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SPECTRAL RADIUS AND PATH-FACTOR CRITICAL GRAPHS

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Abstract. A $P_{\geq d}$-factor of a graph G is a spanning subgraph F of G such that every component of F is a path of order at least d ($d \geq 2$). A graph G is called a $(P_{\geq d},k)$-factor critical graph if after deleting any k vertices of G the remaining graph of G contains a $P_{\geq d}$-factor. Let $\rho(G)$ denote the spectral radius of G. In this paper, we first provide a characterization for a graph to be $(P_{\geq 2},k)$-factor critical; then we prove that an n-vertex connected graph G is a $(P_{\geq 2},k)$-factor critical graph unless $G = K_k \vee (K_{n-k-1} \cup K_1)$ if $\rho(G) \geq \rho(K_k \vee (K_{n-k-1} \cup K_1))$, where k and n are two positive integers with $n \geq k+2$.

***Keywords**:* graph, spectral radius, $P_{\geq 2}$-factor, $(P_{\geq 2},k)$-factor critical graph.
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1. INTRODUCTION

In this paper, we deal only with finite and undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of a graph G is the number $n = |V(G)|$ of its vertices. The number of isolated vertices in G is denoted by $i(G)$. For any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and write $G-S = G[V(G)\setminus S]$. Let G_1 and G_2 be two disjoint graphs. The union $G_1 \cup G_2$ is the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)$. The join $G_1 \vee G_2$ denotes the graph with vertex set $V(G_1)\cup V(G_2)$ and edge set $E(G_1)\cup E(G_2)\cup \{uv : u \in V(G_1), v \in V(G_2)\}$. For a graph G and an integer $k \geq 2$, let kG denote the disjoint union of k copies of G. Let P_n and K_n denote the path and the complete graph of order n, respectively.

Suppose that the vertex set of G is $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G) = (a_{ij})_{n\times n}$ of G is a $(0,1)$-matrix in which the entry $a_{ij} = 1$ if and only if v_i and v_j are adjacent. Note that $A(G)$ is a real nonnegative symmetric matrix. Hence, its eigenvalues are real, which can be arranged in nonincreasing order as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. In particular, the largest eigenvalue $\lambda_1(G)$ is called the adjacency spectral radius (or spectral radius, for short) of G, written as $\rho(G)$.

A path-factor is a spanning subgraph of a graph G in which every component is a path of order at least 2. Let $\mathscr{H}$ be a set of connected graphs. A spanning subgraph H of a graph G is called an $\mathscr{H}$-factor if each component of H is isomorphic to a member of $\mathscr{H}$. An $\mathscr{H}$-factor is also referred as a component factor. Let $d \geq 2$ be an integer. A $\{P_d, P_{d+1}, \ldots\}$-factor is simply denoted by a $P_{\geq d}$-factor. Note that a perfect matching can be regarded as a $\{P_2\}$-factor. A graph G is called a $(P_{\geq d},k)$-factor critical graph if after deleting any k vertices of G the remaining graph of G contains a $P_{\geq d}$-factor. In fact, a $(P_{\geq d},0)$-factor critical graph G is equivalent to G having a $P_{\geq d}$-factor.

In mathematical literature, the study on component factors attracted much attention. Amahashi and Kano [1] provided a necessary and sufficient condition for a graph having a $\{K_{1,j} : 1 \le j \le k\}$-factor, where $k \ge 2$ is an integer. Kano, Lu and Yu [8] proved that a graph G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor if $i(G-S) \le \frac{|S|}{2}$ holds for every $S \subseteq V(G)$. Zhou [24], Zhou, Xu and Sun [31] obtained some sufficient conditions for graphs having component factors. Kano and Saito [9] investigated the existence of $\{K_{1,j} : k \le j \le 2k\}$-factors in graphs. Las Vergnas [10] claimed a necessary and sufficient condition for a graph to contain a $P_{\ge 2}$-factor. Kaneko [7] showed a characterization for a graph with a $P_{\ge 3}$-factor. Dai, Hang, Zhang, Zhang and Wang [4] gave some degree conditions for the existence of $\{P_2, P_5\}$-factors in graphs. Dai [3], Liu [12], Liu and Pan [13], Zhou, Sun and Liu [28], Wu [21] obtained some results on the existence of $P_{\ge 3}$-factors in graphs. Zhou [23] got a binding number condition for the existence of $(P_{\ge 3}, k)$-factor critical graphs. More results on graph factors and factor critical graphs were found in [17–19, 25, 26, 30, 33].

