



TWO NEW q -CONGRUENCES FROM GASPER'S KARLSSON–MINTON TYPE SUMMATION

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Abstract. We give two new q -congruences by using the method of “creative microscoping” and Gasper’s Karlsson–Minton type summation. In particular, we present a q -analogue of a congruence of Barman and Saikia.

Keywords: supercongruence, p -adic Gamma function, cyclotomic polynomials, creative microscoping.

Mathematics Subject Classification (MSC2020): 33D15; 11A07; 11B65.

1. INTRODUCTION

Rodriguez-Villegas [12] studied hypergeometric families of Calabi–Yau manifolds, and found a number of possible supercongruences. For instance, he observed that, for any prime $p > 2$,

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^2}{k!^2} \equiv (-1)^{(p-1)/2} \pmod{p^2}, \quad (1)$$

where $(a)_0 = 1$ and $(a)_n = a(a+1) \cdots (a+n-1)$ ($n \geq 1$) is the *rising factorial*. Mortenson [11] first confirmed the congruence (1). Later, the first author and Zeng [4] obtained a q -analogue of (1):

$$\sum_{k=0}^{p-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} q^{2k} \equiv (-1)^{(p-1)/2} q^{(p^2-1)/4} \pmod{[p]^2} \quad \text{for any odd prime } p.$$

Here and throughout the paper, $(a; q)_0 = 1$ and $(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$ ($n \geq 1$) is the q -shifted factorial, and $[n] = 1 + q + \cdots + q^{n-1}$ is the q -integer. For convenience, we will also adopt the condensed notation $(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$.

In 2020, Barman and Saikia [1] gave a generalization of (1) as follows: for $d \geq 1$ and any prime p satisfying $p \equiv 1 \pmod{d^2 + d}$,

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv (-1)^{d+1} \Gamma_p(\frac{1}{d})^d \Gamma_p(\frac{d}{d+1})^{d+1} \pmod{p^2}, \quad (2)$$

where $\Gamma_p(x)$ denotes the p -adic Gamma function (see [8]).

Let $\Phi_n(q)$ be the n -th cyclotomic polynomial in q , which can be written as

$$\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} (q - \zeta^k),$$

where ζ is an n -th primitive root of unity. The first aim of this note is to give the following q -analogue of Barman and Saikia's congruence (2).

THEOREM 1. *Let d and n be positive integers with $n \equiv 1 \pmod{d^2+d}$. Then, modulo $\Phi_n(q)^2$,*

$$\begin{aligned} & \sum_{k=0}^{(n-1)/(d+1)} \frac{(q^d; q^{d^2+d})_k^{d+1} q^{(d^2+d)k}}{(q^{d+1}; q^{d^2+d})_k^d (q^{d^2+d}; q^{d^2+d})_k} \\ & \equiv \frac{(-1)^{(n-1)/(d+1)} (q^{d^2+d}; q^{d^2+d})_{(n-1)/(d+1)} q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}^d}. \end{aligned} \quad (3)$$

For n prime, letting $q \rightarrow 1$ in Theorem 1, we arrive at the following congruence: for $d \geq 1$ and any prime $p \equiv 1 \pmod{d^2+d}$,

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv \frac{(-1)^{(p-1)/(d+1)} (\frac{p-1}{d+1})!}{(\frac{1}{d})_{(p-1)/(d^2+d)}^d} \pmod{p^2}. \quad (4)$$

In view of properties of the p -adic Gamma function (see [10, Section 2]), it is not hard to show that

$$\frac{(-1)^{(p-1)/(d+1)} (\frac{p-1}{d+1})!}{(\frac{1}{d})_{(p-1)/(d^2+d)}^d} \equiv (-1)^{d+1} \Gamma_p(\frac{1}{d})^d \Gamma_p(\frac{d}{d+1})^{d+1} \pmod{p^2}. \quad (5)$$

Hence, the congruence (4) is equivalent to (2).

We shall also establish the following congruence similar to (2).

