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PURE BRANCHING AND TOTAL MASS PROCESSES

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Abstract. This paper demonstrates that the total mass process of a branching Markov process with a spatially constant branching mechanism behaves identically to that of its corresponding pure branching process: either a continuous time Galton-Watson process or a continuous-state branching (CB) process. The discrete time context is also treated. The study analyses extinction in discrete-time subcritical and supercritical regimes, derives evolution equations for non-local pure branching processes, and examines the one-dimensional CB-process. Using potential theory and stochastic analysis, the research advances branching process theory and population dynamics.

Keywords*:* stochastic processes, branching, Galton-Watson.

Mathematics Subject Classification (MSC2020): primary 60J80; secondary 60J25.

1. INTRODUCTION

Branching processes describe populations where individuals reproduce and die independently, playing a key role in probability, biology, and physics [1, 17]. They are used to model phenomena such as population dynamics and particle systems, with implications for genetics, epidemiology, and queueing theory. Central to these studies is the *total mass process*, which reflects the population size over time and highlights growth, extinction, and stability features. This work examines the total mass processes in branching Markov processes, emphasizing non-local branching and superprocesses under spatially constant branching.

Our first result shows that, when the branching mechanism is spatially constant, the total mass process of a branching Markov process matches that of a pure branching process (Proposition 4). In discrete time, this is a Galton-Watson process; in continuous time it is either a one-dimensional continuous-state branching (CB) process [12, 20], or a Galton-Watson process in continuous time (see for instance another approach in [21]).

We also study branching Markov chains in discrete time, where the total mass process is a Galton-Watson process defined by the initial distribution and offspring probabilities (Proposition 5). Here, extinction occurs almost surely when the expected number of offspring is at most one (Corollary 1), while supercritical cases yield a positive survival probability (Corollary 2) [1].

For non-local pure branching on finite configurations, we derive evolution equations and identify conditions preserving the branching structure (Proposition 1) [4, 7]. This approach also applies to the trivial Markov process, accentuating pure branching behavior.

Lastly, the one-dimensional CB-process, combining linear growth, quadratic competition, and jumps, is analyzed in Section 3. Its Markovian evolution bridges discrete and continuous frameworks, resonating with superprocess theory [14] (see also [2]).

The paper is organized as follows: Section 1 covers mathematical preliminaries. Section 2 details the total mass process in both discrete and continuous frameworks. Section 3 presents non-local pure branching processes and the CB-process with illustrative examples. These results deepen the theoretical insights into branching processes and their mass dynamics.

2. NON-LOCAL PURE BRANCHING PROCESSES ON FINITE CONFIGURATIONS

Based on [11], we consider finite configurations. The space $\widehat{E}$ consists of all finite configurations of a Lusin space E, namely the union $\widehat{E} = \bigcup_{k\geq 0} E^{(k)}$ where for $k \geq 1$, $E^{(k)}$ denotes the unordered k-tuples (i.e. the quotient of E^k by the permutation group) and $E^{(0)} = \{\mathbf{0}\}$ is the empty configuration (see, e.g., [10, 16]).

A branching mechanism on $\widehat{E}$ is specified by a sequence $(b_k, B_k)_{k\geq 0}$ where $(b_k)_{k\geq 0}$ is a sequence of bounded positive Borel functions satisfying $\sum_{k\geq 0} b_k = 1$ and B_k is a Markov kernel from $E^{(k)}$ to E with $B_0 = \delta_{\mathbf{0}}$; one also sets $m_1 = \|\sum_{k\geq 1} k\, b_k\|_\infty$ (with $1 < m_1 < \infty$) and fixes a positive constant a with $0 < a \leq m_1/(m_1 - 1)$.