Many researchers [5, 14, 15, 20, 34] investigated some interesting spectral properties of $A(G)$. O [16], Zhou, Sun and Zhang [29], Zhou and Zhang [32] established some connections between spectral radius and a $\{P_2\}$-factor in a connected graph. Li and Miao [11], Zhou, Zhang and Sun [35], Zhou, Sun and Liu [27] showed some spectral conditions for connected graphs to contain $P_{\ge 2}$-factors. In this paper, we study the existence of a $(P_{\ge 2}, k)$-factor critical graph, and provide a sufficient condition for the existence of a $(P_{\ge 2}, k)$-factor critical graph by using spectral radius.

Theorem 1.1. Let k and n be two positive integers with $n \ge k+2$. If G is an n-vertex connected graph with $\rho(G) \ge \rho(K_k \vee (K_{n-k-1} \cup K_1))$, then G is a $(P_{\ge 2}, k)$-factor critical graph unless $G = K_k \vee (K_{n-k-1} \cup K_1)$.

2. PRELIMINARY LEMMAS

In this section, we provide several necessary preliminary lemmas, which are used to verify the main results in this paper.

Lemma 2.1 (Brouwer and Haemers [2]). Let H be a subgraph of a connected graph G. Then

$$\rho(G) \ge \rho(H)$$

with equality if and only if $G = H$.

Let M be an $n \times n$ real matrix, and let $X = \{1, 2, \ldots, n\}$. Given a partition $\pi : X = X_1 \cup X_2 \cup \cdots \cup X_r$, the matrix M can be correspondingly partitioned as

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1r} \\ M_{21} & M_{22} & \cdots & M_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rr} \end{pmatrix},$$

where M_{ij} denotes the submatrix (block) of M formed by rows in X_i and the columns in X_j. The quotient matrix of M with respect to π is defined by the $r \times r$ matrix $B_\pi = (b_{ij})$, where b_{ij} denotes the average value of all row sums of M_{ij}. The above partition π is equitable if every block M_{ij} of M has constant row sum b_{ij}.

Lemma 2.2 (You, Yang, So and Xi [22]). Let M be a real symmetric matrix with an equitable partition π, and let B_π be the corresponding quotient matrix. Then every eigenvalue of B_π is an eigenvalue of M. Furthermore, if M is nonnegative, then the largest eigenvalues of M and B_π are equal.

Lemma 2.3 (Haemers [6]) Let M be a Hermitian matrix of order s, and let N be a principal submatrix of M of order t. If $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ are the eigenvalues of M and N, respectively. then $\lambda_i \ge \mu_i \ge \lambda_{s-t+i}$ for $i = 1, 2, \ldots, t$.

Las Vergnas [10] provided a characterization for a graph with a $P_{\geq 2}$-factor.

Lemma 2.4 (Las Vergnas [10]). A graph G contains a $P_{\geq 2}$-factor if and only if

$$i(G-S) \leq 2|S|$$

for any subset S of $V(G)$.

Using Lemma 2.4, we verify the following result.

Lemma 2.5. A graph G is a $(P_{\geq 2},k)$-factor critical graph if and only if

$$i(G-S) \leq 2|S|-2k$$

for any $S \subseteq V(G)$ with $|S| \geq k$.

Proof. Suppose $U \subseteq S \subseteq V(G)$ where $|U|=k$, and $S'=S\setminus U$ and $G'=G-U$. Then $G'-S'=G-S$.

