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REMARKS ON LEVI-FLAT REAL HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS

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Abstract. Let M be a Levi-flat real hypersurface in a nonflat complex space form. In this paper, it is proved that M is ruled if and only if the structure Lie operator is η-parallel with respect to the Levi-Civita connection or the GTW connection. Moreover, the GTW η-parallelism on Levi-flat real hypersurfaces was also studied.

Keywords: ruled real hypersurface; complex space form; Levi-flat; structure Lie operator; GTW connection.

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1. INTRODUCTION

Let $\mathbb{C}M^n(c)$ be a complete and simply connected complex space form which is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$,
- a complex Euclidean space $\mathbb{C}^n$ if $c = 0$,
- a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$,

where c denotes the constant holomorphic sectional curvature. A complex space form $\mathbb{C}M^n(c)$ is said to be nonflat if $c \neq 0$. On a real hypersurface M in $\mathbb{C}M^n(c)$ there exists an almost contact metric structure which is denoted by (ϕ, ξ, η, g), where g is the induced metric from the ambient space. A real hypersurface M in $\mathbb{C}M^n(c)$ is said to be Hopf when the structure vector field ξ is principal at every point or ruled if the holomorphic distribution $\ker\eta$ is integrable and its leaves are totally geodesic (see [5, Proposition 8.27]).

For a real hypersurface M in $\mathbb{C}M^n(c)$, the Levi-form *Levi* defined on the holomorphic distribution $\ker\eta$ of M is given by (see [9])

$$Levi(X,Y) = (\mathrm{d}\eta)(X,Y) \tag{1}$$

for any vector fields $X, Y \in \ker\eta$. A real hypersurface is said to be Levi-flat if $Levi = 0$ identically on $\ker\eta$. By (8), Levi-flatness condition $Levi = 0$ is equivalent to

$$g((\phi A + A\phi)X, Y) = 0 \tag{2}$$

for any vector fields $X, Y \in \ker\eta$, namely $\ker\eta$ is integrable. The most known example of a Levi-flat real hypersurface in nonflat complex space forms is a ruled one which is obtained from various points of view (see [11, 23, 30]). In what follows we shall list them from geometry of submanifolds and we refer the reader to [1, 2, 22] for some results from complex analysis.

Kimura and Maeda in [15] proved that a Levi-flat real hypersurface in $\mathbb{C}P^n$ with η-parallel second fundamental form is ruled. Note that Lim and Sohn in [17] considered the case of dimension three under the same conditions. Moreover, by extending the η-parallelism to η-recurrence of the second fundamental form, Hamada in [12] generalized results in [15]. Levi-flatness and weakly Ricci η-parallelism conditions were investigated in [21]. Suh in [23] proved that a Levi-flat real hypersurface M in a nonflat complex space form is ruled if M satisfies $g((S\phi - \phi S)X, Y) = fg(X, Y)$ for certain non-vanishing function f on the holomorphic distribution $\ker\eta$. Levi-flat real hypersurfaces in nonflat complex space forms whose shape operators are weakly ϕ-invariant or satisfies $(\mathscr{L}_\xi \phi)^2 = 0$ were also considered in [19] and [16], respectively. The $*$-Ricci tensor is defined by $\mathrm{Ric}^* := \mathrm{trace}\{Z \to R(X, \phi Y)\phi Z\}$ for any vector fields X, Y, Z on a real hypersurface M in $\mathbb{C}M^n(c)$. It was proved in [13] that a Levi-flat real hypersurface in nonflat complex space forms is ruled provided that its $*$-Ricci tensor is symmetric and it is a constant multiple of the Riemannian metric g over the holomorphic distribution $\ker\eta$. For the three-dimensional case, it was proved in [11] that if a Levi-flat real hypersurface in nonflat complex planes with constant mean curvature is strongly 2-Hopf, then it is minimal and ruled. As seen in the above results, Levi-flatness and some geometrical conditions imply ruled hypersurfaces. But, this is not always true. Non-ruled Levi-flat real hypersurfaces in nonflat complex planes were given in [4] with minimality and was given in [10, 30] where the mean curvature is not necessarily a constant. Levi-flatness on a compact real hypersurface in a complex projective space was studied in [3].

In this paper, we continue to study Levi-flat real hypersurfaces in nonflat complex space forms. In view of [15, 21], we consider η-parallelism of the structure Lie operator and prove that a Levi-flat real hypersurface in a nonflat complex space form is ruled if the structure Lie operator L is η-parallel with respect to the Levi-Civita connection or the GTW connection. The conclusion holds if we replace the structure Lie operator by the shape operator.

