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REDUCED MATRIX mKdV INTEGRABLE MODELS VIA THE DUAL-REDUCTION TECHNIQUE

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Abstract. This paper aims to apply the dual-reduciton technique and derive integrable reductions of matrix modified Korteweg–de Vries (mKdV) models. Using the Lax pair formulation, the study employs two group reductions as the main analytical tool. Two illustrative examples of reduced Ablowitz–Kaup–Newell–Segur matrix spectral problems are presented, showcasing explicit examples of reduced matrix mKdV integrable models generated through these two distinct group reductions.

***Keywords**:* Lax pair, AKNS matrix spectral problem, zero-curvature equation, group reduction
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1. INTRODUCTION

A fundamental challenge in soliton theory is to classify integrable models within the framework of the Lax pair formulation [1]. To gain deeper insights into the structures of integrable models, particularly multi-component ones, it is crucial to explore illustrative examples. A key step in this process is the introduction of matrix spectral matrices, which are formulated using matrix Lie algebras. These Lax pairs ensure the existence of bi-Hamiltonian structures [2], which in turn generate sequences of commuting symmetries and conservation laws. Furthermore, using Lax pairs, the Cauchy problems of the corresponding integrable models can also be solved via the inverse scattering transform [3].

The matrix Ablowitz–Kaup–Newell–Segur (AKNS) spectral problems provide a versatile framework for generating a broad class of integrable models, including the nonlinear Schrödinger (NLS) equation and the modified Korteweg–de Vries (mKdV) equation. Group reductions have been shown to derive reduced matrix spectral problems and their corresponding integrable models [4–7], with illustrative examples including non-local integrable models involving reflection points [8]. Notably, dual group reductions give rise to a new class of reduced integrable models [9]. The key challenge in this approach lies in carefully managing the reductions imposed on the potentials by the two transformations, ensuring the preservation of the invariance of the zero-curvature equations [10]. A comprehensive classification of these lower-order integrable models, associated with matrix AKNS spectral problems, has revealed three types of nonlocal NLS equations and two types of nonlocal mKdV equations [11].

In addition, a variety of powerful methods have been developed to study reduced integrable models, particularly in the context of constructing soliton solutions. The inverse scattering transform remains a highly effective tool for solving Cauchy problems, even for nonlocal integrable models [12, 13]. Traditional techniques such as the Hirota bilinear method, Bäcklund transformations, Darboux transformations, and the Riemann–Hilbert method have also demonstrated significant utility. Furthermore, several novel mathematical frameworks have been proposed to explore nonlocal reduced integrable models (see, e.g., [11, 14–16]).

This paper presents two sets of dual group reductions and constructs reduced integrable mKdV models, starting from the matrix AKNS spectral problems. In Section 2, we establish the foundational background for the analysis by revisiting the matrix AKNS spectral problems and their associated integrable mKdV models. We also outline a general framework for implementing dual group reductions and deriving reduced mKdV integrable models. In Section 3, we introduce two consistent group reductions and apply them to the matrix AKNS spectral problems, resulting in reduced mKdV integrable models. These examples demonstrate the variety and richness of reduced matrix AKNS integrable models. The final section summarizes the main findings and offers some concluding remarks.

2. THE AKNS INTEGRABLE MODELS AND DUAL GROUP REDUCTIONS

2.1. Revisiting matrix AKNS integrable hierarchies

Let us fix two natural numbers: m and n. In the AKNS framework of integrable models, the column vector dependent variable $u = u(p,q)$ consists of two matrix potentials, defined as follows:

$$p = p(x,t) = (p_{jk})_{m\times n},\ q = q(x,t) = (q_{kj})_{n\times m}. \tag{1}$$

