

SUFFICIENT CONDITIONS FOR A GRAPH WITH MINIMUM DEGREE TO HAVE A COMPONENT FACTOR

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Abstract. Let \mathcal{T}_k^r denote the set of trees T such that $i(T - S) \leq \frac{k}{r}|S|$ for any $S \subset V(T)$ and for any $e \in E(T)$ there exists a set $S^* \subset V(T)$ with $i((T - e) - S^*) > \frac{k}{r}|S^*|$, where $r < k$ are two positive integers. A $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k^r\}$ -factor of a graph G is a spanning subgraph of G , in which every component is isomorphic to an element in $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k^r\}$. Let $A(G)$ and $Q(G)$ denote the adjacency matrix and the signless Laplacian matrix of G , respectively. The adjacency spectral radius and the signless Laplacian spectral radius of G , denoted by $\rho(G)$ and $q(G)$, are the largest eigenvalues of $A(G)$ and $Q(G)$, respectively. In this paper, we study the connections between the spectral radius and the existence of a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k^r\}$ -factor in a graph. We first establish a tight sufficient condition involving the adjacency spectral radius to guarantee the existence of a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k^r\}$ -factor in a graph. Then we propose a tight signless Laplacian spectral radius condition for the existence of a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k^r\}$ -factor in a graph.

Keywords: graph, $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k^r\}$ -factor, minimum degree, adjacency spectral radius, signless Laplacian spectral radius.

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1. INTRODUCTION

In this paper, we deal with finite and undirected graphs which have neither loops nor multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G , respectively. The order of G is the number $n = |V(G)|$ of its vertices. The size of G is the number $e(G) = |E(G)|$ of its edges. For $v \in V(G)$, the degree of v in G is denoted by $d_G(v)$. Let $i(G)$ and $\delta(G)$ denote the number of isolated vertices and the minimum degree of G , respectively. For any $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S and $G - S$ is the subgraph of G induced by $V(G) - S$. For any $E' \subseteq E(G)$, let $G - E'$ denote the subgraph obtained from G by deleting E' . For convenience, write $G - v = G - \{v\}$ for $S = \{v\}$ and $G - e = G - \{e\}$ for $E' = \{e\}$. As usual, let P_n , C_n , K_n and $K_{1,n-1}$ denote the path, the circuit, the complete graph and the star of order n , respectively. A connected graph without circuits is called a tree, which is denoted by T . For any two positive integers k and r with $r < k$, let \mathcal{T}_k^r denote the set of trees T such that $i(T - S) \leq \frac{k}{r}|S|$ for any $S \subset V(T)$ and for any $e \in E(T)$ there exists a set $S^* \subset V(T)$ with $i((T - e) - S^*) > \frac{k}{r}|S^*|$. For two given graphs G_1 and G_2 , we denote by $G_1 \cup G_2$ their union. The join $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with each vertex of G_2 by an edge. Let c be a real number. Recall that $\lfloor c \rfloor$ is the greatest integer with $\lfloor c \rfloor \leq c$.

Let \mathcal{H} denote a set of connected graphs. A subgraph H of G is called an \mathcal{H} -factor of G if $V(H) = V(G)$ and each component of H is isomorphic to an element of \mathcal{H} . An \mathcal{H} -factor is also referred as a component

factor. An \mathcal{H} -factor is called a $P_{\geq k}$ -factor if $\mathcal{H} = \{P_k, P_{k+1}, \dots\}$. An \mathcal{H} -factor is called a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor if $\mathcal{H} = \{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$. An \mathcal{H} -factor means a star-factor in which every component is a star. Note that a perfect matching is indeed a $\{P_2\}$ -factor of G .