Suppose first that G is $(P_{\geq 2},k)$-factor critical. Then $G'=G-U$ contains a $P_{\geq 2}$-factor, and so $i(G'-S') \leq 2|S'|$ by Lemma 2.4. Combining this with $S'=S\setminus U$ and $G'-S'=G-S$, we conclude

$$i(G-S)=i(G'-S') \leq 2|S'|=2|S|-2k$$

for any $S \subseteq V(G)$ with $|S| \geq k$.

Suppose conversely that $i(G-S) \leq 2|S|-2k$ for any $S \subseteq V(G)$ with $|S| \geq k$. Together with $U \subseteq S \subseteq V(G)$, $|U|=k$, $S'=S\setminus U$, $G'=G-U$ and $G'-S'=G-S$, we obtain

$$i(G'-S')=i(G-S) \leq 2|S|-2k=2|S\setminus U|=2|S'|$$

for any $S' \subseteq V(G')$. From Lemma 2.4, $G'=G-U$ contains a $P_{\geq 2}$-factor, and so G is a $(P_{\geq 2},k)$-factor critical graph. This completes the proof of Lemma 2.5. □

3. THE PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Suppose, to the contrary, that G is not $(P_{\geq 2},k)$-factor critical. By Lemma 2.5, we obtain

$$i(G-S) \geq 2|S|-2k+1$$

for some subset S of $V(G)$ with $|S| \geq k$. Let $|S|=s$. Then G is a spanning subgraph of $G_1=K_s \vee (K_{n_1} \cup (2s-2k+1)K_1)$ for some nonnegative integer n_1 with $n_1=n-3s+2k-1$. According to Lemma 2.1, we get

$$\rho(G) \leq \rho(G_1) \tag{1}$$

with equality if and only if $G=G_1$. Let $G_*=K_k \vee (K_{n-k-1} \cup K_1)$. In light of the partition $V(G_*)=V(K_k) \cup V(K_{n-k-1}) \cup V(K_1)$, the quotient matrix of $A(G_*)$ is equal to

$$B_*=\begin{pmatrix} k-1 & n-k-1 & 1 \\ k & n-k-2 & 0 \\ k & 0 & 0 \end{pmatrix}.$$

Then the characteristic polynomial of B_* is

$$\varphi_{B_*}(x)=x^3-(n-3)x^2-(n+k-2)x+k(n-k-2).$$

Notice that the partition $V(G_*) = V(K_k) \cup V(K_{n-k-1}) \cup V(K_1)$ is equitable. By Lemma 2.2, the largest root, say ρ_*, of $\varphi_{B_*}(x) = 0$ is equal to $\rho(G_*)$. In what follows, we show that $\rho(G_1) \leq \rho(G_*)$ with equality if and only if $G_1 = G_*$.

Obviously, $G_1 = G_*$ if $s = k$. Hence, it suffices to verify that $\rho(G_1) < \rho(G_*)$ for $s \geq k+1$. The following proof will be divided into three cases.

Case 1. $n_1 = 0$, that is, $n = 3s - 2k + 1$.

In this case, $G_1 = K_s \vee (2s-2k+1)K_1$. In terms of the partition $V(G_1) = V(K_s) \cup V((2s-2k+1)K_1)$, the quotient matrix of $A(G_1)$ is

$$B_1 = \begin{pmatrix} s-1 & 2s-2k+1 \\ s & 0 \end{pmatrix}.$$

Then the characteristic polynomial of B_1 is

$$\varphi_{B_1}(x) = x^2 - (s-1)x - s(2s-2k+1).$$

Notice that the partition $V(G_1) = V(K_s) \cup V((2s-2k+1)K_1)$ is equitable. In view of Lemma 2.2, the largest root, say ρ_1, of $\varphi_{B_1}(x) = 0$ equals $\rho(G_1)$.