For any $\varphi \in p\mathscr{B}(E)$ with $0 \leq \varphi \leq 1$, the integral evolution equation

$$h_t = e^{-at} T_t \varphi + a \int_0^t e^{-a(t-s)} T_{t-s} \sum_{k\geq 0} b_k B_k(h_s^{(k)})\, \mathrm{d}s, \quad t \geq 0, \tag{1}$$

has a unique solution $t \mapsto H_t\varphi$ (jointly measurable in (t,x) and bounded between 0 and 1); here $h^{(k)}(x_1, \ldots, x_k) = h(x_1)\cdots h(x_k)$ (with the convention $h^{(0)}(\mathbf{0}) = 1$). Equation (1) is equivalent to its differential form

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t = (L - a) h_t + a \sum_{k\geq 0} b_k B_k(h_t^{(k)}), \quad t \geq 0, \quad h_0 = \varphi, \tag{2}$$

where L is the generator of the spatial motion (see Remark 4.2(ii) in [7]). The nonlinear semigroup $(H_t)_{t\geq 0}$ induces a branching semigroup of kernels $(\widehat{\mathbf{H}}_t)_{t\geq 0}$ on $\widehat{E}$ via $\widehat{\mathbf{H}}_t \widehat{\varphi} = \widehat{H_t\varphi}$; under additional assumptions (Theorem 4.1 in [7]), one obtains a branching right Markov process $\widehat{X}$ on $\widehat{E}$ whose transitions depend on the spatial motion X, the mechanism $(b_k, B_k)_{k\geq 0}$, and the constant a.

In the case where the spatial motion is trivial (i.e. $X_t^0 = x$ for all $t \geq 0$), the process $\widehat{X^0}$ is a non-local pure branching process and the evolution equation simplifies to

$$h_t^o = e^{-at} \varphi + a \int_0^t e^{-a(t-s)} \sum_{k\geq 0} b_k B_k((h_s^o)^{(k)})\, \mathrm{d}s, \quad t \geq 0, \tag{3}$$

or equivalently, in differential form,

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t^o = -a h_t^o + a \sum_{k\geq 0} b_k B_k((h_t^o)^{(k)}), \quad t \geq 0, \quad h_0^o = \varphi. \tag{4}$$

The solution $H_t^o \varphi$ defines the transition function of $\widehat{X^0}$.

We now state an important result regarding absorbing sets.

PROPOSITION 1. *Let $M \in \mathscr{B}(E)$. If $b_k(x) B_{k,x}(\widehat{M} \cap E^{(k)}) = 0$ for all $k \geq 1$ and $x \in E \setminus M$, so that the measure induced by $b_k B_k$ at $x \notin M$ is supported outside $\widehat{M}$, then $\widehat{E} \setminus \widehat{M}$ is a finely closed absorbing subset of $\widehat{E}$ with respect to $\widehat{X^0}$. Moreover, the restriction of $\widehat{X^0}$ to $\widehat{M}$ remains a pure branching process, induced by the trivial process on M and the restrictions of B_k to M.*

Proof. One applies Theorem 3.2(iii) from [4] with $A = E \setminus M$, noting that A is absorbing for the trivial process X^0. $\square$

For the case of classical kernels (i.e. when $B_kF(x) = F(x,\dots,x)$), the evolution equation reduces to

$$h_t = e^{-at}T_t\varphi + a\int_0^t e^{-a(t-s)}T_{t-s}\Big(\sum_{k\geq 1} b_k h_s^k\Big)\,\mathrm{d}s, \quad t\geq 0,$$

and under this formulation, condition (4.3) holds for every $M\in\mathscr{B}(E)$, so that $\widehat{E}\setminus\widehat{M}$ is absorbing for any such $\widehat{X^0}$ (see [5]). In addition, for the pure branching process $\widehat{X^0} = (\widehat{X_t^0}, \widehat{\mathbb{P}}_\mu^o)$ one may define for $\varphi\in p\mathscr{B}(E)$, $\varphi\leq 1$,

$$h_t(x) = \widehat{\mathbb{E}}^o_{\delta_x}\widehat{\varphi}(\widehat{X_t^0}), \quad x\in E, t\geq 0,$$

so that h_t solves the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}h_t = -ah_t + a\sum_{k\geq 0} b_k B_k(h_t^{(k)}), \quad t\geq 0, \quad h_0=\varphi,$$

which coincides with (4) when $L=0$. Similar equations appear in [18, 19] for locally finite configurations.

