THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume 26, Number yy/2025, pp. xx–xx
DOI:

ON THE SUM OF DIGITS OF PIATETSKI-SHAPIRO SEQUENCES

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Abstract. Let $g \geq 2$ be an integer, and let $S_g(n)$ denote the sum of the digits of a positive integer n in base g. The goal of this work is to study exponential sums of the form $\sum_{n\leq x} e\left(\alpha S_g\left(\lfloor n^c \rfloor\right) + \beta \lfloor n^c \rfloor\right)$ in order to prove some statistical properties of integers n for which $\lfloor n^c \rfloor$ and $S_g\left(\lfloor n^c \rfloor\right)$ belong to given arithmetic progressions.

***Keywords**:* sum of digits function, exponential sum, Piatetski-Shapiro sequence, k-free number.
Mathematics Subject Classification (MSC2020): 11A63, 11K65, 11L03, 11L07, 11N69.

1. INTRODUCTION

Throughout this paper, we use the notation $e(x) = \exp(2\pi i x)$ for any real number x. For two complex-valued functions f and h, we write $f \ll h$ if $f = O(h)$, where the implied constants in the symbols O and $\ll$ are absolute. If these constants depend on specific parameters $\alpha, \beta, \ldots$ (but no others), we denote this as $f = O_{\alpha,\beta,\ldots}(h)$ or $f \ll_{\alpha,\beta,\ldots} h$. For a finite set A, we use $\#A$ to indicate the number of its elements. Given two integers a and b, their greatest common divisor is denoted by (a,b). As usual, $\lfloor t \rfloor$ and $\{t\}$ represent the integer and fractional parts of t, respectively. Finally, $\|t\|$ denotes the distance from t to the nearest integer.

Let $g \geq 2$ be an integer. We can represent every positive integer n in a unique way as

$$n \quad = \sum_{0\leq k\leq \nu} n_k g^k, \quad n_k \in \{0,\ldots,g-1\} \quad \text{and } n_\nu \neq 0. \tag{1}$$

This representation is called the g-ary expansion of n with respect to base g, and the set $\{0,\ldots,g-1\}$ is called the set of digits. The sum-of-digits function is defined by

$$S_g(n) := \sum_{j\geq 0} n_j.$$

Gelfond [7] proved that the sequence $(S_g(n))_n$ is equidistributed in arithmetic progressions. In 1968, he showed that there exists a constant $\lambda_{g,b} > 0$ such that the following holds:

$$\#\{n \leq x : n \equiv \ell \pmod r,\ S_g(n) \equiv a \pmod b\} = \frac{x}{br} + O_g(x^{1-\lambda_{g,b}}), \tag{2}$$

where $b \geq 2$ and r, ℓ, a be integers satisfying $(b, g-1) = 1$. The sum of digits function has been extensively discussed in the literature with respect to their asymptotic distributions (see for instance [2,3,7,8,17]). Gelfond

[7] also raised the question of whether this property remains valid for certain subsequences of $(S_g(n))_n$, a question known as "Gelfond problems". Several results have been achieved in this regard. For example, Gelfond [7] showed in the same paper that the sequence $(S_g(n))_n$ is equidistributed over the set of k-free integers. Later, in 2010, Mauduit and Rivat [10] proved the equidistribution of the sequence $(S_g(p))_{p\geq 2}$, where p is a prime number. Also, in 2011, Drmota, Mauduit, and Rivat [6] were able to prove that the sequence $(S_g(P(n)))_n$, where $P(n)$ is a polynomial, is equidistributed for large bases.