2. REAL HYPERSURFACE

Let M be a real hypersurface immersed in a complex space form $\mathbb{C}M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $\mathbb{C}M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of the metric g, respectively. Then the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,\ \overline{\nabla}_X N = -AX \tag{3}$$

for any vector fields X, Y, where A denotes the shape operator of M in $\mathbb{C}M^n(c)$. For any vector field X, we put

$$JX = \phi X + \eta(X)N,\ JN = -\xi. \tag{4}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{Id} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0, \tag{5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi) \tag{6}$$

for any vector fields X, Y. Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla}J = 0$) of $\mathbb{C}M^n(c)$ and using (3), (4) we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{7}$$

$$\nabla_X \xi = \phi AX \tag{8}$$

for any vector fields X, Y. Let R be the Riemannian curvature tensor of M. Because $\mathbb{C}M^n(c)$ is of constant holomorphic sectional curvature c, the Gauss and Codazzi equations of M in $\mathbb{C}M^n(c)$ are given respectively as the following:

$$R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \tag{9}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\} \tag{10}$$

for any vector fields X,Y.

We refer the reader to [5, 20] for more basic knowledge on differential geometry of real hypersurfaces in nonflat complex space forms.

3. LEVI-CIVITA CONNECTION

As pointed out in [13, Theorem 4.3], a Levi-flat real hypersurface in a nonflat complex space form can not be Hopf. Therefore, throughout this paper we consider a Levi-flat real hypersurface M in a nonflat complex space form such that the structure vector field ξ of M is not an eigenvector field of the shape operator A of M at each point.

We set $\beta = \|A\xi - \eta(A\xi)\xi\| \neq 0$ and hence write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field defined by $U := (A\xi - \alpha\xi)/\beta$. Now we obtain a local orthonormal frame for the tangent space of M by using U, and the frame is called a standard non-Hopf frame for dimension three (see [5, p. 445]). The so-called structure Lie operator L is defined by

$$g(LX,Y) = (\mathscr{L}_\xi g)(X,Y)$$

for any vector fields X,Y. Using (8) in the above equation, we obtain $L = \phi A - A\phi$. The structure Lie operator is important in geometry of real hypersurfaces. One of famous characterization theorems says that a real hypersurface in a nonflat complex space form is an open subset of a type (A) hypersurface if and only if $L = 0$ (see [5, Theorem 8.37]). Some recent results regarding the structure Lie operators can be seen in [26, 27].

In what follows let us assume that the structure Lie operator L is η-parallel with respect to the Levi-Civita connection, i.e.,

$$g((\nabla_X L)Y,Z) = 0 \tag{11}$$

for any $X,Y,Z \in \ker\eta$. Inserting $L = \phi A - A\phi$ in (11), we obtain

$$g((\nabla_X\phi)AY + \phi(\nabla_X A)Y - (\nabla_X A)\phi Y - A(\nabla_X\phi)Y,Z) = 0$$

for any $X,Y,Z \in \ker\eta$. Simplifying the above equation by using (7) we obtain

$$g((\nabla_X A)Y,\phi Z) + g((\nabla_X A)\phi Y,Z) = \eta(AY)g(AX,Z) + \eta(AZ)g(AX,Y) \tag{12}$$

for any $X,Y,Z \in \ker\eta$. The interchange of X,Y,Z cyclicly in (12) twice we obtain

$$g((\nabla_Y A)Z,\phi X) + g((\nabla_Y A)\phi Z,X) = \eta(AZ)g(AY,X) + \eta(AX)g(AY,Z) \tag{13}$$

and

$$g((\nabla_Z A)X,\phi Y) + g((\nabla_Z A)\phi X,Y) = \eta(AX)g(AZ,Y) + \eta(AY)g(AZ,X) \tag{14}$$

for any $X,Y,Z \in \ker\eta$. The addition of (12) to (13) gives an equation. By using the Codazzi equation (10) and the symmetry of ∇A, subtracting this from (14) gives

$$g((\nabla_X A)Y,\phi Z) = \eta(AZ)g(AX,Y) \tag{15}$$

for any $X,Y,Z \in \ker\eta$.

