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SUFFICIENT CONDITIONS FOR A GRAPH WITH MINIMUM DEGREE TO HAVE A COMPONENT FACTOR

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Abstract. Let $\mathscr{T}_{\frac{k}{r}}$ denote the set of trees T such that $i(T-S) \le \frac{k}{r}|S|$ for any $S \subset V(T)$ and for any $e \in E(T)$ there exists a set $S^* \subset V(T)$ with $i((T-e)-S^*) > \frac{k}{r}|S^*|$, where $r < k$ are two positive integers. A $\{C_{2i+1}, T : 1 \le i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor of a graph G is a spanning subgraph of G, in which every component is isomorphic to an element in $\{C_{2i+1}, T : 1 \le i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$. Let $A(G)$ and $Q(G)$ denote the adjacency matrix and the signless Laplacian matrix of G, respectively. The adjacency spectral radius and the signless Laplacian spectral radius of G, denoted by $\rho(G)$ and $q(G)$, are the largest eigenvalues of $A(G)$ and $Q(G)$, respectively. In this paper, we study the connections between the spectral radius and the existence of a $\{C_{2i+1}, T : 1 \le i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor in a graph. We first establish a tight sufficient condition involving the adjacency spectral radius to guarantee the existence of a $\{C_{2i+1}, T : 1 \le i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor in a graph. Then we propose a tight signless Laplacian spectral radius condition for the existence of a $\{C_{2i+1}, T : 1 \le i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor in a graph.

Keywords: graph, $\{C_{2i+1}, T : 1 \le i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor, minimum degree, adjacency spectral radius, signless Laplacian spectral radius.

Mathematics Subject Classification (MSC2020): 05C50, 05C70, 90B99

1. INTRODUCTION

In this paper, we deal with finite and undirected graphs which have neither loops nor multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. The order of G is the number $n = |V(G)|$ of its vertices. The size of G is the number $e(G) = |E(G)|$ of its edges. For $v \in V(G)$, the degree of v in G is denoted by $d_G(v)$. Let $i(G)$ and $\delta(G)$ denote the number of isolated vertices and the minimum degree of G, respectively. For any $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S and $G-S$ is the subgraph of G induced by $V(G)-S$. For any $E' \subseteq E(G)$, let $G-E'$ denote the subgraph obtained from G by deleting E'. For convenience, write $G-v = G-\{v\}$ for $S=\{v\}$ and $G-e = G-\{e\}$ for $E' = \{e\}$. As usual, let P_n, C_n, K_n and $K_{1,n-1}$ denote the path, the circuit, the complete graph and the star of order n, respectively. A connected graph without circuits is called a tree, which is denoted by T. For any two positive integers k and r with $r < k$, let $\mathscr{T}_{\frac{k}{r}}$ denote the set of trees T such that $i(T-S) \le \frac{k}{r}|S|$ for any $S \subset V(T)$ and for any $e \in E(T)$ there exists a set $S^* \subset V(T)$ with $i((T-e)-S^*) > \frac{k}{r}|S^*|$. For two given graphs G_1 and G_2, we denote by $G_1 \cup G_2$ their union. The join $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with each vertex of G_2 by an edge. Let c be a real number. Recall that $\lfloor c \rfloor$ is the greatest integer with $\lfloor c \rfloor \le c$.

Let $\mathscr{H}$ denote a set of connected graphs. A subgraph H of G is called an $\mathscr{H}$-factor of G if $V(H) = V(G)$ and each component of H is isomorphic to an element of $\mathscr{H}$. An $\mathscr{H}$-factor is also referred as a component factor. An $\mathscr{H}$-factor is called a $P_{\geq k}$-factor if $\mathscr{H} = \{P_k, P_{k+1}, \ldots\}$. An $\mathscr{H}$-factor is called a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor if $\mathscr{H} = \{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$. An $\mathscr{H}$-factor means a star-factor in which every component is a star. Note that a perfect matching is indeed a $\{P_2\}$-factor of G.