Another interesting class of sequences is of the form $(S_g(\lfloor n^c \rfloor))_n$, where $(\lfloor n^c \rfloor)_n$ is known as the Piatetski-Shapiro sequence for $c > 1$ and $c \notin \mathbb{N}$ (for more details on this sequence, we refer the reader to [14]). The sequences $(S_g(\lfloor n^c \rfloor))_n$ were studied by Mauduit and Rivat, who proved their equidistribution for $c \in (1, 4/3)$ in [9] and extended this result to $c \in (1, 7/5)$ in [11]. In 2010, Morgenbesser [12] extended these results to all real numbers $c > 0$ that are not integers, enlarging the range of c, but when the base g is large. He then gave a result concerning the uniform distribution modulo 1 for the sequence $(\alpha S_g(\lfloor n^c \rfloor))_n$ when α is irrational, always g is large. In particular, Morgenbesser [12] showed that if $c > 0$ is a real number that is not an integer, and g is large, then there exists a constant $\sigma_{g,b,c} > 0$ satisfying:

$$\#\{n \leq x : S_g(\lfloor n^c \rfloor) \equiv a \pmod b\} = \frac{x}{b} + O_{g,b,c}\left(x^{1-\sigma_{g,b,c}}\right), \tag{3}$$

where $(a,b) \in \mathbb{N} \times \mathbb{N}^*$. To prove his result, Morgenbesser heavily relied on establishing this upper bound:

$$\sum_{n<x} e\left(\alpha S_g(\lfloor n^c \rfloor)\right) \ll_{c,g} (\log x) x^{1-\sigma_{c,g}\|(g-1)\alpha\|^2},$$

where α is a real number, and $\|\cdot\|$ denotes the distance to the nearest integer.

In this paper, we take a further step, we extend (3). Specifically, we aim to provide an asymptotic expansion for the function

$$\#\{1 \leqslant n \leqslant x\,;\, \lfloor n^c \rfloor \equiv \ell \pmod r,\, S_g(\lfloor n^c \rfloor) \equiv a \pmod b\}, \tag{4}$$

where $b \geq 2$ and r, ℓ, a be integers such that $(b, g-1) = 1$. Therefore, our initial focus revolves in establishing an upper bound for the sum $\sum_{n \leq x} e\left(\alpha S_g(\lfloor n^c \rfloor) + \beta \lfloor n^c \rfloor\right)$. Also, we study the distribution of k-free values of $\lfloor n^c \rfloor$ in congruences classes.

2. SOME TOOLS

In this section, we cite several results that will be used later in our proofs. To establish our first result, we require an upper bound for sums of the form $\sum_{1 \leq n \leq x} f(\lfloor n^c \rfloor)$, where f is a g-multiplicative function. Such sums were addressed in [9] and [11], where Mauduit and Rivat used to prove the equidistribution of $S_g(\lfloor n^c \rfloor)$ in congruence classes. Their result is the following:

THEOREM 1. *Let $c \in (1, 7/5)$ and $\gamma = 1/c$. For any $\delta \in (0, (7-5c)/9)$, there exists a constant $C_1(\gamma, \delta) > 0$ such that for any q-multiplicative function f and any $x \geq 1$, we have*

$$\left| \sum_{1 \leq n \leq x} f(\lfloor n^c \rfloor) - \sum_{1 \leq m \leq x^c} \gamma m^{\gamma-1} f(m) \right| \leq C_1(\gamma, \delta) x^{1-\delta}.$$

Furthermore, Spiegelhofer [16] made significant progress in this direction by establishing the bound $1 < c \leq 1.42$ for the Thue–Morse sequence. The core technique involves approximating the nonlinear term $\lfloor n^c \rfloor$ by a Beatty sequence $\lfloor n\alpha + \beta \rfloor$, effectively linearizing the problem. Then, Müllner and Spiegelhofer [13] using the same linearization argument and a Bombieri–Vinogradov type theorem for the Thue–Morse sequence on Beatty sequences, were able to extend this range to $1 < c < 3/2$. In addition, further randomness and non-randomness properties of $(\lfloor n^c \rfloor \bmod m)_{n \in \mathbb{N}}$ have been established in [4].