On the other hand, as we have assumed that the hypersurface M is Levi-flat, so we have (2). The covariant derivative of (2) gives

$$g((\nabla_X(\phi A+A\phi))Y,Z)+g((\phi A+A\phi)\nabla_X Y,Z)+g((\phi A+A\phi)Y,\nabla_X Z)=0$$

for any $X,Y,Z\in\ker\eta$. If we denote by $(\nabla_X Y)_{\mathscr{D}}$ and $(\nabla_X Z)_{\mathscr{D}}$ the $\ker\eta$-components of $\nabla_X Y$ and $\nabla_X Z$, respectively, we write $\nabla_X Y=(\nabla_X Y)_{\mathscr{D}}+g(\nabla_X Y,\xi)\xi$ and $\nabla_X Z=(\nabla_X Z)_{\mathscr{D}}+g(\nabla_X Z,\xi)\xi$. Putting these two into the previous equation and using (2) we obtain

$$g((\nabla_X(\phi A+A\phi))Y,Z)-\eta(\nabla_X Y)\eta((\phi A+A\phi)Z)+\eta(\nabla_X Z)\eta((\phi A+A\phi)Y)=0$$

for any $X,Y,Z\in\ker\eta$. This is reduced to

$$\begin{aligned}&g((\nabla_X\phi)AY+\phi(\nabla_X A)Y+(\nabla_X A)\phi Y+A(\nabla_X\phi)Y,Z)\\&-\eta(\nabla_X Y)\eta(A\phi Z)+\eta(\nabla_X Z)\eta(A\phi Y)=0.\end{aligned}$$

Simplifying the above equation by using (7) we obtain

$$\begin{aligned}&\eta(AY)g(AX,Z)-g((\nabla_X A)Y,\phi Z)+g((\nabla_X A)\phi Y,Z)-\eta(AZ)g(AX,Y)\\&+g(Y,\phi AX)\eta(A\phi Z)-g(Z,\phi AX)\eta(A\phi Y)=0\end{aligned}\tag{16}$$

for any $X,Y,Z\in\ker\eta$. Subtracting this from (12) and eliminating $g((\nabla_X A)\phi Y,Z)$ we obtain

$$2\eta(AY)g(AX,Z)-2g((\nabla_X A)Y,\phi Z)=g(Y,\phi AX)g(\phi A\xi,Z)-g(Z,\phi AX)g(\phi A\xi,Y)$$

for any $X,Y,Z\in\ker\eta$. Putting equation (15) into the above equation and eliminating $g((\nabla_X A)Y,\phi Z)$ we obtain

$$\begin{aligned}&2\eta(AY)g(AX,Z)-2\eta(AZ)g(AX,Y)\\&=g(Y,\phi AX)g(\phi A\xi,Z)-g(Z,\phi AX)g(\phi A\xi,Y)\end{aligned}\tag{17}$$

for any $X,Y,Z\in\ker\eta$.

Putting $A\xi=\alpha\xi+\beta U$ into (17) and using $\beta\neq 0$ we obtain

$$\begin{aligned}&2g(Y,U)g(AX,Z)-2g(Z,U)g(AX,Y)\\&=g(Y,\phi AX)g(\phi U,Z)-g(Z,\phi AX)g(\phi U,Y)\end{aligned}$$

for any $X,Y,Z\in\ker\eta$. In the above equation setting $Z=U$ we obtain

$$2g(Y,U)g(AX,U)-2g(AX,Y)+g(U,\phi AX)g(\phi U,Y)=0.\tag{18}$$

In this equation setting $Y=\phi U$ we obtain $g(AX,\phi U)=0$ for any $Y\in\ker\eta$. This is equivalent to $A\phi U=0$. Using this in (18) we have $g(Y,U)g(AX,U)-g(AX,Y)=0$ for any $X,Y\in\ker\eta$ and this implies that $AX=\beta g(X,U)\xi+g(AX,U)U$ for any $X\in\ker\eta$. In Levi-flatness condition (2), setting $X=U$ and $Y=\phi U$, and using $A\phi U=0$ we obtain $g(AU,U)=0$. This, together the previous equation, implies that $AU=\beta\xi$, and hence we have $AX=\beta g(X,U)\xi$ for $X\in\ker\eta$. This is equivalent to $g(AX,Y)=0$ for any $X,Y\in\ker\eta$. In fact, it is necessary and sufficient condition for a real hypersurface to be ruled (see [5, Proposition 8.27] or [18, Proposition 2]).

THEOREM 1. *A Levi-flat real hypersurface in nonflat complex space forms is ruled if and only if the structure Lie operator is η-parallel.*

The "if" part of Theorem 1 follows from the previous statement and the "only if" part of this theorem is trivial.

As introduced in the first section, the conclusion still holds if we replace the structure Lie operator in

Theorem 1 by the shape operator.