For each $r \geq 0$, a pair of matrix AKNS spectral problems reads

$$-i\phi_x = U\phi,\ -i\phi_t = V^{[r]}\phi, \tag{2}$$

with the Lax pairs being defined by

$$U = U(u,\lambda) = \lambda\Lambda + P,\ V^{[r]} = V^{[r]}(u,\lambda) = \lambda^r\Omega + Q^{[r]}, \tag{3}$$

where

$$\Lambda = \begin{bmatrix} \alpha_1 I_m & 0 \\ 0 & \alpha_2 I_n \end{bmatrix},\ P = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \tag{4}$$

and

$$\Omega = \begin{bmatrix} \beta_1 I_m & 0 \\ 0 & \beta_2 I_n \end{bmatrix},\ Q^{[r]} = \sum_{s=0}^{r-1} \lambda^s \begin{bmatrix} a^{[r-s]} & b^{[r-s]} \\ c^{[r-s]} & d^{[r-s]} \end{bmatrix}. \tag{5}$$

In the above expressions, I_k is the identity matrix of size k, λ denotes the spectral parameter, α_1,α_2 and β_1,β_2 are two pairs of distinct arbitrary constants, and $Q^{[0]}$ is the zero matrix of order $m+n$. The unique Laurent series solution to the stationary zero-curvature equation

$$W_x = i[U,W], \tag{6}$$

with the initial data $W^{[0]} = \Omega$ is given by

$$W = \sum_{s\geq 0} \lambda^{-s} W^{[s]} = \sum_{s\geq 0} \lambda^{-s} \begin{bmatrix} a^{[s]} & b^{[s]} \\ c^{[s]} & d^{[s]} \end{bmatrix}. \tag{7}$$

This series expansion is used to construct hierarchies of commuting integrable models (see, e.g., [17, 18]).

Now, the zero-curvature equations:

$$U_t - V_x^{[r]} + i[U,V^{[r]}] = 0,\ r \geq 0, \tag{8}$$

i.e., the compatibility of the two matrix spectral problems in (2), generate the matrix AKNS hierarchy of integrable models:

$$p_t = i\alpha b^{[r+1]},\ q_t = -i\alpha c^{[r+1]},\ r \geq 0, \tag{9}$$

where $\alpha = \alpha_1 - \alpha_2$. The simplest case, $m = n = 1$, coincides with the classical AKNS integrable hierarchy with scalar potentials p and q [19]. Each system within the matrix AKNS integrable hierarchy possesses a bi-Hamiltonian structure and admits infinitely many symmetries and conserved quantities (see, e.g., [20,21]).

When $r = 2s+1$, with $s \geq 1$, the matrix AKNS integrable hierarchy in (9) gives rise to the matrix mKdV integrable hierarchy. In particular, setting $s = 1$ yields the first nontrivial integrable system – the matrix mKdV integrable model:

$$p_t = -\frac{\beta}{\alpha^3}(p_{xxx} + 3pqp_x + 3p_xqp),\ q_t = -\frac{\beta}{\alpha^3}(q_{xxx} + 3q_xpq + 3qpq_x), \tag{10}$$

where $\beta = \beta_1 - \beta_2$. The corresponding Lax matrix $V^{[3]}$ takes form

$$V^{[3]} = \lambda^3\Omega + \frac{\beta}{\alpha}\lambda^2 P - \frac{\beta}{\alpha^2}\lambda I_{m,n}(P^2 + iP_x) - \frac{\beta}{\alpha^3}(i[P,P_x] + P_{xx} + 2P^3), \tag{11}$$

where $I_{m,n} = \operatorname{diag}(I_m, -I_n)$. These equations form the foundational examples for our subsequent analysis. We also remark that a variety of higher-order matrix AKNS integrable models can be derived in a similar fashion.

2.2. Dual group reductions of different types

We focus on the case $m = n$, which correspond to two square potential matrices, p and q, and assume that

$$\alpha_1 = -\alpha_2 = \frac{1}{2},\ \beta_1 = -\beta_2 = -\frac{1}{2}. \tag{12}$$