We also rely on the following result from [1, Theorem 2.2]

THEOREM 2. *Let $g \geq 2$ be an integer. For any real numbers α and β, we have the bound*

$$\sum_{1 \leq n \leq x} e(\alpha S_g(n) + \beta n) \ll x^{\theta_g(\alpha)},$$

with

$$\theta_g(\alpha) \leq 1 - \frac{4\|(g-1)\alpha\|^2}{g(g+\sqrt{2}-1)^2 \log g}. \tag{5}$$

3. EXPONENTIAL SUM ESTIMATE

THEOREM 3. *Let α and β be real numbers. For any real $c \in (1, 7/5)$, there exists a constant $\vartheta(\alpha, c) > 0$ such that*

$$\sum_{n \leq x} e\left(\alpha S_g\left(\lfloor n^c \rfloor\right) + \beta \lfloor n^c \rfloor\right) \ll x^{1-\vartheta(\alpha,c)},$$

where

$$\vartheta(\alpha, c) = \min\left(\left(1 - \theta_g(\alpha)\right)c, \delta\right),$$

with $\delta \in (0, (7-5c)/9)$ and $\theta_g(\alpha)$ as defined in (5).

Proof: Initially, recall that a function f is g-additive if, for any integer $r \geq 1$, we have $f(ag^r + b) = f(ag^r) + f(b)$, for $1 \leq a \leq g-1$ and $0 \leq b < g^r$. Moreover, we say that f is g-multiplicative if $f(0) = 1$ and, for any $r \geq 1$, we have $f(ag^r + b) = f(ag^r)f(b)$.

Since, the sum-of-digits function S_q is g-additive, then we have

$$\begin{aligned} e\left(\alpha S_g(ag^r + b) + \beta(ag^r + b)\right) &= e\left(\alpha S_g(ag^r) + \alpha S_g(b) + \beta ag^r + \beta b\right) \\ &= e\left(\alpha S_g(ag^r) + \beta ag^r\right) e\left(\alpha S_g(b) + \beta b\right). \end{aligned}$$

Furthermore, $e\left(\alpha S_g(0)\right) = 1$. Hence, the function $e\left(\alpha S_g(ag^r + b) + \beta(ag^r + b)\right)$ is g-multiplicative.

As $e\left(\alpha S_g(ag^r + b) + \beta(ag^r + b)\right)$ is g-multiplicative, then by Theorem 1, we have

$$\sum_{1 \leqslant n \leqslant x} e\left(\alpha S_g \lfloor n^c \rfloor + \beta \lfloor n^c \rfloor\right) = \sum_{1 \leqslant m \leqslant x^c} \gamma m^{\gamma-1} e\left(\alpha S_g(m) + \beta m\right) + O(C_1(\gamma, \delta) x^{1-\delta}). \tag{6}$$

Moving on, we apply Abel's summation formula to the last sum, then

$$\begin{aligned} \sum_{1 \leqslant m \leqslant x^c} m^{\gamma-1} e\left(\alpha S_g(m) + \beta m\right) &= x^{c(\gamma-1)} \sum_{1 \leqslant m \leqslant x^c} e\left(\alpha S_g(m) + \beta m\right) \\ &\quad - (\gamma - 1) \int_1^{x^c} \sum_{1 \leqslant m \leqslant u} e\left(\alpha S_g(m) + \beta m\right) u^{\gamma-2} du \\ &= x^{1-c} \sum_{1 \leqslant m \leqslant x^c} e\left(\alpha S_g(m) + \beta m\right) \\ &\quad - (\gamma - 1) \int_1^{x^c} \sum_{1 \leqslant m \leqslant u} e\left(\alpha S_g(m) + \beta m\right) u^{\gamma-2} du. \end{aligned}$$

Or, by Theorem 2, we have the bound

$$\sum_{1\leqslant m\leqslant x^c} e\left(\alpha S_g(m)+\beta m\right) \ll x^{c\theta_g(\alpha)}.$$

Moreover,

$$\int_1^{x^c} \sum_{1\leqslant m\leqslant u} e\left(\alpha S_g(m)+\beta m\right) u^{\gamma-2}du \ll \int_1^{x^c} u^{\theta_g(\alpha)+\gamma-2}du$$
$$\ll x^{c(\theta_g(\alpha)+\gamma-1)} = x^{c\theta_g(\alpha)+1-c} = x^{1+c(\theta_g(\alpha)-1)}.$$

So, we deduce that

$$\sum_{1\leqslant m\leqslant x^c} \gamma m^{\gamma-1} e\left(\alpha S_g(m)+\beta m\right) \ll x^{1-c+c\theta_g(\alpha)} = x^{1-(1-\theta_g(\alpha))c}. \tag{7}$$

Finally, by inserting (7) in (6), we get

$$\sum_{1\leqslant n\leqslant x} e\left(\alpha S_g\left(\lfloor n^c\rfloor\right)+\beta \lfloor n^c\rfloor\right) \ll x^{1-\vartheta(\alpha,c)},$$

where $\vartheta(\alpha,c)=\min\left((1-\theta_g(\alpha))\,c,\delta\right)$.