Kaneko [6] established a criterion for a graph with a $P_{\geq 3}$ -factor. Liu and Pan [12], Dai [2], Wu [24] provided some sufficient conditions for the existence of $P_{\geq 3}$ -factors in graphs. Ando et al [1] proved that a claw-free graph with minimum degree at least d contains a $P_{\geq d+1}$ -factor. Tutte [18] showed a necessary and sufficient condition for a graph to have a $\{K_2, C_i : i \geq 3\}$ -factor. Klopp and Steffen [9] investigated the existence of $\{K_{1,1}, K_{1,2}, C_i : i \geq 3\}$ -factors in graphs. Zhou, Xu and Sun [37] proposed some sufficient conditions for graphs to contain $\{K_{1,j} : 1 \leq j \leq k\}$ -factors. Kano and Saito [8] verified that a graph G satisfying $i(G-S) \leq \frac{1}{k}|S|$ for any $S \subset V(G)$ has a $\{K_{1,j} : k \leq j \leq 2k\}$ -factor. Kano, Lu and Yu [7] provided sufficient conditions using isolated vertices for component factors with every component of order at least three and proved that a graph G satisfying $i(G-S) \leq \frac{|S|}{2}$ for any $S \subset V(G)$ contains a $\{K_{1,2}, K_{1,3}, K_5\}$ -factor. Wolf [21] claimed a characterization using isolated vertices for a graph with a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor. For other sufficient conditions for the existence of graph factors in graphs, see [4, 19, 22, 25, 28, 30, 32].

Given a graph G with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, the adjacency matrix $A(G) = (a_{ij})_{n \times n}$ of G is a 0–1 matrix in which the entry $a_{ij} = 1$ if and only if $v_i v_j \in E(G)$. Let $D(G)$ denote the diagonal matrix of vertex degrees of G . The signless Laplacian matrix $Q(G)$ of G are defined by $Q(G) = D(G) + A(G)$. The largest eigenvalue of $A(G)$ is called the adjacency spectral radius of G , denoted by $\rho(G)$. The largest eigenvalue of $Q(G)$ is called the signless Laplacian spectral radius of G , denoted by $q(G)$.

O [16], Zhou, Sun and Zhang [35] proved two sharp upper bounds for the adjacency spectral radius in a graph without a $\{P_2\}$ -factor. Zhou and Zhang [38] gave a lower bound on the signless Laplacian spectral radius of G to guarantee that G contains a $\{P_2\}$ -factor. Zhou, Bian and Sun [31], Zhou, Zhang and Sun [39] presented some spectral radius conditions for graphs to possess $P_{\geq 2}$ -factors. Liu, Fan and Shu [11], Wu [23], Wang and Zhang [20], Zhou [27], Zhou, Sun and Liu [34], Zhou and Wu [36] provided some spectral radius conditions for the existence of spanning trees in connected graphs. Zhang and You [26] obtained a lower bound on the A_α -spectral radius of a graph G to guarantee that G has a $\{K_{1,2}, K_{1,3}, K_5\}$ -factor. Zhou [29] proposed two spectral radius conditions for bipartite graphs to have star-factors. Zhou and Liu [33] put forward a lower bound on the A_α -spectral radius for a connected graph to possess a $\{K_{1,j} : m \leq j \leq 2m\}$ -factor. Lv, Li and Xu [13] showed a sufficient condition involving the A_α -spectral radius for a graph to have a $\{K_2, C_{2i+1} : i \geq 1\}$ -factor, and gave a distance signless Laplacian spectral radius condition for a graph to have a $\{K_2, C_{2i+1} : i \geq 1\}$ -factor. Miao and Li [15] obtained some sufficient conditions involving the adjacency spectral radius and the distance spectral radius for the existence of $\{K_{1,j} : 1 \leq j \leq k\}$ -factors in graphs. Lv, Li and Xu [14] proposed a tight sufficient condition, expressed in terms of the A_α -spectral radius, for the existence of $\{K_{1,1}, K_{1,2}, \dots, K_{1,k}, \mathcal{T}(2k+1)\}$ -factors in graphs.

Motivated by [21] directly, we first propose an adjacency spectral radius condition for a connected graph with minimum degree δ to have a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor, then we obtain a signless Laplacian spectral radius condition for a connected graph with minimum degree δ to have a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor.