Kaneko [6] established a criterion for a graph with a $P_{\geq 3}$-factor. Liu and Pan [11], Dai [2], Wu [22] provided some sufficient conditions for the existence of $P_{\geq 3}$-factors in graphs. Ando et al [1] proved that a claw-free graph with minimum degree at least d contains a $P_{\geq d+1}$-factor. Tutte [16] showed a necessary and sufficient condition for a graph to have a $\{K_2, C_i : i \geq 3\}$-factor. Klopp and Steffen [9] investigated the existence of $\{K_{1,1}, K_{1,2}, C_i : i \geq 3\}$-factors in graphs. Zhou, Xu and Sun [32] proposed some sufficient conditions for graphs to contain $\{K_{1,j} : 1 \leq j \leq k\}$-factors. Kano and Saito [8] verified that a graph G satisfying $i(G-S) \leq \frac{1}{k}|S|$ for any $S \subset V(G)$ has a $\{K_{1,j} : k \leq j \leq 2k\}$-factor. Kano, Lu and Yu [7] provided sufficient conditions using isolated vertices for component factors with every component of order at least three and proved that a graph G satisfying $i(G-S) \leq \frac{|S|}{2}$ for any $S \subset V(G)$ contains a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Wolf [19] claimed a characterization using isolated vertices for a graph with a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor. For other sufficient conditions for the existence of graph factors in graphs, see [4, 17, 20, 23, 25, 30, 34].

Given a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the adjacency matrix $A(G) = (a_{ij})_{n\times n}$ of G is a 0–1 matrix in which the entry $a_{ij} = 1$ if and only if $v_iv_j \in E(G)$. Let $D(G)$ denote the diagonal matrix of vertex degrees of G. The signless Laplacian matrix $Q(G)$ of G are defined by $Q(G) = D(G) + A(G)$. The largest eigenvalue of $A(G)$ is called the adjacency spectral radius of G, denoted by $\rho(G)$. The largest eigenvalue of $Q(G)$ is called the signless Laplacian spectral radius of G, denoted by $q(G)$.

O [14], Zhou, Sun and Zhang [29] proved two sharp upper bounds for the adjacency spectral radius in a graph without a $\{P_2\}$-factor. Zhou and Zhang [33] gave a lower bound on the signless Laplacian spectral radius of G to guarantee that G contains a $\{P_2\}$-factor. Zhou, Sun and Liu [28], Zhou, Zhang and Sun [35] presented two spectral radius conditions for graphs to possess $P_{\geq 2}$-factors. Wu [21], Wang and Zhang [18], Zhou, Sun and Liu [27], Zhou and Wu [31] provided some spectral radius conditions for the existence of spanning trees in connected graphs. Zhou [24] proposed two spectral radius conditions for bipartite graphs to have star-factors. Zhou and Liu [26] put forward a lower bound on the A_a-spectral radius for a connected graph to possess a $\{K_{1,j} : m \leq j \leq 2m\}$-factor. Lv, Li and Xu [12] showed a sufficient condition involving the A_α-spectral radius for a graph to have a $\{K_2, C_{2i+1} : i \geq 1\}$-factor, and gave a distance signless Laplacian spectral radius condition for a graph to have a $\{K_2, C_{2i+1} : i \geq 1\}$-factor. Miao and Li [13] obtained some sufficient conditions involving the adjacency spectral radius and the distance spectral radius for the existence of $\{K_{1,j} : 1 \leq j \leq k\}$-factors in graphs.

Motivated by [19] directly, we first propose an adjacency spectral radius condition for a connected graph with minimum degree δ to have a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor, then we obtain a signless Laplacian spectral radius condition for a connected graph with minimum degree δ to have a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor.

Theorem 1.1. Let k and r be two positive integers with $r < k$, and let G be a connected graph of order n with $\delta(G) = \delta$ and $n \geq \max\left\{\frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r}, \frac{2kr\delta^2+(2k^2+kr+2r^2)\delta+k^2+3kr-2r^2}{2r(k-r)}\right\}$. If

$$\rho(G) \geq \rho\Big(K_\delta \vee \Big(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1} \cup \Big(\Big\lfloor\frac{k\delta}{r}\Big\rfloor+1\Big)K_1\Big)\Big),$$

then G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor unless $G = K_\delta \vee (K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1} \cup (\lfloor\frac{k\delta}{r}\rfloor+1)K_1)$.