As an example, when $E=\{a\}$ (a singleton), $\widehat{E}$ is identified with $\mathbb{N}$ and $M(E)$ with $\mathbb{R}_+$; in this setting multiplicative functions on $\mathbb{N}$ have the form $\widehat{s}(k) = s^k$ for $s\in[0,1]$ while on $\mathbb{R}_+$ the exponential functions $e_s(x) = e^{-xs}$ are natural. Fix $a>0$.

The Galton-Watson process in continuous time on $\mathbb{N}$

For $s\in[0,1]$, the differential equation $h' = -ah + a\sum_{k\geq 0} b_k h^k$ with $h(0)=s$ has a unique solution $h_t(s)$ (alternatively written in integral form as

$$h_t = e^{-at}\Big(s + a\int_0^t e^{ar}\sum_{k\geq 0} b_k h_r^k\,\mathrm{d}r\Big), \quad t\geq 0), \tag{5}$$

from which one obtains a unique Markovian branching kernel $\widehat{h}_t$ on $\mathbb{N}$ satisfying $\widehat{h}_t(\widehat{s}) = \widehat{h_t(s)}$ and a branching Markov process $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}^k)$ on $\mathbb{N}$.

The extended weak generator

Following [11] (see also [6], page 4779), define the Markov kernel $\widehat{B}$ on $\mathbb{N}$ by setting $\widehat{B}u(0) = u(0)$ and for $k\geq 1$, $\widehat{B}u(k) = \sum_{j\in\mathbb{N}} b_j u(j+k-1)$ for bounded u. Then we have:

PROPOSITION 2. *If $u=\widehat{s}$ for $s\in(0,1]$, then $u\in\mathscr{D}(\mathscr{L})$, $\mathscr{L}u(\mathbf{0})=0$, and for $k\geq 1$*

$$\mathscr{L}u(k) = ak\Big[\widehat{B}u(k) - u(k)\Big] = ak\int_{\mathbb{N}}[u(j+k-1) - u(k)]\,b(\mathrm{d}j).$$

Indeed, for $k\geq 1$ $\mathscr{L}u(k) = \lim_{t\to 0}\frac{h_t(s)^k - s^k}{t} = ks^{k-1}\big(-as + a\sum_{j\geq 0} b_j s^j\big) = ak\Big[\widehat{B}u(k) - u(k)\Big]$.

Time change representation

Considering the restriction $\widehat{X}^{\mathbf{0}}$ of $\widehat{X}$ to $\mathbb{N}^* := \mathbb{N}\setminus\{0\}$ (assuming $b_0=0$), the extended weak generator $\mathscr{L}^{\mathbf{0}}$ on $\mathbb{N}^*$ satisfies, for $u=\widehat{s}$,

$$\mathscr{L}^{\mathbf{0}}(u|_{\mathbb{N}^*})(k) = ak\Big[\widehat{B}^{\mathbf{0}}u(k) - u(k)\Big], \quad k\geq 1.$$

Defining $d(k) = \frac{1}{ak}$ for $k \geq 1$ and the bounded operator $\mathscr{L}'v = \widehat{B}^{\mathbf{0}}v - v$, by [15] there exists a pure jump process Y on $\mathbb{N}^*$ with generator $\mathscr{L}'$. With the time change $A_t = \int_0^t d(Y_s)\,\mathrm{d}s$ and its right-continuous inverse $\tau_t = \inf\{s > 0 : A_s > t\}$, the process $Y_t^d = Y_{\tau_t}$ has weak generator $\mathscr{L}^{\mathbf{0}}$.

PROPOSITION 3. *The restriction of* $\widehat{X}$ *to* $\mathbb{N}^*$ *is equivalent to* Y^d.