Associated with two invertible block matrices of different types

$$\Delta = \begin{bmatrix} 0 & \Delta_1 \\ \Delta_2 & 0 \end{bmatrix},\ \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \tag{13}$$

where Σ is symmetric, the dual group reductions are defined as

$$\Delta U(\lambda)\Delta^{-1} = -U^T(\lambda) = -(U(\lambda))^T,\ \Sigma U(\lambda)\Sigma^{-1} = -U^T(-\lambda) = -(U(-\lambda))^T, \tag{14}$$

where A^{-1} and A^T denote the inverse and the matrix transpose of matrix A, respectively. These group reductions generate the following relations for the potential matrix P:

$$\Delta P\Delta^{-1} = -P^T,\ \Sigma P\Sigma^{-1} = -P^T, \tag{15}$$

which are equivalent to

$$p^T = -\Delta_2 p\Delta_1^{-1},\ q^T = -\Delta_1 q\Delta_2^{-1}, \tag{16}$$

and

$$p^T = -\Sigma_2 q\Sigma_1^{-1},\ q^T = -\Sigma_1 p\Sigma_2^{-1}, \tag{17}$$

respectively. Under the symmetric condition of Σ and the condition

$$\Delta_1^{-1}\Sigma_1 = \Sigma_2^{-1}\Delta_2, \tag{18}$$

the constraints from the dual group reductions in (14) are compatible. The reduced AKNS matrix spectral problems are given by

$$-i\phi_x = U\phi,\ U = \begin{bmatrix} \frac{1}{2}\lambda I_n & p \\ -\Sigma_2^{-1}p^T\Sigma_1 & -\frac{1}{2}\lambda I_n \end{bmatrix}, \tag{19}$$

where p must satisfy the first constraint in (16), or equivalently,

$$-i\phi_x = U\phi,\ U = \begin{bmatrix} \frac{1}{2}\lambda I_n & -\Sigma_1^{-1}q^T\Sigma_2 \\ q & -\frac{1}{2}\lambda I_n \end{bmatrix}, \tag{20}$$

where q must satisfy the second constraint in (16).

2.3. Reduced matrix mKdV integrable hierarchies

Based on the same initial values:

$$\Delta W(\lambda)\Delta^{-1}|_{\lambda=\infty} = -(W(\lambda))^T|_{\lambda=\infty} = -\Omega,\ \Sigma W(\lambda)\Sigma^{-1}|_{\lambda=\infty} = (W(-\lambda))^T|_{\lambda=\infty} = \Omega, \tag{21}$$

the uniqueness of solutions to the stationary zero-curvature equation implies

$$\Delta W(\lambda)\Delta^{-1} = -W^T(\lambda) = -(W(\lambda))^T,\ \Sigma W(\lambda)\Sigma^{-1} = W^T(-\lambda) = (W(-\lambda))^T, \tag{22}$$

where W solves the stationary zero-curvature equation (6).

As a result, for each $s \geq 0$, we obtain:

$$\begin{cases} \Delta V^{[2s+1]}(\lambda)\Delta^{-1} = -V^{[2s+1]T}(\lambda) = -(V^{[2s+1]}(\lambda))^T, \\ \Sigma V^{[2s+1]}(\lambda)\Sigma^{-1} = -V^{[2s+1]T}(-\lambda) = -(V^{[2s+1]}(-\lambda))^T, \end{cases} \tag{23}$$

where $V^{[2s+1]} = (\lambda^{2s+1}W)_+$ defined as the polynomial part of $\lambda^{2s+1}W$, as previously stated. This guarantees the following invariance property:

$$\Delta(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Delta^{-1} = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda))^T, \tag{24}$$

and

$$\Sigma(U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(\lambda)\Sigma^{-1} = -((U_t - V_x^{[2s+1]} + i[U, V^{[2s+1]}])(-\lambda))^T. \tag{25}$$

Consequently, the matrix AKNS integrable models, as described by (9) with $r = 2s+1$, leads to the following integrable mKdV models:

$$p_t = 2ib^{[2s+2]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}, s \geq 0, \tag{26}$$

where p satisfies the first constraint in (16), or alternatively,

$$q_t = -2ic^{[2s+2]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2},\ s \geq 0, \tag{27}$$

where q satisfies the second constraint in (16). The corresponding Lax pair of matrix spectral problems consists of (19) and

$$-i\phi_t = V^{[2s+1]}|_{q=-\Sigma_2^{-1}p^T\Sigma_1}\phi,\ s \geq 0, \tag{28}$$

and alternatively, (20) and

$$-i\phi_t = V^{[2s+1]}|_{p=-\Sigma_1^{-1}q^T\Sigma_2}\phi,\ s \geq 0. \tag{29}$$

These equations describe the above reduced integrable mKdV hierarchy.