Theorem 1.2. Let k and r be two positive integers with $r < k$, and let G be a connected graph of order n with $\delta(G) = \delta$ and $n \geq \max\left\{\frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r}, \frac{(k^2+2kr)\delta^2+(2k^2+3kr+2r^2)\delta+k^2+3kr}{2r(k-r)}\right\}$. If

$$q(G) \geq q\Big(K_\delta \vee \Big(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1} \cup \Big(\Big\lfloor\frac{k\delta}{r}\Big\rfloor+1\Big)K_1\Big)\Big),$$

then G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor unless $G = K_\delta \vee (K_{n-\lfloor \frac{k\delta}{r} \rfloor - \delta - 1} \cup (\lfloor \frac{k\delta}{r} \rfloor + 1)K_1)$.

2. PRELIMINARY LEMMAS

In this section, we show some lemmas, which will be used to verify our main results. Wolf [19] claimed a characterization for a graph with a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor.

Lemma 2.1 (Wolf [19]). Let k and r be two positive integers with $r < k$, and let G be a graph. Then G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor if and only if

$$i(G-S) \leq \frac{k}{r}|S|$$

for any $S \subset V(G)$.

Lemma 2.2 (Li and Feng [10]). Let G be a connected graph and let H be a subgraph of G. Then

$$\rho(G) \geq \rho(H),$$

with equality if and only if $G = H$.

Lemma 2.3 (Hong [5]). Let G be a graph with n vertices. Then

$$\rho(G) \leq \sqrt{2e(G) - n + 1},$$

where the equality holds if and only if G is a star or a complete graph.

Lemma 2.4 (Shen, You, Zhang and Li [15]). Let G be a connected graph. If H is a subgraph of G, then

$$q(G) \geq q(H),$$

with equality holding if and only if $G = H$.

Lemma 2.5 (Das [3]). Let G be a graph of order n. Then

$$q(G) \leq \frac{2e(G)}{n-1} + n - 2.$$

3. THE PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Assume that G has no $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor. By virtue of Lemma 2.1, there exists some nonempty subset S of $V(G)$ such that

$$i(G-S) > \frac{k}{r}|S|.$$

In terms of the integrity of $i(G-S)$, we possess

$$i(G-S) \geq \left\lfloor \frac{k}{r}|S| \right\rfloor + 1.$$

Let $|S|=s$. Then G is a spanning subgraph of $G_1=K_s\vee(K_{n-\lfloor\frac{ks}{r}\rfloor-s-1}\cup(\lfloor\frac{ks}{r}\rfloor+1)K_1)$. Together with Lemma 2.2, we deduce

$$\rho(G)\leq\rho(G_1), \tag{1}$$

where the equality holds if and only if $G=G_1$. Notice that $\delta(G)=\delta$ and $\delta(G_1)\geq\delta(G)$. Thus, we get $s=\delta(G_1)\geq\delta(G)=\delta$. The following proof will be divided into two cases according to the value of s.

Case 1. $s=\delta$.

In this case, $G_1=K_\delta\vee(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup(\lfloor\frac{k\delta}{r}\rfloor+1)K_1)$. Together with (1), we conclude

$$\rho(G)\leq\rho\Big(K_\delta\vee\Big(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup\Big(\Big\lfloor\frac{k\delta}{r}\Big\rfloor+1\Big)K_1\Big)\Big),$$

with equality holding if and only if $G=K_\delta\vee(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup(\lfloor\frac{k\delta}{r}\rfloor+1)K_1)$. Observe that $K_\delta\vee(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup(\lfloor\frac{k\delta}{r}\rfloor+1)K_1)$ has no $\{C_{2i+1},T:1\leq i<\frac{r}{k-r},T\in\mathscr{T}_{\frac{k}{r}}\}$-factor. Thus, we can get a contradiction.

Case 2. $s\geq\delta+1$.

Recall that $G_1=K_s\vee(K_{n-\lfloor\frac{ks}{r}\rfloor-s-1}\cup(\lfloor\frac{ks}{r}\rfloor+1)K_1)$. By virtue of Lemma 2.3, $\frac{ks}{r}-1<\lfloor\frac{ks}{r}\rfloor\leq\frac{ks}{r}$ and $n\geq\lfloor\frac{ks}{r}\rfloor+s+1>\frac{ks}{r}+s$, we obtain