THEOREM 1.1. *Let k and r be two positive integers with $r < k$, and let G be a connected graph of order n with $\delta(G) = \delta$ and $n \geq \max \left\{ \frac{(k+r)(k+2r)(k\delta+k+r)}{k^2 r}, \frac{2kr\delta^2 + (2k^2 + kr + 2r^2)\delta + k^2 + 3kr - 2r^2}{2r(k-r)} \right\}$. If*

$$\rho(G) \geq \rho \left(K_\delta \vee \left(K_{n - \lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup \left(\lfloor \frac{k\delta}{r} \rfloor + 1 \right) K_1 \right) \right),$$

then G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor unless $G = K_\delta \vee (K_{n - \lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1) K_1)$. S

THEOREM 1.2. *Let k and r be two positive integers with $r < k$, and let G be a connected graph of order n*

with $\delta(G) = \delta$ and $n \geq \max \left\{ \frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r}, \frac{(k^2+2kr)\delta^2+(2k^2+3kr+2r^2)\delta+k^2+3kr}{2r(k-r)} \right\}$. If

$$q(G) \geq q\left(K_\delta \vee \left(K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup \left(\lfloor \frac{k\delta}{r} \rfloor + 1\right)K_1\right)\right),$$

then G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor unless $G = K_\delta \vee (K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$.

2. PRELIMINARY LEMMAS

In this section, we show some lemmas, which will be used to verify our main results. Wolf [21] claimed a characterization for a graph with a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor.

LEMMA 2.1 (Wolf [21]). *Let k and r be two positive integers with $r < k$, and let G be a graph. Then G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor if and only if*

$$i(G-S) \leq \frac{k}{r}|S|$$

for any $S \subset V(G)$.

LEMMA 2.2 (Li and Feng [10]). *Let G be a connected graph and let H be a subgraph of G . Then*

$$\rho(G) \geq \rho(H),$$

with equality if and only if $G = H$.

LEMMA 2.3 (Hong [5]). *Let G be a graph with n vertices. Then*

$$\rho(G) \leq \sqrt{2e(G) - n + 1},$$

where the equality holds if and only if G is a star or a complete graph.

LEMMA 2.4 (Shen, You, Zhang and Li [17]). *Let G be a connected graph. If H is a subgraph of G , then*

$$q(G) \geq q(H),$$

with equality holding if and only if $G = H$.

LEMMA 2.5 (Das [3]). *Let G be a graph of order n . Then*

$$q(G) \leq \frac{2e(G)}{n-1} + n - 2.$$

3. THE PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Assume that G has no $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor. By virtue of Lemma 2.1, there exists some nonempty subset S of $V(G)$ such that

$$i(G-S) > \frac{k}{r}|S|.$$

In terms of the integrity of $i(G-S)$, we possess

$$i(G-S) \geq \left\lfloor \frac{k}{r} |S| \right\rfloor + 1.$$

Let $|S| = s$. Then G is a spanning subgraph of $G_1 = K_s \vee (K_{n-\lfloor \frac{ks}{r} \rfloor - s - 1} \cup (\lfloor \frac{ks}{r} \rfloor + 1)K_1)$. Together with Lemma 2.2, we deduce

$$\rho(G) \leq \rho(G_1), \quad (1)$$

where the equality holds if and only if $G = G_1$. Notice that $\delta(G) = \delta$ and $\delta(G_1) \geq \delta(G)$. Thus, we get $s = \delta(G_1) \geq \delta(G) = \delta$. The following proof will be divided into two cases according to the value of s .

Case 1. $s = \delta$.

In this case, $G_1 = K_\delta \vee (K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$. Together with (1), we conclude

$$\rho(G) \leq \rho\left(K_\delta \vee \left(K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup \left(\left\lfloor \frac{k\delta}{r} \right\rfloor + 1\right)K_1\right)\right),$$

with equality holding if and only if $G = K_\delta \vee (K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$. Observe that $K_\delta \vee (K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$ has no $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor. Thus, we can get a contradiction.

Case 2. $s \geq \delta + 1$.