As a direct consequence of the Lax operator algebras, these reduced integrable models exhibit an infinite sequence of commuting symmetries. It is important to note that, since Δ_1, Δ_2 and Σ_1 are arbitrary, selecting specific forms for these matrices allows the construction of a diverse range of integrable mKdV models. One key requirement is that the two matrices Σ_1 and Σ_2 must be symmetric. These models serve as concrete examples of reduced matrix AKNS systems. However for $r = 2s$ with $s \geq 0$, the similarity properties outlined in (23) no longer apply, meaning that such reductions are not valid in this case.

3. DEMONSTRATIVE EXAMPLES

In this section, we investigate two representative cases by selecting different sets of dual group reductions. Each case serves to exemplify reduced matrix AKNS spectral problems and their corresponding mKdV integrable models. We focus on the specific setting where $m = n = 3$, with the spectral matrix defined as

$$U = U(u,\lambda) = \begin{bmatrix} \frac{1}{2}\lambda I_3 & p \\ q & -\frac{1}{2}\lambda I_3 \end{bmatrix}, \tag{30}$$

where the potential p satisfies the first constraint in (16), and q is determined by either the first or the second constraint in (17).

Example 3.1: We begin by examining the first case through the selection of the following specific pairs of matrices:

$$\Delta_1 = \Delta_2 = \begin{bmatrix} \delta_1 & \delta_3 & 0 \\ \delta_3 & \delta_2 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}; \ \Sigma_1 = \begin{bmatrix} 0 & 0 & \sigma_1 \\ 0 & \sigma_2 & 0 \\ \sigma_1 & 0 & 0 \end{bmatrix}, \ \Sigma_2 = \begin{bmatrix} \frac{\delta_3^2}{\sigma_2} & \frac{\delta_2\delta_3}{\sigma_2} & \frac{\delta_1^2}{\sigma_1} \\ \frac{\delta_2\delta_3}{\sigma_2} & \frac{\delta_2^2}{\sigma_2} & \frac{\delta_1\delta_3}{\sigma_1} \\ \frac{\delta_1^2}{\sigma_1} & \frac{\delta_1\delta_3}{\sigma_1} & 0 \end{bmatrix}; \tag{31}$$

where $\delta_i,\ 1 \leq i \leq 3$, and σ_1, σ_2 are arbitrary constants but need to satisfy

$$\delta_1(\delta_1\delta_2 - \delta_3^2)\sigma_1\sigma_2 \neq 0, \tag{32}$$

which confirms the invertibility of all involved matrices. After specifying these matrices and assuming $\delta_2 \neq 0$, the dual group reductions in (14) yield the explicit expressions for p and q:

$$p = \begin{bmatrix} \frac{\delta_3 p_2}{\delta_2} & p_2 & p_1 \\ -\frac{\delta_1 p_2}{\delta_2} & -\frac{\delta_3 p_2}{\delta_2} & p_3 \\ -\frac{\delta_1 p_1 + \delta_3 p_3}{\delta_1} & -\frac{\delta_3 p_1 + \delta_2 p_3}{\delta_1} & 0 \end{bmatrix}, \tag{33}$$

$$q = \begin{bmatrix} -\frac{\delta_3\sigma_1\sigma_2 p_3}{\delta_1\gamma} & -\frac{\delta_2\sigma_1\sigma_2 p_3}{\delta_1\gamma} & \frac{\sigma_1(\delta_1\delta_3\sigma_2 p_2 - \delta_2^2\sigma_1 p_1)}{\delta_1\delta_2\gamma} \\ \frac{\sigma_1\sigma_2 p_3}{\gamma} & \frac{\delta_3\sigma_1\sigma_2 p_3}{\delta_1\gamma} & \frac{\sigma_1(\delta_2\delta_3\sigma_1 p_1 - \delta_1^2\sigma_2 p_2)}{\delta_1\delta_2\gamma} \\ \frac{\sigma_1^2 p_1}{\delta_1^2} & \frac{\sigma_1\sigma_2 p_2}{\delta_1\delta_2} & 0 \end{bmatrix}, \tag{34}$$