$$\begin{aligned}\rho(G_1)\leq&\sqrt{2e(G_1)-n+1}\\=&\sqrt{2\binom{n-\lfloor\frac{ks}{r}\rfloor-1}{2}+2s\Big(\Big\lfloor\frac{ks}{r}\Big\rfloor+1\Big)-n+1}\\=&\sqrt{\Big(n-\Big\lfloor\frac{ks}{r}\Big\rfloor-1\Big)\Big(n-\Big\lfloor\frac{ks}{r}\Big\rfloor-2\Big)+2s\Big(\Big\lfloor\frac{ks}{r}\Big\rfloor+1\Big)-n+1}\\<&\sqrt{\Big(n-\Big(\frac{ks}{r}-1\Big)-1\Big)\Big(n-\Big(\frac{ks}{r}-1\Big)-2\Big)+2s\Big(\Big\lfloor\frac{ks}{r}\Big\rfloor+1\Big)-n+1}\\=&\frac{1}{r}\sqrt{(k^2+2kr)s^2-(2krn-2r^2-kr)s+r^2n^2-2r^2n+r^2}.\end{aligned}\tag{2}$$

Let $f(s)=(k^2+2kr)s^2-(2krn-2r^2-kr)s+r^2n^2-2r^2n+r^2$. Since $n\geq\lfloor\frac{ks}{r}\rfloor+s+1>\frac{ks}{r}+s$, we possess $\delta+1\leq s<\frac{rn}{k+r}$. By a direct computation, we get

$$\begin{aligned}f(\delta+1)-f\Big(\frac{rn}{k+r}\Big)=&(k^2+2kr)(\delta+1)^2-(2krn-2r^2-kr)(\delta+1)+r^2n^2-2r^2n+r^2\\&-\Big((k^2+2kr)\Big(\frac{rn}{k+r}\Big)^2-(2krn-2r^2-kr)\Big(\frac{rn}{k+r}\Big)+r^2n^2-2r^2n+r^2\Big)\\=&\Big(\frac{rn}{k+r}-\delta-1\Big)\Big(\frac{k^2rn}{k+r}-(k+2r)(k\delta+k+r)\Big)\\>&0,\end{aligned}$$

where the inequality holds from the fact that

$$\begin{aligned}n>&\max\Big\{\frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r},\frac{2kr\delta^2+(2k^2+kr+2r^2)\delta+k^2+3kr-2r^2}{2r(k-r)}\Big\}\\\geq&\frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r}\\>&\frac{(k+r)(\delta+1)}{r}.\end{aligned}$$

This implies that, for $\delta+1\leq s<\frac{rn}{k+r}$, the function $f(s)$ attains its maximum value at $s=\delta+1$. Combining this with (1), (2) and $n>\max\left\{\frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r},\frac{2kr\delta^2+(2k^2+kr+2r^2)\delta+k^2+3kr-2r^2}{2r(k-r)}\right\}\geq\frac{2kr\delta^2+(2k^2+kr+2r^2)\delta+k^2+3kr-2r^2}{2r(k-r)}$, we obtain

$$\begin{aligned}\rho(G)\leq&\rho(G_1)\\<&\frac{1}{r}\sqrt{f(\delta+1)}\\=&\frac{1}{r}\sqrt{(k^2+2kr)(\delta+1)^2-(2krn-2r^2-kr)(\delta+1)+r^2n^2-2r^2n+r^2}\\=&\frac{1}{r}\sqrt{(rn-k\delta-2r)^2-2r(k-r)n+2kr\delta^2+(2k^2+kr+2r^2)\delta+k^2+3kr-2r^2}\\<&\frac{1}{r}(rn-k\delta-2r).\end{aligned}\tag{3}$$

Since $K_{n-\lfloor\frac{k\delta}{r}\rfloor-1}$ is a proper subgraph of $K_\delta\vee(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup(\lfloor\frac{k\delta}{r}\rfloor+1)K_1)$, it follows from Lemma 2.2, $\lfloor\frac{k\delta}{r}\rfloor\leq\frac{k\delta}{r}$ and the hypothesis of the theorem that

$$\begin{aligned}\rho(G)\geq&\rho\Big(K_\delta\vee\Big(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup\Big(\Big\lfloor\frac{k\delta}{r}\Big\rfloor+1\Big)K_1\Big)\Big)\\>&\rho(K_{n-\lfloor\frac{k\delta}{r}\rfloor-1})\\=&n-\Big\lfloor\frac{k\delta}{r}\Big\rfloor-2\\\geq&n-\frac{k\delta}{r}-2\\=&\frac{1}{r}(rn-k\delta-2r),\end{aligned}$$