Recall that $G_1 = K_s \vee (K_{n-\lfloor \frac{ks}{r} \rfloor - s - 1} \cup (\lfloor \frac{ks}{r} \rfloor + 1)K_1)$. By virtue of Lemma 2.3, $\frac{ks}{r} - 1 < \lfloor \frac{ks}{r} \rfloor \leq \frac{ks}{r}$ and $n \geq \lfloor \frac{ks}{r} \rfloor + s + 1 > \frac{ks}{r} + s$, we obtain

$$\begin{aligned} \rho(G_1) &\leq \sqrt{2e(G_1) - n + 1} \\ &= \sqrt{2\binom{n - \lfloor \frac{ks}{r} \rfloor - 1}{2} + 2s\left(\lfloor \frac{ks}{r} \rfloor + 1\right) - n + 1} \\ &= \sqrt{\left(n - \lfloor \frac{ks}{r} \rfloor - 1\right)\left(n - \lfloor \frac{ks}{r} \rfloor - 2\right) + 2s\left(\lfloor \frac{ks}{r} \rfloor + 1\right) - n + 1} \\ &< \sqrt{\left(n - \left(\frac{ks}{r} - 1\right) - 1\right)\left(n - \left(\frac{ks}{r} - 1\right) - 2\right) + 2s\left(\lfloor \frac{ks}{r} \rfloor + 1\right) - n + 1} \\ &= \frac{1}{r} \sqrt{(k^2 + 2kr)s^2 - (2krn - 2r^2 - kr)s + r^2n^2 - 2r^2n + r^2}. \end{aligned} \quad (2)$$

Let $f(s) = (k^2 + 2kr)s^2 - (2krn - 2r^2 - kr)s + r^2n^2 - 2r^2n + r^2$. Since $n \geq \lfloor \frac{ks}{r} \rfloor + s + 1 > \frac{ks}{r} + s$, we possess $\delta + 1 \leq s < \frac{rn}{k+r}$. By a direct computation, we get

$$\begin{aligned} f(\delta + 1) - f\left(\frac{rn}{k+r}\right) &= (k^2 + 2kr)(\delta + 1)^2 - (2krn - 2r^2 - kr)(\delta + 1) + r^2n^2 - 2r^2n + r^2 \\ &\quad - \left((k^2 + 2kr)\left(\frac{rn}{k+r}\right)^2 - (2krn - 2r^2 - kr)\left(\frac{rn}{k+r}\right) + r^2n^2 - 2r^2n + r^2\right) \\ &= \left(\frac{rn}{k+r} - \delta - 1\right)\left(\frac{k^2rn}{k+r} - (k+2r)(k\delta + k+r)\right) \\ &> 0, \end{aligned}$$

where the inequality holds from the fact that

$$n > \max\left\{\frac{(k+r)(k+2r)(k\delta + k+r)}{k^2r}, \frac{2kr\delta^2 + (2k^2 + kr + 2r^2)\delta + k^2 + 3kr - 2r^2}{2r(k-r)}\right\}$$

$$\begin{aligned} &\geq \frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r} \\ &> \frac{(k+r)(\delta+1)}{r}. \end{aligned}$$

This implies that, for $\delta+1 \leq s < \frac{rn}{k+r}$, the function $f(s)$ attains its maximum value at $s = \delta+1$. Combining this with (1), (2) and $n > \max \left\{ \frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r}, \frac{2kr\delta^2+(2k^2+kr+2r^2)\delta+k^2+3kr-2r^2}{2r(k-r)} \right\} \geq \frac{2kr\delta^2+(2k^2+kr+2r^2)\delta+k^2+3kr-2r^2}{2r(k-r)}$, we obtain

$$\begin{aligned} \rho(G) &\leq \rho(G_1) \\ &< \frac{1}{r} \sqrt{f(\delta+1)} \\ &= \frac{1}{r} \sqrt{(k^2+2kr)(\delta+1)^2 - (2krn-2r^2-kr)(\delta+1) + r^2n^2 - 2r^2n + r^2} \\ &= \frac{1}{r} \sqrt{(rn-k\delta-2r)^2 - 2r(k-r)n + 2kr\delta^2 + (2k^2+kr+2r^2)\delta + k^2 + 3kr - 2r^2} \\ &< \frac{1}{r} (rn - k\delta - 2r). \end{aligned} \tag{3}$$