where

$$\gamma = \delta_1\delta_2 - \delta_3^2. \tag{35}$$

Consequently, the reduced matrix mKdV integrable model system, with $u = (p_1, p_2, p_3)^T$, can be expressed as follows:

$$\begin{cases} p_{1,t} = p_{1,xxx} + \dfrac{3\sigma_1}{\delta_1^2\delta_2}\left[2(\delta_2\sigma_1 p_1^2 + \delta_1\sigma_2 p_2 p_3)p_{1,x} + \delta_1\sigma_2 p_1(p_2p_3)_x\right], \\ p_{2,t} = p_{2,xxx} + \dfrac{3\sigma_1}{\delta_1^2\delta_2}\left[\delta_2\sigma_1 p_1 p_2 p_{1,x} + (\delta_2\sigma_1 p_1^2 + 3\delta_1\sigma_2 p_2 p_3)p_{2,x} + \delta_1\sigma_2 p_2^2 p_{3,x}\right], \\ p_{3,t} = p_{3,xxx} + \dfrac{3\sigma_1}{\delta_1^2\delta_2}\left[\delta_2\sigma_1 p_1 p_3 p_{1,x} + \delta_1\sigma_2 p_3^2 p_{2,x} + (\delta_2\sigma_1 p_1^2 + 3\delta_1\sigma_2 p_2 p_3)p_{3,x}\right], \end{cases} \tag{36}$$

where $\delta_1, \delta_2, \sigma_1$ and σ_2 are arbitrary nonzero constants. It is noteworthy that the resulting integrable model is independent of the parameter δ_3.

Selecting

$$\delta_1 = 1,\ \delta_2 = \varepsilon = \pm 1,\ \delta_3 = 0,\ \sigma_1 = \sigma_2 = 1, \tag{37}$$

leads to the two potential matrices:

$$p = \begin{bmatrix} 0 & p_2 & p_1 \\ -\varepsilon p_2 & 0 & p_3 \\ -p_1 & -\varepsilon p_3 & 0 \end{bmatrix},\ q = \begin{bmatrix} 0 & -p_3 & -p_1 \\ \varepsilon p_3 & 0 & -p_2 \\ p_1 & \varepsilon p_2 & 0 \end{bmatrix}, \tag{38}$$

and the following simplified mKdV integrable model system:

$$\begin{cases} p_{1,t} = p_{1,xxx} + 6(p_1^2 + \varepsilon p_2 p_3)p_{1,x} + 3\varepsilon p_1 p_3 p_{2,x} + 3\varepsilon p_1 p_2 p_{3,x}, \\ p_{2,t} = p_{2,xxx} + 3p_1 p_2 p_{1,x} + 3p_1^2 p_{2,x} + 3\varepsilon(3p_2 p_3 p_{2,x} + p_2^2 p_{3,x}), \\ p_{3,t} = p_{3,xxx} + 3p_1 p_3 p_{1,x} + 3\varepsilon(p_3^2 p_{2,x} + 3p_2 p_3 p_{3,x}) + 3p_1^2 p_{3,x}, \end{cases} \tag{39}$$

where $\varepsilon = \pm 1$.

We note that the case $\delta_2 = 0$ does not yield new mKdV integrable models, but rather new Lax pairs.

Example 3.2: Secondly, we investigate the other scenario by introducing the following two specific matrix pairs:

$$\Delta_1 = \Delta_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & \delta_3 \\ 0 & \delta_3 & \delta_1 \end{bmatrix};\ \Sigma_1 = \begin{bmatrix} 0 & 0 & \sigma_1 \\ 0 & \sigma_2 & 0 \\ \sigma_1 & 0 & 0 \end{bmatrix},\ \Sigma_2 = \begin{bmatrix} 0 & \dfrac{\delta_1\delta_3}{\sigma_1} & \dfrac{\delta_1^2}{\sigma_1} \\ \dfrac{\delta_1\delta_3}{\sigma_1} & \dfrac{\delta_2^2}{\sigma_2} & \dfrac{\delta_2\delta_3}{\sigma_2} \\ \dfrac{\delta_1^2}{\sigma_1} & \dfrac{\delta_2\delta_3}{\sigma_2} & \dfrac{\delta_3^2}{\sigma_2} \end{bmatrix}; \tag{40}$$