Note that $n = 3s - 2k + 1$ and $\varphi_{B_1}(\rho_1) = 0$. By plugging the value ρ_1 into x of $\varphi_{B_*}(x) - x\varphi_{B_1}(x)$, we get

$$\begin{aligned}\varphi_{B_*}(\rho_1) =& \varphi_{B_*}(\rho_1) - \rho_1\varphi_{B_1}(\rho_1) \\ =& -(2s-2k-1)\rho_1^2 + (2s^2 - 2ks - 2s + k + 1)\rho_1 + k(3s - 3k - 1).\end{aligned}$$

Since $\rho_1 = \frac{s-1+\sqrt{(s-1)^2+4s(2s-2k+1)}}{2} = \frac{s-1+\sqrt{9s^2-8ks+2s+1}}{2}$, we obtain

$$\begin{aligned}\varphi_{B_*}(\rho_1) =& -(2s-2k-1)\rho_1^2 + (2s^2 - 2ks - 2s + k + 1)\rho_1 + k(3s - 3k - 1) \\ =& \frac{s-k}{2}(-8s^2 + (8k+1)s + 6k + 1 + \sqrt{9s^2 - 8ks + 2s + 1}).\end{aligned} \tag{2}$$

Claim 1. $8s^2 - (8k+1)s - 6k - 1 > \sqrt{9s^2 - 8ks + 2s + 1}$ for $s \geq k+1$.

Proof. Write $M_1 = 8s^2 - (8k+1)s - 6k - 1$ and $N_1 = \sqrt{9s^2 - 8ks + 2s + 1}$, where $s \geq k+1$. Then

$$M_1^2 - N_1^2 = 64s^4 - 16(8k+1)s^3 + (64k^2 - 80k - 24)s^2 + (96k^2 + 36k)s + 36k^2 + 12k. \tag{3}$$

Let $f_1(x) = 64x^4 - 16(8k+1)x^3 + (64k^2 - 80k - 24)x^2 + (96k^2 + 36k)x + 36k^2 + 12k$ be a real function in x with $x \in [k+1, +\infty)$. Then we have

$$f_1'(x) = 256x^3 - 48(8k+1)x^2 + 2(64k^2 - 80k - 24)x + 96k^2 + 36k$$

and

$$f_1''(x) = 768x^2 - 96(8k+1)x + 2(64k^2 - 80k - 24).$$

Notice that

$$\frac{96(8k+1)}{2 \times 768} = \frac{8k+1}{16} < k+1 \leq s.$$

Then $f_1''(x)$ is increasing in the interval $[k+1, +\infty)$, and so

$$f_1''(x) \geq f_1''(k+1) = 128k^2 + 512k + 624 > 0$$

for $x \geq k+1$, which implies that $f_1'(x)$ is increasing in the interval $[k+1, +\infty)$. Thus, we conclude

$$f_1'(x) \geq f_1'(k+1) = 16k^2 + 116k + 160 > 0$$

for $x \geq k+1$, which yields that $f_1(x)$ is increasing in the interval $[k+1, +\infty)$. Recall that $s \geq k+1$. Then we obtain

$$f_1(s) \geq f_1(k+1) = 24 > 0$$

for $s \geq k+1$. Combining this with (3), we get $M_1 > N_1$ for $s \geq k+1$. Claim 1 is verified. □

According to (2), Claim 1 and $s \geq k+1$, we have $\varphi_{B_*}(\rho_1) < 0$, which leads to $\rho(G_1) = \rho_1 < \rho_* = \rho(G_*)$.

Case 2. $n_1 = 1$, that is, $n = 3s - 2k + 2$.

In this case, $G_1 = K_s \vee (2s-2k+2)K_1$. By virtue of the partition $V(G_1) = V(K_s) \cup V((2s-2k+2)K_1)$, the quotient matrix of $A(G_1)$ is

$$B_2 = \begin{pmatrix} s-1 & 2s-2k+2 \\ s & 0 \end{pmatrix}.$$

Then the characteristic polynomial of B_2 equals

$$\varphi_{B_2}(x) = x^2 - (s-1)x - s(2s-2k+2).$$

Obviously, the partition $V(G_1) = V(K_s) \cup V((2s-2k+2)K_1)$ is equitable. According to Lemma 2.2, the largest root, say $\rho_2 = \frac{s-1+\sqrt{9s^2-8ks+6s+1}}{2}$, of $\varphi_{B_2}(x) = 0$ equals $\rho(G_1)$.