4. DISTRIBUTION OF $\lfloor n^c\rfloor$ IN CONGRUENCE CLASSES

THEOREM 4. *Let $g,b\geq 2$ integers such that $(b,g-1)=1$ and $r,\ell,a\in\mathbb{Z}$. For any real $c\in(1,7/5)$, there exists a constant $\vartheta>0$ depending on b and c, such that*

$$\#\{n\leq x : \lfloor n^c\rfloor \equiv \ell \pmod r, S_g(\lfloor n^c\rfloor)\equiv a \pmod b\} = \frac{x}{rb}+O\left(x^{1-\vartheta}\right).$$

Proof: We have by the orthogonality relation

$$\begin{aligned}\sum_{\substack{n\leq x\\ \lfloor n^c\rfloor\equiv\ell \pmod r\\ S_g(\lfloor n^c\rfloor)\equiv a \pmod b}} 1 &= \frac{1}{rb}\sum_{i=0}^{r-1}\sum_{j=0}^{b-1}\sum_{n\leq x} e\left(\frac{i}{r}\left(\lfloor n^c\rfloor-\ell\right)+\frac{j}{b}\left(S_g(\lfloor n^c\rfloor)-a\right)\right)\\
&= \frac{1}{rb}\sum_{i=0}^{r-1}\sum_{j=0}^{b-1} e\left(\frac{-i\ell}{r}-\frac{ja}{b}\right)\sum_{n\leq x} e\left(\frac{jS_g\left(\lfloor n^c\rfloor\right)}{b}+\frac{i\lfloor n^c\rfloor}{r}\right)\\
&= \frac{x}{rb}+\frac{1}{rb}\underbrace{\sum_{i=0}^{r-1}\sum_{j=0}^{b-1}}_{i+j\neq 0} e\left(\frac{-i\ell}{r}-\frac{ja}{b}\right)\sum_{n\leq x} e\left(\frac{jS_g\left(\lfloor n^c\rfloor\right)}{b}+\frac{i\lfloor n^c\rfloor}{r}\right).\end{aligned} \tag{8}$$

Next, applying Theorem 3 to the last sum in (8) with $\alpha=j/b$ and $\beta=i/r$, we get

$$\sum_{n\leqslant x} e\left(\frac{jS_g\left(\lfloor n^c\rfloor\right)}{b}+\frac{i\lfloor n^c\rfloor}{r}\right) \ll x^{1-\vartheta(j/b,c)}.$$

Hence,

$$\frac{1}{rb}\underbrace{\sum_{i=0}^{r-1}\sum_{j=0}^{b-1}}_{i+j\neq 0} e\left(\frac{-i\ell}{r}-\frac{ja}{b}\right)\sum_{n\leq x} e\left(\frac{jS_g\left(\lfloor n^c\rfloor\right)}{b}+\frac{i\lfloor n^c\rfloor}{r}\right) \ll x^{1-\vartheta},$$

where $\vartheta = \min_{1\le j\le b-1} \vartheta(j/b,c)$ and the result follows.

THEOREM 5. *Let $g,b\ge 2$ integers such that $(b,g-1)=1$. For any real $c\in(1,7/5)$ and $a\in\mathbb{Z}$, we define the set $\mathbb{B}=\{n\in\mathbb{N}: S_g(\lfloor n^c\rfloor)\equiv a \pmod b\}$. Then the sequence $(\beta\lfloor n^c\rfloor)_{n\in\mathbb{B}}$ is uniformly distributed modulo 1 if and only if $\beta\in\mathbb{R}\setminus\mathbb{Q}$.*

