THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A
Volume xx, Number yy/2025, pp. xx–xx
DOI:

INFINITE FAMILIES OF CONGRUENCES FOR 4- AND 6-REGULAR PARTITIONS

Liu Xin JIN[1], Jing JIN[2], Olivia X.M. YAO[3]

[1]Jiangsu University, School of Mathematical Sciences, Zhenjiang, Jiangsu, 212013, P.R. China,
E-mail: liuxj@ujs.edu.cn
[2]Jiangsu Agri-animal Husbandry Vocational College, College of Agricultural Information, Taizhou, 225300, Jiangsu, P.R. China,
E-mail: jinjing19841@126.com
[3]Suzhou University of Science and Technology, School of Mathematical Sciences, Suzhou, 215009, Jiangsu, P.R. China,
E-mail: yaoxiangmei@163.cn
Corresponding author: Olivia X.M. YAO, E-mail: yaoxiangmei@163.cn

Abstract. Recently, Ballantine and Merca proved some congruences modulo powers of 2 for $b_4(n)$ and congruences modulo 3 for $b_6(n)$, where $b_t(n)$ denotes the number of t-regular partitions of n. Motivated by Ballantine and Merca's works on congruences of $b_t(n)$, we present a characterization of congruences modulo 8 for $b_4(n)$, from which, we obtain infinite families of congruences modulo 8 for $b_4(n)$. Furthermore, we also prove infinite families of congruences modulo 3 for $b_6(n)$ based on Newman's identities. Those congruences involve primes which are congruent to 1 modulo 24.

***Keywords**:* partition, congruences, regular partition.
Mathematics Subject Classification (MSC2020): 11P83, 05A17.

1. INTRODUCTION

Recall that a partition of n is a non-increasing sequence of positive integers, called parts, whose sum is n. If $t \geq 2$ is an integer, then a partition is called a t-regular partition if there is no part divisible by t. As usual, let $b_t(n)$ denote the number of t-regular partitions of n and set $b_t(0) = 1$. The generating function of $b_t(n)$ is

$$\sum_{n=0}^{\infty} b_t(n)q^n = \frac{(q^t;q^t)_\infty}{(q;q)_\infty}, \tag{1}$$

where here and throughout this paper, $(q;q)_\infty := \prod_{n=1}^{\infty}(1-q^n)$.

In recent years, congruence properties for $b_t(n)$ are investigated in many interesting papers by Andrews, Hirschhorn and Sellers [1], Ballantine and Merca [2], Chen [4], Cui and Gu [5,6], Keith [7], Keith and Zanello [8], Lin and Wang [9], Merca [10, 11], Xia [14] and Yao [15, 16]. For example, Andrews, Hirschhorn and Sellers [1] proved that for $n \geq 0$,

$$\begin{aligned} b_4(9n+4) &\equiv 0 \pmod{4}, \\ b_4(9n+7) &\equiv 0 \pmod{12}. \end{aligned}$$

They also proved the following infinite families of congruences modulo 2 for $b_4(n)$: for $n,\alpha \geq 0$,

$$b_4\left(3^{2\alpha+2}n+\frac{j\times 3^{2\alpha+1}-1}{8}\right)\equiv 0 \pmod 2,$$

where $j\in\{11,17,19\}$. Merca [10] proved that $b_4(n)$ is odd if and only if n is a triangular number. He also established some relations between $b_4(n)$ and the number of partitions into parts not congruent to 2 modulo 4. Chen [4] proved that for $n,\alpha\geq 1$,

$$b_4\left(5^{2\alpha}n+\frac{r\cdot 5^{2\alpha-1}-1}{8}\right)\equiv 0 \pmod 4,$$

where $r\in\{13,21,29,37\}$. Xia [13] proved that for $n,\alpha\geq 1$,

$$b_4\left(3^{4\alpha}n+\frac{j\cdot 3^{4\alpha-1}-1}{8}\right)\equiv 0 \pmod 8,$$

where $j\in\{11,19\}$. In [2], Ballantine and Merca proved that for $n\geq 0$,

$$b_4(25n+8)\equiv b_4(25n+13)\equiv b_4(25n+18)\equiv b_4(25n+23)\equiv 0 \pmod{16}.$$

Very recently, Ballantine and Merca [3] proved infinite families of congruences modulo 3 for $b_6(n)$. More precisely, they proved the following theorem.

THEOREM 1. *[3] Let α be a nonnegative integer and let p_i $(1\leq i\leq \alpha+1)$ be primes. If $p_{\alpha+1}\equiv 3 \pmod 4$ and $j\not\equiv 0 \pmod{p_{\alpha+1}}$, then for all $n\geq 0$,*

$$b_6\left(p_1^2\cdots p_{\alpha+1}^2 n+\frac{p_1^2\cdots p_\alpha^2 p_{\alpha+1}(24j+5p_{\alpha+1})-5}{24}\right)\equiv 0 \pmod 3.$$

Motivated by Ballantine and Merca's works on congruences of $b_4(n)$ and $b_6(n)$, we investigate congruences modulo 8 for $b_4(n)$ and congruences modulo 3 for $b_6(n)$ in this paper.