where $\delta_i,\ 1 \le i \le 3$, and σ_1, σ_2 are arbitrary constants satisfying

$$\delta_1(\delta_1\delta_2 - \delta_3^2)\sigma_1\sigma_2 \neq 0. \tag{41}$$

With these matrices in place, the dual group reductions outlined in (14) provide the explicit formulas for for p and q:

$$p = \begin{bmatrix} 0 & p_2 & p_1 \\ -\dfrac{\delta_1(\delta_1 p_2 - \delta_3 p_1)}{\gamma} & \dfrac{\delta_3 p_3}{\delta_1} & p_3 \\ -\dfrac{\delta_1(\delta_2 p_1 - \delta_3 p_2)}{\gamma} & -\dfrac{\delta_2 p_3}{\delta_1} & -\dfrac{\delta_3 p_3}{\delta_1} \end{bmatrix}, \tag{42}$$

$$q=\begin{bmatrix} 0 & -\frac{\sigma_1\sigma_2 p_3}{\delta_1^2} & \frac{\sigma_1^2(\delta_3 p_2-\delta_2 p_1)}{\delta_1\gamma} \\ \frac{\sigma_1[\sigma_2\gamma p_3-\sigma_1\delta_3(\delta_2 p_1-\delta_3 p_2)]}{\gamma^2} & \frac{\sigma_1\sigma_2\delta_3(\delta_3 p_1-\delta_1 p_2)}{\gamma^2} & \frac{\sigma_1\sigma_2\delta_1(\delta_3 p_1-\delta_1 p_2)}{\gamma^2} \\ \frac{\sigma_1[\sigma_1\delta_1\delta_2(\delta_2 p_1-\delta_3 p_2)-\sigma_2\delta_3\gamma p_3]}{\delta_1\gamma^2} & \frac{\sigma_1\sigma_2\delta_2(\delta_1 p_2-\delta_3 p_1)}{\gamma^2} & \frac{\sigma_1\sigma_2\delta_3(\delta_1 p_2-\delta_3 p_1)}{\gamma^2} \end{bmatrix}, \tag{43}$$

where $\gamma=\delta_1\delta_2-\delta_3^2$, as defined in (35). As a result, the corresponding reduced matrix mKdV integrable model system, with $u=(p_1,p_2,p_3)^T$, assumes the following form:

$$\begin{cases} p_{1,t}=p_{1,xxx}+\frac{3\sigma_1}{\delta_1\gamma^2}\big\{\big[2\delta_1\delta_2^2\sigma_1 p_1^2-3\delta_3(\delta_1\delta_2\sigma_1 p_2+\gamma\sigma_2 p_3)p_1+\delta_1(\delta_3^2\sigma_1 p_2+2\gamma\sigma_2 p_3)p_2\big]p_{1,x} \\ \quad -p_1\big[\delta_1(\delta_2\delta_3\sigma_1 p_1-\delta_3^2\sigma_1 p_2-\gamma\sigma_2 p_3)p_{2,x}+\gamma\sigma_2(\delta_3 p_1-\delta_1 p_2)p_{3,x}\big]\big\}, \\ p_{2,t}=p_{2,xxx}+\frac{3\sigma_1}{\delta_1\gamma^2}\big\{\big[2\delta_1\delta_3^2\sigma_1 p_2^2-3\delta_1(\delta_2\delta_3\sigma_1 p_1-\gamma\sigma_2 p_3)p_2+p_1(\delta_1\delta_2^2\sigma_1 p_1-\delta_3\gamma\sigma_2 p_3)\big]p_{2,x} \\ \quad +p_2\big[(\delta_1\delta_2^2\sigma_1 p_1-\delta_1\delta_2\delta_3\sigma_1 p_2-\delta_3\gamma\sigma_2 p_3)p_{1,x}-\gamma\sigma_2(\delta_3 p_1-\delta_1 p_2)p_{3,x}\big]\big\}, \\ p_{3,t}=p_{3,xxx}+\frac{3\sigma_1}{\delta_1\gamma^2}\big\{\big[\delta_1\sigma_1(\delta_2 p_1-\delta_3 p_2)^2-3\gamma\sigma_2(\delta_3 p_1-\delta_1 p_2)\big]p_{3,x} \\ \quad +\big[\delta_1\delta_2\sigma_1(\delta_2 p_1-\delta_3 p_2)-\delta_3\gamma\sigma_2 p_3\big]p_3 p_{1,x}+\delta_1\big[\gamma\sigma_2 p_3-\delta_3\sigma_1(\delta_2 p_1-\delta_3 p_2)\big]p_3 p_{2,x}\big\}, \end{cases} \tag{44}$$