THEOREM 2. *Let $d \geq 1$ and let p be a prime with $p \equiv 1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{k(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv \frac{(-1)^{d+2}}{2(d^2+d)} \Gamma_p(\frac{1}{d})^d \Gamma_p(\frac{d}{d+1})^{d+1} \pmod{p^2}. \quad (6)$$

Since $\Gamma_p(1) = -1$ and $\Gamma_p(\frac{1}{2})^2 = (-1)^{(p+1)/2}$, for $d = 1$, the congruence (6) reduces to

$$\sum_{k=0}^{p-1} \frac{k(\frac{1}{2})_k^2}{k!^2} \equiv \frac{(-1)^{(p+1)/2}}{4} \pmod{p^2},$$

of which a generalization modulo p^3 for $p > 3$ has already been given by Sun [13, Theorem 1.2, (1.8) and (1.10)].

It is easy to see that, for any prime $p > 2$,

$$\sum_{k=0}^{(p+1)/2} \frac{(-\frac{1}{2})_k^2}{k!^2} = \frac{2p+3}{4^{p+1}} \left(\frac{p+1}{(p+1)/2} \right)^2 \equiv 0 \pmod{p^2}. \quad (7)$$

The last aim of this note is to give the following generalization of (7).

THEOREM 3. *Let d and n be positive integers with $n \equiv 2d + 1 \pmod{d^2 + d}$. Then*

$$\sum_{k=0}^{(n+1)/(d+1)} \frac{(q^{-d}; q^{d^2+d})_k^{d+1} q^{(d^2+d)k}}{(q^{d+1}; q^{d^2+d})_k^d (q^{d^2+d}; q^{d^2+d})_k} \equiv 0 \pmod{\Phi_n(q)^2}. \quad (8)$$

In particular, letting n be a prime and taking $q \rightarrow 1$ in Theorem 3, we are led to the conclusion.

COROLLARY 1. *Let $d \geq 1$ and let p be a prime with $p \equiv 2d + 1 \pmod{d^2 + d}$. Then*

$$\sum_{k=0}^{(p+1)/(d+1)} \frac{\left(-\frac{1}{d+1}\right)_k^{d+1}}{\left(\frac{1}{d}\right)_k^d k!} \equiv 0 \pmod{p^2}.$$

Like the proof of Theorem 2, we can also deduce the following congruence from Theorem 3.

COROLLARY 2. *Let $d \geq 1$ and let p be a prime with $p \equiv 2d + 1 \pmod{d^2 + d}$. Then*

$$\sum_{k=0}^{(p+1)/(d+1)} \frac{k \left(-\frac{1}{d+1}\right)_k^{d+1}}{\left(\frac{1}{d}\right)_k^d k!} \equiv 0 \pmod{p^2}.$$

2. PROOF OF THEOREM 1

We will make use of Gasper's Karlsson–Minton type summation (see [2, (1.9.9)]; and see [3, (5.13)] for a generalization): for all non-negative integers n_1, \dots, n_m ,

$$\sum_{k=0}^N \frac{(q^{-N}, b_1 q^{n_1}, \dots, b_m q^{n_m}; q)_k}{(q, b_1, \dots, b_m; q)_k} q^k = (-1)^N \frac{(q; q)_N b_1^{n_1} \cdots b_m^{n_m}}{(b_1; q)_{n_1} \cdots (b_m; q)_{n_m}} q^{\binom{n_1}{2} + \cdots + \binom{n_m}{2}}, \quad (9)$$

where $N = n_1 + \cdots + n_m$. For some recent congruences and q -congruences related to (9), see [5, 7, 9].

We first build the following generalization of Theorem 1 with an extra parameter a by employing the “creative microscoping” method devised in [6].