Proof: If $\beta\in\mathbb{Q}$, then the sequence $(\beta\lfloor n^c\rfloor)_{n\in\mathbb{B}}$ takes only a finite number of values modulo 1. Consequently, it is not uniformly distributed modulo 1. Conversely, if $\beta\in\mathbb{R}\setminus\mathbb{Q}$, then for every $h\in\mathbb{Z}\setminus\{0\}$, we have $h\beta\in\mathbb{R}\setminus\mathbb{Z}$. By Weyl's criterion from [5], we must show that

$$\frac{1}{x}\sum_{\substack{n\le x\\ S_g(\lfloor n^c\rfloor)\equiv a \pmod b}} e(h\beta\lfloor n^c\rfloor)=o(1),$$

for every integer $h\neq 0$. By the orthogonality relation we have

$$\begin{aligned}\sum_{\substack{1\le n\le x\\ S_g(n)\equiv a \pmod b}} e((h\beta\lfloor n^c\rfloor) &= \frac{1}{b}\sum_{n=1}^{x} e((h\beta\lfloor n^c\rfloor)\sum_{i=0}^{b-1} e\left(\frac{i}{b}\left(S_g(\lfloor n^c\rfloor)-a\right)\right)\\ &=\frac{1}{b}\sum_{i=0}^{b-1} e\left(-\frac{ia}{b}\right)\sum_{n=1}^{x} e\left(\frac{i}{b}S_g(\lfloor n^c\rfloor)+h\beta\lfloor n^c\rfloor\right).\end{aligned}$$

Applying Theorem 3 to the inner sum, we obtain the desired result.

5. DISTRIBUTION OF k- FREE VALUES OF $\lfloor n^c\rfloor$ IN CONGRUENCE CLASSES

Let k and n be integers with $k\ge 2$. We say that n is k-free if there exists no prime p such that p^k divides n (i.e., n is not divisible by any k-th power of a prime). We denote by μ_k the characteristic function of the k-free numbers. It is well known that the density of k-free integers is $1/\zeta(k)$, and an elementary sieve shows

$$\sum_{n\le X}\mu_k(n)=\frac{X}{\zeta(k)}+O\left(X^{1/k}\right),$$

with $\zeta(.)$ is the Riemann zeta function.

The distribution of k-free numbers in sequences of the form $\lfloor n^c\rfloor$ has been extensively studied in the literature. For instance, Rieger [15] showed that for any fixed $1<c<\frac{3}{2}$ the asymptotic formula

$$\sum_{n\le X}\mu_2(\lfloor n^c\rfloor)=\frac{6}{\pi^2}X+O\left(X^{\frac{2c+1}{4}+\varepsilon}\right)$$

holds for any $\varepsilon>0$.

In this context, we prove the following result.

THEOREM 6. *Let $g,b\ge 2$ integers such that $(b,g-1)=1$. For any real $c\in(1,7/5)$ and $a\in\mathbb{Z}$. Then, we have the following asymptotic estimate:*

$$\#\{n\le x: \lfloor n^c\rfloor \text{ is } k\text{-free},\ S_g(\lfloor n^c\rfloor)\equiv a \pmod b\}=\frac{x}{b\zeta(k)}+O\left(x^{1-\vartheta(1-\frac{c}{k})}\right).$$

Proof: To simplify the writing, set the function

$$A(x):=\sum_{\substack{n\le x\\ \lfloor n^c\rfloor \text{ is } k\text{-free}\\ S_g(\lfloor n^c\rfloor)\equiv a \pmod b}} 1.$$

We have

$$A(x)=\sum_{n\leqslant x}\mu_k\left(\lfloor n^c\rfloor\right)\nu\left(\lfloor n^c\rfloor\right),$$

where

$$\mu_k\left(\lfloor n^c\rfloor\right)=\begin{cases}1, & \text{if } \lfloor n^c\rfloor \text{ is } k\text{-free},\\ 0, & \text{otherwise.}\end{cases} \quad \text{and} \quad \nu\left(\lfloor n^c\rfloor\right)=\begin{cases}1, & \text{if } S_g\left(\lfloor n^c\rfloor\right)\equiv a \pmod b,\\ 0, & \text{otherwise.}\end{cases}.$$