4. GTW CONNECTION

The Tanaka-Webster connection, introduced independently by Tanaka in [24] and Webster in [31], is a unique affine connection on a non-degenerate pseudo-Hermitian *CR*-manifold. Tanno in [25] introduced the notion of the generalized Tanaka-Webster connection (in short, GTW connection) on a contact Riemannian manifold and such a connection is the same with the Tanaka-Webster connection when the associated *CR*-structure is integrable. On real hypersurfaces in Kähler manifolds there exists an almost contact metric structure (ϕ,ξ,η,g). Cho in [6, 7] introduced the generalized Tanaka-Webster connection on a real hypersurface in Kähler manifolds, that is,

$$\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX,Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{19}$$

for any vector fields X,Y and certain non-zero constant k, where ∇ and A are the Levi-Civita connection and the shape operator of the hypersurface, respectively. The generalized Tanaka-Webster connection on real hypersurfaces coincides with the Tanaka-Webster connection when $\phi A + A\phi = 2k\phi$.

A tensor field T of type $(1,1)$ on real hypersurfaces is called η-parallel with respect to the GTW connection if $g((\widehat{\nabla}^{(k)}_X T)Y,Z)=0$ for any vector fields $X,Y,Z\in\ker\eta$. Cho in [6] proved that the shape operator of a real hypersurface in a nonflat complex space form is GTW-parallel (i.e., $\widehat{\nabla}^{(k)}_X A = 0$) if and only if the hypersurface is locally congruent to one of real hypersurfaces of type (A) or (B). GTW-parallelism for some other operators on real hypersurfaces in nonflat complex space forms were considered in [14, 28] for the structure Jacobi operator, in [8] for the Ricci operator and in [29] for the *h*-operator.

In this section, we consider GTW η-parallel shape operator which is much weaker than GTW-parallelism used in [6].

Now suppose that the shape operator A of a Levi-flat real hypersurface M in a nonflat complex space form is η-parallel with respect to the GTW connection. From a direct calculation, we have

$$\begin{aligned}(\widehat{\nabla}^{(k)}_X A)Y =&(\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX\\ &- k\eta(X)\phi AY - g(\phi AX,Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y\end{aligned}$$

for any vector fields X,Y. By this, the GTW η-parallelism of the shape operator A is equivalent to

$$g((\nabla_X A)Y,Z) = -\eta(AY)g(AX,\phi Z) - \eta(AZ)g(AX,\phi Y) \tag{20}$$

for any $X,Y,Z\in\ker\eta$. The interchanging of X and Y in the above equation gives

$$g((\nabla_Y A)X,Z) = -\eta(AX)g(AY,\phi Z) - \eta(AZ)g(AY,\phi X) \tag{21}$$

for any $X,Y,Z\in\ker\eta$. Subtracting (21) from (20) gives an equation. Simplifying this equation by the Codazzi equation (10) gives

$$-\eta(AY)g(AX,\phi Z) + \eta(AX)g(AY,\phi Z) - \eta(AZ)(g(AX,\phi Y) - g(AY,\phi X)) = 0$$

for any $X,Y,Z\in\ker\eta$. As we have assumed that the hypersurface M is Levi-flat, we have (2). Simplifying the above equation by (2) we get

$$-\eta(AY)g(AX,\phi Z) + \eta(AX)g(AY,\phi Z) = 0$$

for any $X,Y,Z\in\ker\eta$. This reduces to

$$-\eta(AY)AX + \eta(AX)AY = 0$$

for any $X,Y\in\ker\eta$. As seen in proof of Theorem 1, Levi-flatness implies that M is not Hopf. Setting $A\xi=\alpha\xi+\beta U$ in the above equation and using $\beta\neq 0$ we obtain

$$-g(U,Y)AX+g(U,X)AY=0$$

for any $X,Y\in\ker\eta$. Setting $Y\in\{\xi,U\}^{\perp}$ in the above equation, we obtain $AY=0$ for any $Y\in\{\xi,U\}^{\perp}$. Setting $X=U$ and $Y=\phi U$ in (2) and using $A\phi U=0$, we obtain $AU=\beta\xi$. Now we obtain $g(AX,Y)=0$ for any $X,Y\in\ker\eta$.