THEOREM 4. *Let $d, n > 1$ be integers with $n \equiv 1 \pmod{d^2 + d}$. Let a be an indeterminate. Then, modulo $(1 - aq^n)(a - q^n)$,*

$$\begin{aligned} & \sum_{k=0}^{(n-1)/(d+1)} \frac{(a^d q^d, a^{d-2} q^d, \dots, a q^d; q^{d^2+d})_k}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \dots, a^2 q^{d+1}, q^{d+1}; q^{d^2+d})_k} \\ & \times \frac{(a^{-d} q^d, a^{2-d} q^d, \dots, a^{-1} q^d; q^{d^2+d})_k q^{(d^2+d)k}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \dots, a^{-2} q^{d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k} \\ & \equiv \frac{(-1)^{(n-1)/(d+1)} (q^{d^2+d}; q^{d^2+d})_{(n-1)/(d+1)}}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \dots, a^2 q^{d+1}, q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}} \\ & \times \frac{q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \dots, a^{-2} q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}} \end{aligned} \quad (10)$$

if d is odd, and

$$\begin{aligned}
& \sum_{k=0}^{(n-1)/(d+1)} \frac{(a^d q^d, a^{d-2} q^d, \dots, a^2 q^d, q^d; q^{d^2+d})_k}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \dots, a q^{d+1}; q^{d^2+d})_k} \\
& \times \frac{(a^{-d} q^d, a^{2-d} q^d, \dots, a^{-2} q^d; q^{d^2+d})_k q^{(d^2+d)k}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \dots, a^{-1} q^{d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k} \\
& \equiv \frac{(-1)^{(n-1)/(d+1)} (q^{d^2+d}; q^{d^2+d})_{(n-1)/(d+1)}}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \dots, a q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}} \\
& \times \frac{q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \dots, a^{-1} q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}} \quad (11)
\end{aligned}$$

if d is even.

Proof. It is obvious that $\gcd(d, n) = 1$, and therefore none of the numbers $d, 2d, \dots, (n-1)d$ are divisible by n . This indicates that the denominators on the left-hand side of (10) do not have the factor $1 - aq^n$ nor $1 - a^{-1}q^n$. Thus, for $a = q^{-n}$ or $a = q^n$, the left-hand side of (10) may be written as

$$\begin{aligned}
& \sum_{k=0}^{(n-1)/(d+1)} \frac{(q^{-(n-1)d}, q^{-(n-1)d+2n}, \dots, q^{-n+d}; q^{d^2+d})_k}{(q^{-(d-1)n+d+1}, q^{-(d-3)n+d+1}, \dots, q^{-2n+d+1}, q^{d+1}; q^{d^2+d})_k} \\
& \times \frac{(q^{(n+1)d}, q^{(n+1)d-2n}, \dots, q^{n+d}; q^{d^2+d})_k q^{(d^2+d)k}}{(q^{(d-1)n+d+1}, q^{(d-3)n+d+1}, \dots, q^{2n+d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k}. \quad (12)
\end{aligned}$$

Letting $q \mapsto q^{d^2+d}$, $N = (n-1)/(d+1)$, $m = d$, $b_j = q^{-(d-1)n+d+1+(2j-2)n}$ and $n_j = (n-1)/(d^2+d)$ ($1 \leq j \leq d$) in (9), we conclude that (12) is equal to

$$\begin{aligned}
& \frac{(-1)^{(n-1)/(d+1)} (q^{d^2+d}; q^{d^2+d})_{(n-1)/(d+1)}}{(q^{-(d-1)n+d+1}, q^{-(d-3)n+d+1}, \dots, q^{-2n+d+1}, q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}} \\
& \times \frac{q^{(n-1)+d(d^2+d)\binom{(n-1)}{2}}}{(q^{(d-1)n+d+1}, q^{(d-3)n+d+1}, \dots, q^{2n+d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}},
\end{aligned}$$

which is just the $a = q^{-n}$ or $a = q^n$ case of the right-hand side (10). This proves the q -congruence (10).