Moreover, we utilize the identity $\mu_k\left(\lfloor n^c\rfloor\right)=\sum_{d^k|\lfloor n^c\rfloor}\mu(d)$, where $\mu(\cdot)$ is the Möbius function. Therefore, we have for $x_1=x^{c/k}$, and $x_2<x_1$ that we choose later

$$\begin{aligned} A(x)=\sum_{n\leq x}\nu\left(\lfloor n^c\rfloor\right)\sum_{d^k|\lfloor n^c\rfloor}\mu(d) &= \sum_{d=1}^{x_1}\mu(d)\sum_{\substack{n\leq x\\ d^k|\lfloor n^c\rfloor}}\nu\left(\lfloor n^c\rfloor\right)\\ &= \sum_{d=1}^{x_2}\mu(d)\sum_{\substack{n\leq x\\ d^k|\lfloor n^c\rfloor}}\nu\left(\lfloor n^c\rfloor\right)+\sum_{d=x_2+1}^{x_1}\mu(d)\sum_{\substack{n\leq x\\ d^k|\lfloor n^c\rfloor}}\nu\left(\lfloor n^c\rfloor\right)\\ &= \sum_1+\sum_2. \end{aligned} \tag{9}$$

Given that $(b,g-1)=1$, we apply Theorem 4 with $\ell=0$ and $r=d^k$ to derive

$$\sum_{\substack{n\leq x\\ d^k|\lfloor n^c\rfloor}}\nu\left(\lfloor n^c\rfloor\right)=\frac{x}{d^kb}+O(x^{1-\vartheta}).$$

Therefore, we estimate $\sum_1$ as follows

$$\begin{aligned} \sum_1 &= \sum_{d=1}^{x_2}\mu(d)\left[\frac{x}{d^kb}+O(x^{1-\vartheta})\right]=\frac{x}{b}\sum_{d=1}^{x_2}\frac{\mu(d)}{d^k}+O\left(x^{1-\vartheta}\sum_{d=1}^{x_2}\mu(d)\right)\\ &= \frac{x}{b}\sum_{d=1}^{+\infty}\frac{\mu(d)}{d^k}+O\left(x^{1-\vartheta}x_2\right)-\frac{x}{b}\sum_{d=x_2+1}^{+\infty}\frac{\mu(d)}{d^k}\\ &= \frac{x}{b\zeta(k)}+O\left(x^{1-\vartheta}x_2\right)-\frac{x}{b}\sum_{d=x_2+1}^{+\infty}\frac{\mu(d)}{d^k}. \end{aligned}$$

The last term can be bounded as follows. Indeed, since the series and the integral are convergent, we get for $R\longrightarrow\infty$

$$\left|\sum_{d=x_2+1}^{R}\frac{\mu(d)}{d^k}\right|\leq\sum_{d=x_2+1}^{R}\frac{1}{d^k}\leq\int_{x_2}^{R}\frac{1}{t^k}\,dt=\frac{x_2^{1-k}-R^{1-k}}{1-k}.$$

Thus,

$$\sum_1=\frac{x}{b\zeta(k)}+O\left(x^{1-\vartheta}x_2\right)+O\left(xx_2^{1-k}\right). \tag{10}$$

Moreover, applying Theorem 4 once again, we get

$$\left|\sum_{2}\right| = \left|\sum_{d=x_2+1}^{x_1} \mu(d) \sum_{\substack{n\le x \\ d^k | \lfloor n^c \rfloor}} \nu\left(\lfloor n^c \rfloor\right)\right| \le \sum_{d=x_2+1}^{x_1} \frac{x}{d^k b} \le \frac{x}{b}\int_{x_2+1}^{x_1} \frac{1}{t^k}dt$$
$$\ll x x_2^{1-k}. \tag{11}$$

Now, by inserting equations (10) and (11) in (9), we obtain

$$A(x) = \frac{x}{b\zeta(k)} + O\left(x^{1-\vartheta} x_2\right) + O\left(x x_2^{1-k}\right).$$

Finally, setting $x_2 = \lfloor x^c \rfloor^{\vartheta/k}$, we obtain the result.

ACKOWLEDGEMENTS

The authors express their deep gratitude to the referee for the meticulous review of the text, which enhanced the quality of this work.

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Received March 29, 2025