Proof. Since the weak generator of Y^d equals $\frac{1}{d}\mathscr{L}'$ and for $u = \widehat{s}$ we have $u \in \mathscr{D}(\mathscr{L})$ with $t \mapsto \mathscr{L}\widehat{h}_t(u)(k)$ continuous, it follows that $u_t := \widehat{h}_t(u)$ solves $\frac{\mathrm{d}u_t}{\mathrm{d}t} = \mathscr{L}u_t$ with $u_0 = u$ (see [3], Proposition 3.3). □

3. THE ONE-DIMENSIONAL CB-PROCESS

The one-dimensional continuous-state branching process (CB-process) models population dynamics with continuous sizes. Its branching mechanism is given by (see [20] for more details)

$$\Psi(\lambda) = -b\lambda - c\lambda^2 + \int_0^\infty (1 - e^{-\lambda u} - \lambda u)N(\mathrm{d}u), \tag{6}$$

where $b \in \mathbb{R}$, $c \in \mathbb{R}_+$, and N is a measure on $(0,\infty)$ with $\int_0^\infty (u \wedge u^2)N(\mathrm{d}u) < \infty$. The evolution of the CB-process is described by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \Psi(v), \quad v(0) = s, \tag{7}$$

which has a unique solution $v_t(s)$ (alternatively, one may write an equivalent integral form by adding a linear adjustment using a constant $a > 0$). The CB-process $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}^x)$ on $\mathbb{R}_+$ is Markovian with transition kernel characterized by

$$\widehat{v}_t(e_s) = e_{v_t(s)}, \quad \text{with } e_s(x) = e^{-xs},$$

so that for $x \in \mathbb{R}_+$ and $A \in \mathscr{B}(\mathbb{R}_+)$,

$$\widehat{\mathbb{P}}^x(\widehat{X}_t \in A) = \widehat{v}_t(1_A)(x).$$

4. NON-LOCAL PURE BRANCHING PROCESSES ON FINITE CONFIGURATIONS

A branching mechanism on $\widehat{E}$ is defined by a sequence $(b_k, B_k)_{k\geq 0}$ where $(b_k)_{k\geq 0}$ are bounded positive Borel functions with $\sum_{k\geq 0} b_k = 1$, and each B_k is a Markov kernel from $E^{(k)}$ to E with $B_0 = \delta_{\mathbf{0}}$. With $m_1 = \left\|\sum_{k\geq 1} k\, b_k\right\|_\infty$ satisfying $1 < m_1 < \infty$ and a constant $a > 0$ with $a \leq m_1/(m_1 - 1)$, for any $\varphi \in \mathrm{p}\mathscr{B}(E)$ (with $0 \leq \varphi \leq 1$) the evolution is given by

$$h_t = e^{-at}T_t\varphi + a\int_0^t e^{-a(t-s)}T_{t-s}\sum_{k\geq 0} b_k B_k\big(h_s^{(k)}\big)\,\mathrm{d}s, \quad t \geq 0, \tag{8}$$

which has a unique measurable solution $t \mapsto H_t\varphi$ with $0 \leq H_t\varphi \leq 1$ (see Proposition 4.1 in [7] and Theorem 3.1 in [9]); here $h^{(k)}(x_1,\dots,x_k) = h(x_1)\cdots h(x_k)$ (with $h^{(0)}(\mathbf{0}) = 1$) and $\widehat{h}$ denotes the corresponding function on $\widehat{E}$. Equivalently, one has the differential form

$$\frac{\mathrm{d}}{\mathrm{d}t}h_t = (L-a)h_t + a\sum_{k\geq 0} b_k B_k\big(h_t^{(k)}\big), \quad t \geq 0, \quad h_0 = \varphi,$$

with L the generator of the spatial motion (Remark 4.2 (ii) in [7]). The nonlinear semigroup $(H_t)_{t\geq 0}$ induces a branching semigroup $(\widehat{\mathbf{H}}_t)_{t\geq 0}$ on $\widehat{E}$ by $\widehat{\mathbf{H}}_t\widehat{\varphi} = \widehat{H_t\varphi}$, and under suitable conditions Theorem 4.1 in [7] ensures the existence of a branching right Markov process $\widehat{X}$ on $\widehat{E}$ determined by the spatial motion X, the branching mechanism $(b_k, B_k)_{k\geq 0}$, and the constant a. In the special case where the spatial motion is trivial (i.e. $X_t^0 = x$ for all $t \geq 0$), the evolution reduces to

$$h_t^o = e^{-at}\varphi + a\int_0^t e^{-a(t-s)} \sum_{k\geq 0} b_k B_k\big((h_s^o)^{(k)}\big)\,\mathrm{d}s, \quad t \geq 0,$$

or in differential form,

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t^o = -ah_t^o + a\sum_{k\geq 0} b_k B_k\big((h_t^o)^{(k)}\big), \quad t\geq 0, \quad h_0^o = \varphi.$$