Since $K_{n-\lfloor \frac{k\delta}{r} \rfloor - 1}$ is a proper subgraph of $K_\delta \vee (K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$, it follows from Lemma 2.2, $\lfloor \frac{k\delta}{r} \rfloor \leq \frac{k\delta}{r}$ and the hypothesis of the theorem that

$$\begin{aligned} \rho(G) &\geq \rho\left(K_\delta \vee \left(K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup \left(\left\lfloor \frac{k\delta}{r} \right\rfloor + 1\right)K_1\right)\right) \\ &> \rho(K_{n-\lfloor \frac{k\delta}{r} \rfloor - 1}) \\ &= n - \left\lfloor \frac{k\delta}{r} \right\rfloor - 2 \\ &\geq n - \frac{k\delta}{r} - 2 \\ &= \frac{1}{r} (rn - k\delta - 2r), \end{aligned}$$

which leads to a contradiction to (3). Theorem 1.1 is proved. \square

4. THE PROOF OF THEOREM 1.2

Proof of Theorem 1.2. Assume that G has no $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_{\frac{k}{r}}\}$ -factor. Then using Lemma 2.1, there exists some nonempty subset S of $V(G)$ such that

$$i(G-S) > \frac{k}{r}|S|.$$

According to the integrity of $i(G-S)$, we obtain

$$i(G-S) \geq \left\lfloor \frac{k}{r}|S| \right\rfloor + 1.$$

Let $|S| = s$. Then G is a spanning subgraph of $G_1 = K_s \vee (K_{n-\lfloor \frac{ks}{r} \rfloor - s - 1} \cup (\lfloor \frac{ks}{r} \rfloor + 1)K_1)$. Together with Lemma 2.4, we possess

$$q(G) \leq q(G_1), \tag{4}$$

where the equality holds if and only if $G = G_1$. Note that $\delta(G) = \delta$ and $\delta(G_1) = s \geq \delta(G)$. Thus, we get $s \geq \delta$. In what follows, we shall consider two cases by the value of s .

Case 1. $s = \delta$.

In this case, $G_1 = K_\delta \vee (K_{n - \lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$. In terms of (4), we obtain

$$q(G) \leq q\left(K_\delta \vee \left(K_{n - \lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup \left(\left\lfloor \frac{k\delta}{r} \right\rfloor + 1\right)K_1\right)\right),$$

where the equality holds if and only if $G = K_\delta \vee (K_{n - \lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$. Observe that $K_\delta \vee (K_{n - \lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$ contains no $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor. Thus, we can obtain a contradiction.

Case 2. $s \geq \delta + 1$.

Recall that $G_1 = K_s \vee (K_{n - \lfloor \frac{ks}{r} \rfloor - s - 1} \cup (\lfloor \frac{ks}{r} \rfloor + 1)K_1)$. It follows from Lemma 2.5, $\frac{ks}{r} - 1 < \lfloor \frac{ks}{r} \rfloor \leq \frac{ks}{r}$ and $n \geq \lfloor \frac{ks}{r} \rfloor + s + 1 > \frac{ks}{r} + s$ that

$$\begin{aligned} q(G_1) &\leq \frac{2e(G_1)}{n-1} + n - 2 \\ &= \frac{2^{(n - \lfloor \frac{ks}{r} \rfloor - 1)} + 2s \left(\left\lfloor \frac{ks}{r} \right\rfloor + 1\right)}{n-1} + n - 2 \\ &= \frac{\left(n - \left\lfloor \frac{ks}{r} \right\rfloor - 1\right) \left(n - \left\lfloor \frac{ks}{r} \right\rfloor - 2\right) + 2s \left(\left\lfloor \frac{ks}{r} \right\rfloor + 1\right)}{n-1} + n - 2 \\ &< \frac{\left(n - \left(\frac{ks}{r} - 1\right) - 1\right) \left(n - \left(\frac{ks}{r} - 1\right) - 2\right) + 2s \left(\frac{ks}{r} + 1\right)}{n-1} + n - 2 \\ &= \frac{\left(n - \frac{ks}{r}\right) \left(n - \frac{ks}{r} - 1\right) + 2s \left(\frac{ks}{r} + 1\right)}{n-1} + n - 2 \\ &= \frac{(k^2 + 2kr)s^2 - (2krn - kr - 2r^2)s + 2r^2n^2 - 4r^2n + 2r^2}{r^2(n-1)}. \end{aligned} \quad (5)$$