where $\gamma=\delta_1\delta_2-\delta_3^2$, and $\delta_1,\delta_2,\delta_3,\sigma_1,\sigma_2$ are arbitrary constants subject to the condition given in (41).

When taking

$$\delta_1=-\delta_3=1,\ \delta_2=0,\ \sigma_1=-\sigma_2=1, \tag{45}$$

we arrive at

$$p=\begin{bmatrix} 0 & p_2 & p_1 \\ p_1+p_2 & -p_3 & p_3 \\ p_2 & 0 & p_3 \end{bmatrix},\ q=\begin{bmatrix} 0 & p_3 & p_2 \\ p_2+p_3 & -p_1-p_2 & p_1+p_2 \\ p_3 & 0 & p_1+p_2 \end{bmatrix}, \tag{46}$$

and the simplified mKdV integrable model system:

$$\begin{cases} p_{1,t}=p_{1,xxx}+3(3p_1p_3+p_2^2+2p_2p_3)p_{1,x}+3p_1\big[(p_2+p_3)p_{2,x}+(p_1+p_2)p_{3,x}\big], \\ p_{2,t}=p_{2,xxx}+3(2p_1p_3+2p_2^2+3p_2p_3)p_{2,x}+3p_2\big[p_3p_{1,x}+(p_1+p_2)p_{3,x}\big], \\ p_{3,t}=p_{3,xxx}+3(3p_1p_3+p_2^2+3p_2p_3)p_{3,x}+3p_3\big[p_3p_{1,x}+(p_2+p_3)p_{2,x}\big]. \end{cases} \tag{47}$$

It is worth noting that computing the Lax matrix $V^{[3]}$, as defined in (11), is straightforward in both scenarios. This matrix constitutes the temporal component of the Lax pairs for the corresponding reduced mKdV integrable models. Moreover, one can derive the full integrable hierarchies described by (26) (or (27)), along with their associated Lax pairs, given by (19) and (28) (or (20) and (29)).

4. CONCLUSIONS

This paper explores two classes of dual group reductions and applies them to matrix AKNS spectral problems to generate reduced matrix mKdV integrable models. The corresponding reduced matrix AKNS spectral problems and reduced mKdV integrable models are explicitly derived. Dual group reductions provide a novel approach for generating integrable models, and the newly formulated matrix spectral problems offer valuable insights into the classification of mKdV-type integrable models (see, e.g., [22–24]).

The examples presented in this study highlight the versatility and depth of reduced Lax pairs in constructing integrable models. By applying various dual group reductions to the zero-curvature equations, a wide range of integrable reductions can be obtained (see, e.g., [25–27]). The selection of diagonal and off-diagonal block matrices in these transformations is crucial in shaping the structure of the resulting systems. Furthermore, these transformations contribute to the ongoing advancement of integrable models linked to higher-order matrix spectral problems, as explored in [28–33]. Comparative studies of these models with other integrable systems could offer valuable insights into their algebraic and geometric foundations. An exciting avenue for future research lies in exploring rich solution phenomena, such as rogue waves, lump solutions, and soliton waves (see, e.g., [34–38]). Techniques such as the inverse scattering transform and the Riemann-Hilbert method provide powerful tools for deriving these nonlinear wave solutions. In particular, Darboux transformations open up new possibilities for investigating intriguing nonlinear wave phenomena, with promising applications in applied mathematics and engineering sciences.

In summary, this research presents a robust framework for the formulation and detailed analysis of integrable models. The models developed offer valuable new insights into the classification of multi-component integrable systems within the Lax pair framework. These findings are expected to make significant contributions to future advancements and applications in both the physical and engineering sciences.

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