The solution $(H_t^o\varphi)_{t\geq 0}$ then defines the transition function $(\widehat{\mathbf{H}}_t^o)_{t\geq 0}$ of the pure branching process $\widehat{X^0}$. Moreover, if for some $M \in \mathscr{B}(E)$ the condition

$$b_k(x)\,B_{k,x}\Big(\widehat{M}\cap E^{(k)}\Big) = 0 \quad \text{for all } k\geq 1 \text{ and } x\in E\setminus M,$$

holds (meaning that for $x \notin M$ the measure $b_k B_k$ is supported outside $\widehat{M}$), then $\widehat{E}\setminus\widehat{M}$ is a finely closed absorbing subset for $\widehat{X^0}$ and its restriction to $\widehat{M}$ remains a pure branching process (see Proposition 1 and Theorem 3.2(iii) in [4]). Classical kernels, for instance when $B_k F(x) = F(x,\dots,x)$, yield the alternative evolution

$$h_t = e^{-at}T_t\varphi + a\int_0^t e^{-a(t-s)}T_{t-s}\Big(\sum_{k\geq 1} b_k h_s^k\Big)\,\mathrm{d}s, \quad t\geq 0,$$

and probabilistically $b_k(x)$ gives the probability that a particle at x produces k offspring, while $B_{k,x}$ describes their distribution. When $E = \{a\}$, so that $\widehat{E}\equiv\mathbb{N}$ represents particle counts and $M(E)\equiv\mathbb{R}_+$ total mass, multiplicative functions reduce to $\widehat{s}(k) = s^k$ and exponential functions to $e_s(x) = e^{-xs}$, yielding, for example, the Galton-Watson process in continuous time described by

$$h' = -ah + a\sum_{k\geq 0} b_k h^k, \quad h(0) = s,$$

or equivalently

$$h_t = e^{-at}\Big(s + a\int_0^t e^{ar}\sum_{k\geq 0} b_k h_r^k\,\mathrm{d}r\Big), \quad t\geq 0. \tag{9}$$

One can then define a branching kernel $\widehat{h}_t$ on $\mathbb{N}$ by $\widehat{h}_t(\widehat{s}) = \widehat{h_t(s)}$, and a Markov process $\widehat{X}$ on $\mathbb{N}$ with transition function $(\widehat{h}_t)_{t\geq 0}$; related work on extended weak generators and time change representations can be found in [11,13,15], with Proposition 2 and Proposition 3 characterizing the generator and showing that the restriction of $\widehat{X}$ to $\mathbb{N}^*$ is equivalent to an appropriate time-changed pure jump process.

5. THE TOTAL MASS OF A BRANCHING PROCESS

Define the *total mass process* of a branching process $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}^\mu)$ by

$$|\widehat{X}|_t := \langle \widehat{X}_t, 1\rangle, \quad t\geq 0.$$

Thus, if $\widehat{X}$ is a non-local branching process on $\widehat{E}$ its total mass takes values in $\mathbb{N}$, while for a superprocess on $M(E)$ it takes values in $\mathbb{R}_+$. Notice that in [8] the total mass process was used in solving a nonlinear Dirichlet problem (with discontinuous boundary data) related to non-local branching processes.

5.1. Spatially constant branching mechanism

Suppose that $\widehat{X}$ is either a non-local branching process on $\widehat{E}$ or a superprocess on $M(E)$. We say that the branching mechanism is *spatially constant* if either (a) $\widehat{X}$ is a non-local branching Markov process with b_k independent of the spatial variable (for all $k \geq 0$) or (b) $\widehat{X}$ is a superprocess whose branching mechanism Ψ is spatially constant with $c = 0$.

