 0$. Thus, in $\overline{\Omega}_{\delta}$, relation (5) implies that for $\lambda \gg 1$:

$$-\left(\frac{p}{p-1+\alpha} \right)^{p-1} \left[\gamma |\nabla w|^p - \left(c_v \delta_0 + \|\zeta\|_{L^{\infty}}^{-(p-1+\alpha)} \right) w^{1-\alpha} \right] \leq -m \leq \lambda^{1-r-\frac{r\alpha}{p-1}} f_{\mathsf{min}},$$

where $f_{\mathsf{min}} := \min_{s \geq 0} f(s) < 0$. Therefore, we obtain in $\overline{\Omega}_{\delta}$:

$$-\Delta_p \psi + V|\psi|^{p-2}\psi \leq \lambda \frac{f_{\mathsf{min}}}{(\lambda^{\frac{r}{p-1}} w^{\frac{p}{p-1+\alpha}})^{\alpha}} \leq \lambda \frac{f(\psi)}{\psi^{\alpha}}. \tag{7}$$

In the domain $\Omega \setminus \overline{\Omega}_{\delta}$, we know that $w \geq \mu > 0$. According to hypothesis (H2), we can assert the following inequality for sufficiently large λ:

$$f\left(\lambda^{\frac{r}{p-1}} w^{\frac{p}{p-1+\alpha}}\right) \geq B\left(\lambda^{\frac{r}{p-1}} w^{\frac{p}{p-1+\alpha}}\right)^{\beta}, \tag{8}$$

where $\lambda \gg 1$. This leads us to the following bound, noting that the condition $1+\frac{r(\beta-\alpha)}{p-1}-\frac{r}{p-1}>0$ holds:

$$\left(\frac{p}{p-1+\alpha}\right)^{p-1}\left(c_v\delta_0+\|\zeta\|_{L^\infty}^{-(p-1+\alpha)}\right) \leq B\lambda^{1+\frac{r(\beta-\alpha)}{p-1}-\frac{r}{p-1}}.$$

To proceed, we apply the condition (H2), which implies $p\beta-p+1-2\alpha+\alpha p+\alpha^2<0$. Additionally, since $\mu \leq w \leq 1$, we derive the following inequality:

$$\left(w^{\frac{p}{p-1+\alpha}}\right)^{\beta-\alpha-1} w^{\frac{\alpha^2+2\alpha(p-1)+1}{p-1+\alpha}} = w^{\frac{p\beta-p+1-2\alpha+\alpha p+\alpha^2}{p-1+\alpha}} \geq 1.$$

This result leads to the conclusion that:

$$B\lambda^{1+\frac{r(\beta-\alpha)}{p-1}-\frac{r}{p-1}} \leq B\lambda^{1+\frac{r(\beta-\alpha)}{p-1}-\frac{r}{p-1}}\left(w^{\frac{p}{p-1+\alpha}}\right)^{\beta-\alpha-1} w^{\frac{\alpha^2+2\alpha(p-1)+1}{p-1+\alpha}}.$$

Consequently, we obtain the following inequality:

$$\begin{aligned}\lambda^{\frac{r}{p-1}}\left(\frac{p}{p-1+\alpha}\right)^{p-1}\left(c_v\delta_0+\|\zeta\|_{L^\infty}^{-(p-1+\alpha)}\right) w^{\frac{p}{p-1+\alpha}-\frac{\alpha^2+2\alpha(p-1)+1}{p-1+\alpha}}&\\ \leq B\lambda^{1+\frac{r(\beta-\alpha)}{p-1}}\left(w^{\frac{p}{p-1+\alpha}}\right)^{\beta-\alpha}&.\end{aligned} \tag{9}$$