Let $g(s) = (k^2 + 2kr)s^2 - (2krn - kr - 2r^2)s + 2r^2n^2 - 4r^2n + 2r^2$. Since $n \geq \lfloor \frac{ks}{r} \rfloor + s + 1 > \frac{ks}{r} + s$, we deduce $\delta + 1 \leq s < \frac{m}{k+r}$. By a simple computation, we obtain

$$\begin{aligned} g(\delta + 1) - g\left(\frac{rn}{k+r}\right) &= (k^2 + 2kr)(\delta + 1)^2 - (2krn - kr - 2r^2)(\delta + 1) + 2r^2n^2 - 4r^2n + 2r^2 \\ &\quad - \left((k^2 + 2kr)\left(\frac{rn}{k+r}\right)^2 - (2krn - kr - 2r^2)\left(\frac{rn}{k+r}\right) + 2r^2n^2 - 4r^2n + 2r^2\right) \\ &= \left(\frac{rn}{k+r} - \delta - 1\right) \left(\frac{k^2rn}{k+r} - (k+2r)(k\delta + k+r)\right) \\ &> 0, \end{aligned}$$

where the inequality holds from the fact that

$$\begin{aligned} n &> \max \left\{ \frac{(k+r)(k+2r)(k\delta + k+r)}{k^2r}, \frac{(k^2 + 2kr)\delta^2 + (2k^2 + 3kr + 2r^2)\delta + k^2 + 3kr}{2r(k-r)} \right\} \\ &\geq \frac{(k+r)(k+2r)(k\delta + k+r)}{k^2r} \\ &> \frac{(k+r)(\delta + 1)}{r}. \end{aligned}$$

This implies that, for $\delta + 1 \leq s < \frac{rn}{k+r}$, the function $g(s)$ attains its maximum value at $s = \delta + 1$. Combining this with (4), (5) and $n > \max \left\{ \frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r}, \frac{(k^2+2kr)\delta^2+(2k^2+3kr+2r^2)\delta+k^2+3kr}{2r(k-r)} \right\} \geq \frac{(k^2+2kr)\delta^2+(2k^2+3kr+2r^2)\delta+k^2+3kr}{2r(k-r)}$, we conclude

$$\begin{aligned}
q(G) &\leq q(G_1) \\
&< \frac{g(\delta+1)}{r^2(n-1)} \\
&= \frac{(k^2+2kr)(\delta+1)^2 - (2krn - kr - 2r^2)(\delta+1) + 2r^2n^2 - 4r^2n + 2r^2}{r^2(n-1)} \\
&= \frac{2(rn - k\delta - 2r)}{r} - \frac{2r(k-r)n - (k^2+2kr)\delta^2 - (2k^2+3kr+2r^2)\delta - k^2 - 3kr}{r^2(n-1)} \\
&< \frac{2(rn - k\delta - 2r)}{r}.
\end{aligned} \tag{6}$$

Note that $K_\delta \vee (K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$ contains $K_{n-\lfloor \frac{k\delta}{r} \rfloor - 1}$ as a proper subgraph. Together with Lemma 2.4, $\lfloor \frac{k\delta}{r} \rfloor \leq \frac{k\delta}{r}$ and the assumption of the theorem, we possess

$$\begin{aligned}
q(G) &\geq q\left(K_\delta \vee \left(K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup \left(\lfloor \frac{k\delta}{r} \rfloor + 1\right)K_1\right)\right) \\
&> q(K_{n-\lfloor \frac{k\delta}{r} \rfloor - 1}) \\
&= 2\left(n - \left\lfloor \frac{k\delta}{r} \right\rfloor - 2\right) \\
&\geq 2\left(n - \frac{k\delta}{r} - 2\right) \\
&= \frac{2(rn - k\delta - 2r)}{r},
\end{aligned}$$

which is to a contradiction to (6). This completes the proof of Theorem 1.2. \square

5. CONCLUDING REMARKS

In this paper, we provide two sufficient conditions to ensure that a connected graph G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor in terms of its adjacency spectral radius and signless Laplacian spectral radius. It is natural and interesting to propose some other spectral sufficient conditions to guarantee that a connected graph G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathcal{T}_k\}$ -factor. It is also natural and interesting to put forward some spectral sufficient conditions to ensure that a connected graph G has some other substructure.

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