PROPOSITION 4. *If $\widehat{X}$ is a branching Markov process with a spatially constant mechanism, then*

(i) The total mass processes of $\widehat{X}$ and of the associated pure branching process $\widehat{X}^o$ are equal in distribution; that is,

$$\widehat{\mathbb{P}}^\mu \circ |\widehat{X}|_t^{-1} = \widehat{\mathbb{P}}^o_\mu \circ |\widehat{X}^o|_t^{-1} \quad \textit{for every } t \geq 0 \textit{ and } \mu.$$

(ii) Moreover, $|\widehat{X}|$ is itself a pure branching Markov process. Under (a) its transition function is $(\widehat{h}_t)_{t\geq 0}$ of a continuous time Galton-Watson process on $\mathbb{N}$ with mechanism $(b_k)_{k\geq 0}$; under (b) it is that of a one-dimensional CB-process on $\mathbb{R}_+$ with branching mechanism Ψ.

To prove this, we first show the following.

LEMMA 1. *If φ is constant (say, $\varphi = s$), then $H_t\varphi$ is spatially constant and equals the solution $H_t^o\varphi$ arising from the pure branching evolution (i.e. using the trivial spatial semigroup). In particular, $H_t\varphi$ solves*

$$H_t = e^{-at}\left(s + a\int_0^t e^{ar}\sum_{k\geq 0} b_k\,(H_r)^k\,\mathrm{d}r\right),$$

which is equivalent to (9) (or the corresponding equation for superprocesses).

Proof. When (a) holds, one defines the approximating sequences H_t^n and $H_t^{o,n}$ (for the full and pure processes, respectively) by

$$H_t^0 = e^{-ct}T_t\varphi, \quad H_t^{n+1} = e^{-ct}T_t\varphi + c\int_0^t e^{-c(t-u)}T_{t-u}\sum_{k\geq 0} b_k B_k((H_u^n)^{(k)})\,\mathrm{d}u,$$

and

$$H_t^{o,0} = e^{-ct}\varphi, \quad H_t^{o,n+1} = e^{-ct}\varphi + c\int_0^t e^{-c(t-u)}\sum_{k\geq 0} b_k B_k((H_u^{o,n})^{(k)})\,\mathrm{d}u.$$

Since $T_t 1 = B_k 1 = 1$ and φ is constant, one checks inductively that $H_t^n = H_t^{o,n}$ (and constant) for all n. Passing to the limit yields $H_t\varphi = H_t^o\varphi$, proving the lemma. The superprocess case is analogous. □

Proof of Proposition 4. (i) For any positive function f on $\mathbb{N}$, by a monotone class argument it suffices to take $f = \widehat{s}$ with $s \in [0,1]$. Then $f \circ l_1 = \widehat{\varphi_s}$ (with φ_s constant), so by Lemma 1 we have

$$\widehat{\mathbb{E}}^\mu\left[f(|\widehat{X}|_t)\right] = \widehat{H}_t\varphi_s(\mu) = \widehat{H_t^o\varphi_s}(\mu) = \widehat{\mathbb{E}}^o_\mu\left[f(|\widehat{X}^o|_t)\right].$$

(ii) To show that $|\widehat{X}|$ is Markovian with transition function $(\widehat{h}_t)_{t\geq 0}$, one verifies that for $f = \widehat{s}$,

$$\widehat{\mathbb{E}}^\mu\left[f(|\widehat{X}|_{t+t'})\middle|\mathscr{F}_t\right] = \widehat{H}_{t'}\widehat{\varphi}_s(\widehat{X}_t) = \widehat{h}_{t'}f(|\widehat{X}|_t).$$

This completes the proof. □

5.2. The total mass of a branching Markov chain

Now assume that $(b_k, B_k)_{k\geq 0}$ is a branching mechanism on $\widehat{E}$ with spatially constant b_k's. Define the Markovian kernel B from $\widehat{E}$ to E by

$$BF := \sum_{k\geq 0} b_k B_k \Big(F|_{E^{(k)}} \Big), \quad F \in \mathrm{bp}\mathscr{B}(\widehat{E}),$$

so that any such kernel B can be written in this form; it induces a branching kernel $\widehat{B}$ on $\widehat{E}$ via convolution.