Note that $n = 3s - 2k + 2$ and $\varphi_{B_2}(\rho_2) = 0$. By plugging the value ρ_2 into x of $\varphi_{B_*}(x) - x\varphi_{B_2}(x)$, we possess

$$\begin{aligned}\varphi_{B_*}(\rho_2) =& \varphi_{B_*}(\rho_2) - \rho_2\varphi_{B_2}(\rho_2)\\ =&(s-k)(-2\rho_2^2 + (2s-1)\rho_2 + 3k)\\ =&\frac{s-k}{2}(-8s^2 + (8k-7)s + 6k - 1 + \sqrt{9s^2-8ks+6s+1}).\end{aligned} \tag{4}$$

Claim 2. $8s^2 - (8k-7)s - 6k + 1 > \sqrt{9s^2-8ks+6s+1}$ for $s \geq k+1$.

Proof. Let $M_2 = 8s^2 - (8k-7)s - 6k + 1$ and $N_2 = \sqrt{9s^2-8ks+6s+1}$, where $s \geq k+1$. Then

$$M_2^2 - N_2^2 = 64s^4 - 16(8k-7)s^3 + (64k^2 - 208k + 56)s^2 + (96k^2 - 92k + 8)s + 36k^2 - 12k. \tag{5}$$

Let $f_2(x) = 64x^4 - 16(8k-7)x^3 + (64k^2 - 208k + 56)x^2 + (96k^2 - 92k + 8)x + 36k^2 - 12k$ be a real function in x, where $x \in [k+1, +\infty)$. By a direct computation, we get

$$f_2'(x) = 256x^3 - 48(8k-7)x^2 + 2(64k^2 - 208k + 56)x + 96k^2 - 92k + 8$$

and

$$f_2''(x) = 768x^2 - 96(8k-7)x + 2(64k^2 - 208k + 56).$$

Note that

$$\frac{96(8k-7)}{2 \times 768} = \frac{8k-7}{16} < k+1 \leq s.$$

Then $f_2''(x)$ is increasing in the interval $[k+1, +\infty)$, and so

$$f_2''(x) \geq f_2''(k+1) = 128k^2 + 1024k + 1552 > 0$$

for $x \geq k+1$. Obviously, $f_2'(x)$ is increasing in the interval $[k+1, +\infty)$, and so

$$f_2'(x) \geq f_2'(k+1) = 144k^2 + 660k + 712 > 0$$

for $x \geq k+1$, which implies that $f_2(x)$ is increasing in the interval $[k+1, +\infty)$. Recall that $s \geq k+1$. Then we conclude

$$f_2(s) \geq f_2(k+1) = 80k^2 + 272k + 240 > 0$$

for $s \geq k+1$. Together with (5), we have $M_2 > N_2$ for $s \geq k+1$. This completes the proof of Claim 2. □

In terms of (4), Claim 2 and $s \geq k+1$, we obtain $\varphi_{B_*}(\rho_2) < 0$, which yields $\rho(G_1) = \rho_2 < \rho_* = \rho(G_*)$.

Case 3. $n_1 \geq 2$, that is, $n \geq 3s-2k+3$.

Recall that $G_1 = K_s \vee (K_{n_1} \cup (2s-2k+1)K_1)$, where $n_1 = n-3s+2k-1$. The quotient matrix of $A(G_1)$ according to the partition $V(G_1) = V(K_s) \cup V(K_{n-3s+2k-1}) \cup V((2s-2k+1)K_1)$ equals

$$B_3 = \begin{pmatrix} s-1 & n-3s+2k-1 & 2s-2k+1 \\ s & n-3s+2k-2 & 0 \\ s & 0 & 0 \end{pmatrix}.$$

By a simple computation, we conclude that the characteristic polynomial of B_3 is

$$\begin{aligned}\varphi_{B_3}(x) =& x^3 - (n-2s+2k-3)x^2 - (n+2s^2-2ks-s+2k-2)x \\ &+ s(2s-2k+1)(n-3s+2k-2).\end{aligned}$$