By employing an estimate in $\Omega\setminus\overline{\Omega}_\delta$ and neglecting the non-dominant term $\gamma|\nabla w|^p$, inequality (6) yields:

$$\begin{aligned}-\Delta_p\psi+V|\psi|^{p-2}\psi \leq \lambda^{\frac{r}{p-1}}&\left(\frac{p}{p-1+\alpha}\right)^{p-1}\\ &\times\left(c_v\delta_0+\|\zeta\|_{L^\infty}^{-(p-1+\alpha)}\right) w^{\frac{p}{p-1+\alpha}-\frac{\alpha^2+2\alpha(p-1)+1}{p-1+\alpha}}.\end{aligned} \tag{10}$$

By equations (9) and (10), we deduce that in the region $\Omega\setminus\overline{\Omega}_\delta$:

$$-\Delta_p\psi+V|\psi|^{p-2}\psi \leq B\lambda^{1+\frac{r(\beta-\alpha)}{p-1}}\left(w^{\frac{p}{p-1+\alpha}}\right)^{\beta-\alpha}.$$

From this expression and equation (8), we obtain the final estimate in the region $\Omega\setminus\overline{\Omega}_\delta$:

$$-\Delta_p\psi+V|\psi|^{p-2}\psi \leq \lambda\frac{B\left(\lambda^{\frac{r}{p-1}} w^{\frac{p}{p-1+\alpha}}\right)^{\beta}}{\left(\lambda^{\frac{r}{p-1}} w^{\frac{p}{p-1+\alpha}}\right)^{\alpha}} \leq \lambda\frac{f(\psi)}{\psi^\alpha}. \tag{11}$$

The function ψ, constructed from w, aligns with the boundary requirements. Consequently, from equations (7) and (11), it follows that ψ serves as a positive subsolution to equation (1) in the domain Ω for large values of λ.

Now, we construct the supersolution Z using the solution ξ of the auxiliary problem (2), while ensuring compatibility with the subsolution $\psi=\lambda^{\frac{r}{p-1}} w^{\frac{p}{p-1+\alpha}}$. By assumption (H1), we have $p-1+\alpha-\gamma>0$ and

$\gamma-\alpha>0$. Define the supersolution $Z(x)=M_\lambda \xi(x)$, where

$$M_\lambda=\left(\lambda A\|\xi\|_{L^\infty(\Omega)}^{\gamma-\alpha}\right)^{\frac{1}{p-1-\gamma+\alpha}}.$$

Using $\xi \leq \|\xi\|_{L^\infty(\Omega)}$, this reduces to:

$$M_\lambda^{p-1}=\lambda A M_\lambda^{\gamma-\alpha}\|\xi\|_{L^\infty(\Omega)}^{\gamma-\alpha} \geq \lambda \frac{A(M_\lambda \xi)^\gamma}{(M_\lambda \xi)^\alpha}.$$

By (H1), $f(Z) \leq A Z^\gamma$. This yields:

$$-\Delta_p Z+V(x)|Z|^{p-2}Z=M_\lambda^{p-1} \geq \lambda \frac{A(M_\lambda \xi)^\gamma}{(M_\lambda \xi)^\alpha} \geq \lambda \frac{f(Z)}{Z^\alpha}.$$

The function $Z=M_\lambda \xi$ inherits the strict positivity of ξ in Ω and vanishes on $\partial\Omega$. Therefore, Z serves as a positive supersolution to equation (1) in the domain Ω.

For $\lambda \gg 1$, M_λ dominates the subsolution $\psi=\lambda^{\frac{r}{p-1}} w^{\frac{p}{p-1+\alpha}}$, as the exponent of λ in M_λ satisfies:

$$\frac{1}{p-1-\gamma+\alpha}>\frac{r}{p-1},$$

by the interval $r \in \left(\frac{p-1}{p-1+\alpha}, \frac{p-1}{p-1+\alpha-\beta}\right)$ and (H2). Consequently, $Z \geq \psi$ in Ω for sufficiently large λ.

By the sub- and supersolution method, there exists a positive solution $u_\lambda \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ to (1) satisfying $\psi \leq u_\lambda \leq Z$ in Ω for $\lambda \gg 1$. This completes the proof of Theorem 1.

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