If φ is constant, then

$$B\widehat{\varphi} \text{ is constant on } E, \tag{10}$$

since for $\varphi = s$ one has $B\widehat{\varphi} = \sum_{k\geq 0} b_k s^k$.

Given a probability measure ν on $\widehat{E}$, Kolmogorov's theorem yields a Markov chain $\widehat{X} = (\widehat{X}_k)_{k\geq 0}$ with transition kernel $\widehat{B}$. Define the total mass process by

$$|\widehat{X}|_k := \langle \widehat{X}_k, 1 \rangle, \quad k \geq 0,$$

which takes values in $\mathbb{N}$. For a fixed $x \in E$, let $P_k := \widehat{B}_{kx} \circ l_1^{-1}$ and define P on $\mathbb{N}$ by $Pf(k) = \int f \, \mathrm{d}P_k$.

PROPOSITION 5. *The following hold:*

(i) P is a branching kernel on $\mathbb{N}$.

(ii) The total mass process $|\widehat{X}|$ is a Markov chain on $\mathbb{N}$ with transition kernel P, so it is a branching process on $\mathbb{N}$.

(iii) Its law coincides with that of a Galton-Watson process having ancestral distribution $\nu_o = \nu \circ l_1^{-1}$ and offspring distribution P_1.

Proof. (i) One shows that for every $k \geq 2$ and $s \in [0,1]$, $P_k(\widehat{s}) = (P_1(\widehat{s}))^k$; indeed, since $\widehat{s} \circ l_1 = \widehat{\varphi}$ for $\varphi \equiv s$, we have $P_k(\widehat{s}) = \widehat{B}_{kx}(\widehat{\varphi}) = (P_1(\widehat{s}))^k$. (ii) For $f = \widehat{s}$, using that $B\widehat{\varphi}$ is constant by (10), one verifies that

$$\mathbb{E}^\nu \big[f(|\widehat{X}|_{n+1}) \mid \mathscr{F}_n \big] = \widehat{B\widehat{\varphi}}(\widehat{X}_n) = Pf(|\widehat{X}|_n).$$

(iii) Since P is a branching kernel, $P_k = p^{*k}$ with $p = P_1$; the initial law of $|\widehat{X}|$ is ν_o, so the process is a Galton-Watson process with offspring distribution p. □

Assuming $\sum_{k\geq 1} k^2 \nu(E^{(k)}) < \infty$, one obtains classical extinction properties. Define the extinction probability $\eta = \mathbb{P}^\nu(\widehat{X}_k = \mathbf{0} \text{ for some } k \geq 0) = \lim_k \mathbb{P}^\nu(\widehat{X}_k = \mathbf{0})$.

By Proposition 5, we cane describe now the asymptotic behaviour of the branching Markov chain $\widehat{X}$, using the classical results for the Galton-Watson process.

COROLLARY 1. ***(The subcritical case.)*** *Assume $\sum_{k\geq 1} k \widehat{B}_x(E^{(k)}) \leq 1$ and $b_0 \neq 0$; then $\eta = 1$ (a.s. extinction). If $b_0 = 0$ then $\eta = \nu(\{\mathbf{0}\})$.*

COROLLARY 2. ***(The supercritical case.)*** *If $x \in E$ satisfies $1 < m := \sum_{k\geq 1} k \widehat{B}_x(E^{(k)}) < \infty$, then (i) the sequence $\left(\frac{1}{m^n} |\widehat{X}|_n \right)_{n\geq 0}$ converges $\mathbb{P}^\nu$-a.s. and in L^2 to a random variable $|\widehat{X}|_\infty$ which is not identically zero, and (ii) on the event $\{|\widehat{X}|_\infty > 0\}$ one has $\liminf_n |\widehat{X}|_n > 0$.*

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