Since the partition $V(G_1) = V(K_s) \cup V(K_{n-3s+2k-1}) \cup V((2s-2k+1)K_1)$ is equitable, by Lemma 2.2, the largest root, say ρ_3, of $\varphi_{B_3}(x) = 0$ equals $\rho(G_1)$. Let $\rho_3 = \rho(G_1) \geq \rho_4 \geq \rho_5$ be the three roots of $\varphi_{B_3}(x) = 0$ and $Q = \mathrm{diag}(s, n-3s+2k-1, 2s-2k+1)$. It is easy to check that

$$Q^{\frac{1}{2}} B_3 Q^{-\frac{1}{2}} = \begin{pmatrix} s-1 & s^{\frac{1}{2}}(n-3s+2k-1)^{\frac{1}{2}} & s^{\frac{1}{2}}(2s-2k+1)^{\frac{1}{2}} \\ s^{\frac{1}{2}}(n-3s+2k-1)^{\frac{1}{2}} & n-3s+2k-2 & 0 \\ s^{\frac{1}{2}}(2s-2k+1)^{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

is symmetric, and also contains

$$\begin{pmatrix} n-3s+2k-2 & 0 \\ 0 & 0 \end{pmatrix}$$

as its submatrix. Since $Q^{\frac{1}{2}} B_3 Q^{-\frac{1}{2}}$ and B_3 have the same eigenvalues, by the Cauchy Interlacing Theorem (see Lemma 2.3), we have

$$\rho_4 \leq n-3s+2k-2 < n-k-2. \tag{6}$$

Note that K_{n-1} is a proper subgraph of $G_* = K_k \vee (K_{n-k-1} \cup K_1)$. According to Lemma 2.1, we conclude

$$\rho_* = \rho(G_*) > \rho(K_{n-1}) = n-2 > n-k-2 > \rho_4. \tag{7}$$

Note that $\varphi_{B_*}(\rho_*) = 0$. By plugging the value ρ_* into x of $\varphi_{B_3}(x) - \varphi_{B_*}(x)$, we obtain

$$\begin{aligned}\varphi_{B_3}(\rho_*) =& \varphi_{B_3}(\rho_*) - \varphi_{B_*}(\rho_*) \\ =& (s-k)(2\rho_*^2 - (2s-1)\rho_* + 2sn + n - 6s^2 + 4ks - 7s - k - 2).\end{aligned} \tag{8}$$

Let $h(\rho_*) = 2\rho_*^2 - (2s-1)\rho_* + 2sn + n - 6s^2 + 4ks - 7s - k - 2$. Note that

$$\frac{2s-1}{4} < s+1 < 3s-2k+1 \leq n-2 < \rho_*$$

by (7). Combining this with $s \geq k+1$ and $n \geq 3s-2k+3$, we obtain

$$\begin{aligned}h(\rho_*) >& h(n-2) \\ =& 2(n-2)^2 + 2n - 6s^2 + 4ks - 3s - k - 4 \\ \geq& 2(3s-2k+1)^2 + 2(3s-2k+3) - 6s^2 + 4ks - 3s - k - 4\end{aligned}$$

$$
\begin{aligned}
&=12s^2-(20k-15)s+8k^2-13k+4\\
&\geq 12(k+1)^2-(20k-15)(k+1)+8k^2-13k+4\\
&=6k+31\\
&>0. \qquad (9)
\end{aligned}
$$

It follows from (8), (9) and $s \geq k+1$ that

$$\varphi_{B_3}(\rho_*) = (s-k)h(\rho_*) > 0.$$

As $\rho_4 < n-2 < \rho(G_*) = \rho_*$ (see (7)), we infer $\rho(G_1) = \rho_3 < \rho_* = \rho(G_*)$.

From Cases 1–3, we have $\rho(G_1) < \rho(G_*)$ for $s \geq k+1$. Thus, we conclude $\rho(G_1) \leq \rho(G_*)$ with equality if and only if $G_1 = G_*$. Combining this with (1), we have $\rho(G) \leq \rho(G_*)$ with equality if and only if $G = G_*$, where $G_* = K_k \vee (K_{n-k-1} \cup K_1)$. This contradicts the condition of Theorem 1.1. This completes the proof of Theorem 1.1. □

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