which leads to a contradiction to (3). Theorem 1.1 is proved. □

4. THE PROOF OF THEOREM 1.2

Proof of Theorem 1.2. Assume that G has no $\{C_{2i+1},T:1\leq i<\frac{r}{k-r},T\in\mathscr{T}_{\frac{k}{r}}\}$-factor. Then using Lemma 2.1, there exists some nonempty subset S of $V(G)$ such that

$$i(G-S)>\frac{k}{r}|S|.$$

According to the integrity of $i(G-S)$, we obtain

$$i(G-S)\geq\Big\lfloor\frac{k}{r}|S|\Big\rfloor+1.$$

Let $|S|=s$. Then G is a spanning subgraph of $G_1=K_s\vee(K_{n-\lfloor\frac{ks}{r}\rfloor-s-1}\cup(\lfloor\frac{ks}{r}\rfloor+1)K_1)$. Together with Lemma 2.4, we possess

$$q(G)\leq q(G_1),\tag{4}$$

where the equality holds if and only if $G=G_1$. Note that $\delta(G)=\delta$ and $\delta(G_1)=s\geq\delta(G)$. Thus, we get $s\geq\delta$. In what follows, we shall consider two cases by the value of s.

Case 1. $s=\delta$.

In this case, $G_1=K_\delta\vee(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup(\lfloor\frac{k\delta}{r}\rfloor+1)K_1)$. In terms of (4), we obtain

$$q(G)\le q\Big(K_\delta\vee\Big(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup\Big(\Big\lfloor\frac{k\delta}{r}\Big\rfloor+1\Big)K_1\Big)\Big),$$

where the equality holds if and only if $G=K_\delta\vee(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup(\lfloor\frac{k\delta}{r}\rfloor+1)K_1)$. Observe that $K_\delta\vee(K_{n-\lfloor\frac{k\delta}{r}\rfloor-\delta-1}\cup(\lfloor\frac{k\delta}{r}\rfloor+1)K_1)$ contains no $\{C_{2i+1},T:1\le i<\frac{r}{k-r},T\in\mathscr{T}_{\frac{k}{r}}\}$-factor. Thus, we can obtain a contradiction.

Case 2. $s\ge\delta+1$.

Recall that $G_1=K_s\vee(K_{n-\lfloor\frac{ks}{r}\rfloor-s-1}\cup(\lfloor\frac{ks}{r}\rfloor+1)K_1)$. It follows from Lemma 2.5, $\frac{ks}{r}-1<\lfloor\frac{ks}{r}\rfloor\le\frac{ks}{r}$ and $n\ge\lfloor\frac{ks}{r}\rfloor+s+1>\frac{ks}{r}+s$ that

$$\begin{aligned}
q(G_1)\le&\frac{2e(G_1)}{n-1}+n-2\\
=&\frac{2\binom{n-\lfloor\frac{ks}{r}\rfloor-1}{2}+2s\left(\left\lfloor\frac{ks}{r}\right\rfloor+1\right)}{n-1}+n-2\\
=&\frac{\left(n-\left\lfloor\frac{ks}{r}\right\rfloor-1\right)\left(n-\left\lfloor\frac{ks}{r}\right\rfloor-2\right)+2s\left(\left\lfloor\frac{ks}{r}\right\rfloor+1\right)}{n-1}+n-2\\
<&\frac{\left(n-\left(\frac{ks}{r}-1\right)-1\right)\left(n-\left(\frac{ks}{r}-1\right)-2\right)+2s\left(\frac{ks}{r}+1\right)}{n-1}+n-2\\
=&\frac{\left(n-\frac{ks}{r}\right)\left(n-\frac{ks}{r}-1\right)+2s\left(\frac{ks}{r}+1\right)}{n-1}+n-2\\
=&\frac{(k^2+2kr)s^2-(2krn-kr-2r^2)s+2r^2n^2-4r^2n+2r^2}{r^2(n-1)}.
\end{aligned}\tag{5}$$