THEOREM 2. *The shape operator of a Levi-flat real hypersurface M in nonflat complex space forms is η-parallel with respect to the GTW connection if and only if M is ruled.*

Now suppose that the structure Lie operator L of a Levi-flat real hypersurface M in a nonflat complex space form is η-parallel with respect to the GTW connection, i.e.,

$$g((\widehat{\nabla}^{(k)}_X L)Y,Z)=0$$

for any $X,Y,Z\in\ker\eta$. By a direct calculation, the above equation transforms into

$$\begin{aligned}&g((\nabla_X\phi)AY+\phi(\nabla_XA)Y-(\nabla_XA)\phi Y-A(\nabla_X\phi)Y,Z)\\&-g(\phi A\xi,Y)g(\phi AX,Z)-g(\phi AX,Y)g(\phi A\xi,Z)=0\end{aligned}$$

for any $X,Y,Z\in\ker\eta$. Simplifying the above equation by using (7) we obtain

$$\begin{aligned}&g((\nabla_XA)Y,\phi Z)+g((\nabla_XA)\phi Y,Z)=\eta(AY)g(AX,Z)\\&+g(AX,Y)\eta(AZ)-g(\phi A\xi,Y)g(\phi AX,Z)-g(\phi AX,Y)g(\phi A\xi,Z)\end{aligned}\tag{22}$$

for any $X,Y,Z\in\ker\eta$. The interchange of X,Y,Z cyclicly in (12) twice we obtain

$$\begin{aligned}&g((\nabla_YA)Z,\phi X)+g((\nabla_YA)\phi Z,X)=\eta(AZ)g(AY,X)\\&+g(AY,Z)\eta(AX)-g(\phi A\xi,Z)g(\phi AY,X)-g(\phi AY,Z)g(\phi A\xi,X)\end{aligned}\tag{23}$$

and

$$\begin{aligned}&g((\nabla_ZA)X,\phi Y)+g((\nabla_ZA)\phi X,Y)=\eta(AX)g(AZ,Y)\\&+g(AZ,X)\eta(AY)-g(\phi A\xi,X)g(\phi AZ,Y)-g(\phi AZ,X)g(\phi A\xi,Y)\end{aligned}\tag{24}$$

for any $X,Y,Z\in\ker\eta$. The addition of (22) to (23) gives an equation. By using the Codazzi equation (10) and the symmetry of ∇A, subtracting this from (24) gives

$$\begin{aligned}&2g((\nabla_XA)Y,\phi Z)=2\eta(AZ)g(AX,Y)-g(\phi A\xi,Y)(g(\phi AX,Z)-g(\phi AZ,X))\\&-g(\phi A\xi,Z)(g(\phi AX,Y)+g(\phi AY,X))-g(\phi A\xi,X)(g(\phi AY,Z)-g(\phi AZ,Y))\end{aligned}$$

for any $X,Y,Z\in\ker\eta$. On the other hand, as we have assumed that the hypersurface M is Levi-flat, so we have (2). Simplifying the above equation by using (2) gives

$$2g((\nabla_XA)Y,\phi Z)=2\eta(AZ)g(AX,Y)-g(\phi A\xi,Z)(g(\phi AX,Y)+g(\phi AY,X))\tag{25}$$

for any $X,Y,Z\in\ker\eta$. Is is remarked that equation (16) holds in this situation and in fact this equation depends only on Levi-flatness condition (2). Putting (25) into (16) we obtain

$$\begin{aligned}&4\eta(AY)g(AX,Z)-4\eta(AZ)g(AX,Y)\\&+g(\phi A\xi,Z)(g(\phi AY,X)-g(\phi AX,Y))-g(\phi A\xi,X)(g(\phi AZ,X)-g(\phi AX,Z))=0\end{aligned}$$

for any $X,Y,Z \in \ker\eta$. Simplifying the above equation by using the Levi-flatness condition (2) we obtain

$$\eta(AY)g(AX,Z) - \eta(AZ)g(AX,Y) = 0 \tag{26}$$

for any $X,Y,Z \in \ker\eta$. The remaining analysis is similar to that of Theorem 2, so here we omit it and obtain the following theorem.

THEOREM 3. *The structure Lie operator of a Levi-flat real hypersurface M in nonflat complex space forms is η-parallel with respect to the GTW connection if and only if M is ruled.*

The combination of Theorems 1, 2, 3 and results in [16, 17] give the following corollary.

COROLLARY 1. *On a Levi-flat real hypersurface M in nonflat complex space forms, the following statements are equivalent mutually.*

- *The real hypersurface M is ruled.*
- *The shape operator is η-parallel with respect to the Levi-Civita connection or the GTW connection.*
- *The structure Lie operator is η-parallel with respect to the Levi-Civita connection or the GTW connection.*

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