Similarly, for $a = q^{-n}$ or $a = q^n$, the left-hand side of (11) may be expressed as

$$\begin{aligned}
& \sum_{k=0}^{(n-1)/(d+1)} \frac{(q^{-(n-1)d}, q^{-(n-1)d+2n}, \dots, q^{-2n+d}, q^d; q^{d^2+d})_k}{(q^{-(d-1)n+d+1}, q^{-(d-3)n+d+1}, \dots, q^{-n+d+1}; q^{d^2+d})_k} \\
& \times \frac{(q^{(n+1)d}, q^{(n+1)d-2n}, \dots, q^{2n+d}, q^{d^2+d})_k q^{(d^2+d)k}}{(q^{(d-1)n+d+1}, q^{(d-3)n+d+1}, \dots, q^{n+d+1}, q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k}. \quad (13)
\end{aligned}$$

Letting $q \mapsto q^{d^2+d}$, $N = (n-1)/(d+1)$, $m = d$, $b_j = q^{-(d-1)n+d+1+(2j-2)n}$ and $n_j = (n-1)/(d^2+d)$ ($1 \leq j \leq d$) in (9), we deduce that (13) is equal to the $a = q^{-n}$ or $a = q^n$ case of the right-hand side (11). This establishes (11). \square

Proof of Theorem 1. Note that $\Phi_n(q)$ is a factor of $1 - q^m$ if and only if m is divisible by n . Hence, when $a = 1$ the denominators of (10) are all coprime with $\Phi_n(q)$. Meanwhile, when $a = 1$ the polynomial $(1 - aq^n)(a - q^n) = (1 - q^n)^2$ incorporates the factor $\Phi_n(q)^2$. The proof of (3) then follows immediately from the $a = 1$ case of (10) and (11). \square

3. PROOF OF THEOREM 2

Let $n > 1$ be an integer with $n \equiv 1 \pmod{d^2 + d}$. Performing the substitution $q \mapsto q^{-1}$ in (3), we get its dual form: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^d; q^{d^2+d})_k^{d+1}}{(q^{d+1}; q^{d^2+d})_k^d (q^{d^2+d}; q^{d^2+d})_k} \equiv \frac{(-1)^{(n-1)/(d+1)} (q^{d^2+d}; q^{d^2+d})_{(n-1)/(d+1)} q^{\frac{(1-n)(nd+d^2)}{2(d+1)}}}{(q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}^d}. \quad (14)$$

Subtracting (3) from (14) and dividing both sides by $1 - q$, we are led to

$$\begin{aligned} & \sum_{k=0}^{(n-1)/(d+1)} \frac{(q^d; q^{d^2+d})_k^{d+1} (1 - q^{(d^2+d)k})}{(q^{d+1}; q^{d^2+d})_k^d (q^{d^2+d}; q^{d^2+d})_k (1 - q)} \\ & \equiv \frac{(-1)^{(n-1)/(d+1)} (q^{d^2+d}; q^{d^2+d})_{(n-1)/(d+1)} q^{\frac{(1-n)(nd+d^2)}{2(d+1)}} (1 - q^{\frac{n^2-1}{2}})}{(q^{d+1}; q^{d^2+d})_{(n-1)/(d^2+d)}^d (1 - q)} \pmod{\Phi_n(q)^2}. \end{aligned}$$

Letting $n = p$ be a prime and taking the limit as $q \rightarrow 1$ in the above q -supercongruence, we obtain the following result: for any positive integer d and prime $p \equiv 1 \pmod{d^2 + d}$,

$$\begin{aligned} \sum_{k=0}^{(p-1)/(d+1)} \frac{k \left(\frac{1}{d+1}\right)_k^{d+1}}{\left(\frac{1}{d}\right)_k^d k!} & \equiv \frac{(-1)^{(p-1)/(d+1)} \left(\frac{p-1}{d+1}\right)! (p^2 - 1)}{2 \left(\frac{1}{d}\right)_{(p-1)/(d^2+d)}^d (d^2 + d)} \\ & \equiv \frac{(-1)^{(p+d)/(d+1)} \left(\frac{p-1}{d+1}\right)!}{2 \left(\frac{1}{d}\right)_{(p-1)/(d^2+d)}^d (d^2 + d)} \pmod{p^2}. \end{aligned}$$

The proof then follows from the congruence (5).