Let $g(s)=(k^2+2kr)s^2-(2krn-kr-2r^2)s+2r^2n^2-4r^2n+2r^2$. Since $n\ge\lfloor\frac{ks}{r}\rfloor+s+1>\frac{ks}{r}+s$, we deduce $\delta+1\le s<\frac{rn}{k+r}$. By a simple computation, we obtain

$$\begin{aligned}
g(\delta+1)-g\Big(\frac{rn}{k+r}\Big)=&(k^2+2kr)(\delta+1)^2-(2krn-kr-2r^2)(\delta+1)+2r^2n^2-4r^2n+2r^2\\
&-\Big((k^2+2kr)\Big(\frac{rn}{k+r}\Big)^2-(2krn-kr-2r^2)\Big(\frac{rn}{k+r}\Big)+2r^2n^2-4r^2n+2r^2\Big)\\
=&\Big(\frac{rn}{k+r}-\delta-1\Big)\Big(\frac{k^2rn}{k+r}-(k+2r)(k\delta+k+r)\Big)\\
>&0,
\end{aligned}$$

where the inequality holds from the fact that

$$\begin{aligned}
n>&\max\Big\{\frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r},\frac{(k^2+2kr)\delta^2+(2k^2+3kr+2r^2)\delta+k^2+3kr}{2r(k-r)}\Big\}\\
\ge&\frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r}\\
>&\frac{(k+r)(\delta+1)}{r}.
\end{aligned}$$

This implies that, for $\delta+1\le s<\frac{rn}{k+r}$, the function $g(s)$ attains its maximum value at $s=\delta+1$. Combining this with (4), (5) and $n>\max\Big\{\frac{(k+r)(k+2r)(k\delta+k+r)}{k^2r},\frac{(k^2+2kr)\delta^2+(2k^2+3kr+2r^2)\delta+k^2+3kr}{2r(k-r)}\Big\}\ge\frac{(k^2+2kr)\delta^2+(2k^2+3kr+2r^2)\delta+k^2+3kr}{2r(k-r)}$, we conclude

$$\begin{aligned} q(G) \leq & q(G_1) \\ < & \frac{g(\delta+1)}{r^2(n-1)} \\ = & \frac{(k^2+2kr)(\delta+1)^2-(2krn-kr-2r^2)(\delta+1)+2r^2n^2-4r^2n+2r^2}{r^2(n-1)} \\ = & \frac{2(rn-k\delta-2r)}{r}-\frac{2r(k-r)n-(k^2+2kr)\delta^2-(2k^2+3kr+2r^2)\delta-k^2-3kr}{r^2(n-1)} \\ < & \frac{2(rn-k\delta-2r)}{r}. \end{aligned} \tag{6}$$

Note that $K_\delta \vee (K_{n-\lfloor \frac{k\delta}{r} \rfloor-\delta-1} \cup (\lfloor \frac{k\delta}{r} \rfloor+1)K_1)$ contains $K_{n-\lfloor \frac{k\delta}{r} \rfloor-1}$ as a proper subgraph. Together with Lemma 2.4, $\lfloor \frac{k\delta}{r} \rfloor \leq \frac{k\delta}{r}$ and the assumption of the theorem, we possess

$$\begin{aligned} q(G) \geq & q\Big(K_\delta \vee \Big(K_{n-\lfloor \frac{k\delta}{r} \rfloor-\delta-1} \cup \Big(\Big\lfloor \frac{k\delta}{r} \Big\rfloor+1\Big)K_1\Big)\Big) \\ > & q(K_{n-\lfloor \frac{k\delta}{r} \rfloor-1}) \\ = & 2\Big(n-\Big\lfloor \frac{k\delta}{r} \Big\rfloor-2\Big) \\ \geq & 2\Big(n-\frac{k\delta}{r}-2\Big) \\ = & \frac{2(rn-k\delta-2r)}{r}, \end{aligned}$$

which is to a contradiction to (6). This completes the proof of Theorem 1.2. □

5. CONCLUDING REMARKS

In this paper, we provide two sufficient conditions to ensure that a connected graph G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor in terms of its adjacency spectral radius and signless Laplacian spectral radius. It is natural and interesting to propose some other spectral sufficient conditions to guarantee that a connected graph G has a $\{C_{2i+1}, T : 1 \leq i < \frac{r}{k-r}, T \in \mathscr{T}_{\frac{k}{r}}\}$-factor. It is also natural and interesting to put forward some spectral sufficient conditions to ensure that a connected graph G has some other substructure.

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