4. PROOF OF THEOREM 3

We will utilize another Karlsson–Minton type summation due to Gasper (see [2, (1.9.11)]): for all non-negative integers n_1, \dots, n_m ,

$$\sum_{k=0}^N \frac{(q^{-N}, b_1 q^{n_1}, \dots, b_m q^{n_m}; q)_k}{(q, b_1, \dots, b_m; q)_k} q^k = 0, \quad (15)$$

where $N > n_1 + \dots + n_m$.

We first establish the following parametric generalization of Theorem 3.

THEOREM 5. *Let $d, n > 1$ be integers with $n \equiv 2d + 1 \pmod{d^2 + d}$. Let a be an indeterminate. Then, modulo $(1 - aq^n)(a - q^n)$,*

$$\begin{aligned} & \sum_{k=0}^{(n+1)/(d+1)} \frac{(a^d q^{-d}, a^{d-2} q^{-d}, \dots, a q^{-d}; q^{d^2+d})_k}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \dots, a^2 q^{d+1}, q^{d+1}; q^{d^2+d})_k} \\ & \times \frac{(a^{-d} q^{-d}, a^{2-d} q^{-d}, \dots, a^{-1} q^{-d}; q^{d^2+d})_k q^{(d^2+d)k}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \dots, a^{-2} q^{d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k} \equiv 0 \end{aligned} \quad (16)$$

if d is odd, and

$$\begin{aligned} & \sum_{k=0}^{(n+1)/(d+1)} \frac{(a^d q^{-d}, a^{d-2} q^{-d}, \dots, a^2 q^{-d}, q^{-d}; q^{d^2+d})_k}{(a^{d-1} q^{d+1}, a^{d-3} q^{d+1}, \dots, a q^{d+1}; q^{d^2+d})_k} \\ & \times \frac{(a^{-d} q^{-d}, a^{2-d} q^{-d}, \dots, a^{-2} q^{-d}; q^{d^2+d})_k q^{(d^2+d)k}}{(a^{1-d} q^{d+1}, a^{3-d} q^{d+1}, \dots, a^{-1} q^{d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k} \equiv 0 \end{aligned} \quad (17)$$

if d is even.

Proof. It is easy to see that $\gcd(d, n) = 1$ and so none of the numbers $d, 2d, \dots, (n-1)d$ are multiples of n . This implies that the denominators of the left-hand sides of (16) have no factors $1 - aq^n$ and $1 - a^{-1}q^n$. Therefore, for $a = q^{-n}$ or $a = q^n$, the left-hand side of (16) can be expressed as

$$\begin{aligned} & \sum_{k=0}^{(n+1)/(d+1)} \frac{(q^{-(n+1)d}, q^{-(n+1)d+2n}, \dots, q^{-n+d}; q^{d^2+d})_k}{(q^{-(d-1)n+d+1}, q^{-(d-3)n+d+1}, \dots, q^{-2n+d+1}, q^{d+1}; q^{d^2+d})_k} \\ & \times \frac{(q^{(n-1)d}, q^{(n-1)d-2n}, \dots, q^{n-d}; q^{d^2+d})_k q^{(d^2+d)k}}{(q^{(d-1)n+d+1}, q^{(d-3)n+d+1}, \dots, q^{2n+d+1}; q^{d^2+d})_k (q^{d^2+d}; q^{d^2+d})_k}. \end{aligned} \quad (18)$$

Letting $q \mapsto q^{d^2+d}$, $N = (n+1)/(d+1)$, $m = d$, $b_j = q^{-(d-1)n+d+1+(2j-2)n}$ and $n_j = (n-2d-1)/(d^2+d)$ ($1 \leq j \leq d$) in (15), we conclude that (18) is equal to 0, which is just the $a = q^{-n}$ or $a = q^n$ case of the right-hand side of (16). Namely, the congruence (16) holds. Exactly in the same way, we can prove the q -congruence (17). \square

Proof of Theorem 3. When $a = 1$, the polynomial $(1 - aq^n)(a - q^n)$ contains the factor $\Phi_n(q)^2$, which is coprime with the denominators of the left-hand sides of (16) and (17). Hence, the congruence (8) immediately follows from the $a = 1$ case of (16) and (17). \square

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Received January 1, 2025