The fist goal of this paper is to present a characterization of congruences modulo 8 for $b_4(n)$. To state the main results on congruences modulo 8 for $b_4(n)$, define

$$\mu_1(n):=\begin{cases}1, & \text{if } n=k(k-1)/2 \text{ for some positive integer } k,\\ 0, & \text{otherwise,}\end{cases} \tag{2}$$

$$V_{1,2}(n):=\sum_{\substack{m,k\geq 1,\\ 2m^2+k(k-1)/2=n}}(-1)^m, \tag{3}$$

$$V_{1,4}(n):=\sum_{\substack{m,k\geq 1,\\ 4m^2+k(k-1)/2=n}}1. \tag{4}$$

The main results on congruences modulo 8 for $b_4(n)$ can be stated as follows.

THEOREM 2. *For $n\geq 1$,*

$$b_4(n)\equiv \mu_1(n)-2V_{1,2}(n)+4V_{1,4}(n) \pmod 8. \tag{5}$$

For example, setting $n=200$ in (5), we deduce that $\mu_1(200)=0$, $V_{1,2}=1$, $V_{1,4}=1$ and

$$b_4(200)\equiv 0-2\times 1+4\times 1\equiv 2 \pmod 8.$$

In fact, $b_4(200)=122730022082$.

Based on Theorem 2, we obtain the following corollary.

COROLLARY 1. *Let p be a prime with $p \equiv 7 \pmod 8$. If n, α are nonnegative integers with $p \nmid n$, then*

$$b_4\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod 8. \tag{6}$$

The second goal of this paper is to establish infinite families of congruences modulo 3 for $b_6(n)$ involving other choices of primes.

THEOREM 3. *Let p be a prime with $p \equiv 1 \pmod{24}$. If $b_6(5(p-1)/24) \equiv 0 \pmod 3$, then for $n, \alpha \geq 0$ with $p \nmid (24n+5)$, then*

$$b_6\left(p^{2\alpha+1}n + \frac{5(p^{2\alpha+1}-1)}{24}\right) \equiv 0 \pmod 3. \tag{7}$$

If $b_6(5(p-1)/24) \not\equiv 0 \pmod 3$, then for $n, \alpha \geq 0$ with $p \nmid (24n+5)$, then

$$b_6\left(p^{3\alpha+2}n + \frac{5(p^{3\alpha+2}-1)}{24}\right) \equiv 0 \pmod 3. \tag{8}$$

For example, setting $p = 73$ in Theorem 3 and using the fact that $b_6(15) = 143$, we deduce that for $\alpha \geq 0$,

$$b_6\left(73^{3k+2}n + \frac{5(73^{3k+2}-1)}{24}\right) \equiv 0 \pmod 3,$$

where $73 \nmid (24n+5)$.

2. PROOFS OF THEOREM 2 AND COROLLARY 1

It is easy to check that

$$\begin{aligned}\sum_{m,n=1}^{\infty}(-1)^{m+n}q^{m^2+n^2} &= \sum_{\substack{m,n=1,\\ m>n}}^{\infty}(-1)^{m+n}q^{m^2+n^2} + \sum_{\substack{m,n=1,\\ n>m}}^{\infty}(-1)^{m+n}q^{m^2+n^2} + \sum_{n=1}^{\infty}q^{2n^2}\\ &= 2\sum_{\substack{m,n=1,\\ m>n}}^{\infty}(-1)^{m+n}q^{m^2+n^2} + \sum_{n=1}^{\infty}q^{2n^2}.\end{aligned} \tag{9}$$

To prove the main results of this paper, we require the following two identities due to Gauss:

$$1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} \tag{10}$$

and

$$\sum_{k=1}^{\infty} q^{k(k-1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{11}$$

In light of (1), (10) and (11),

$$\sum_{n=0}^{\infty} b_4(n)q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2}\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}$$

$$
\begin{aligned}
&= \frac{1}{1+2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}} \sum_{k=1}^{\infty} q^{k(k-1)/2} \\
&= \left(1+\sum_{j=1}^{\infty}(-2)^j \left(\sum_{t=1}^{\infty}(-1)^t q^{2t^2}\right)^j\right) \sum_{k=1}^{\infty} q^{k(k-1)/2} \\
&\equiv \left(1-2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}+4\sum_{m,n=1}^{\infty}(-1)^{m+n} q^{2m^2+2n^2}\right) \sum_{k=1}^{\infty} q^{k(k-1)/2} \\
&\equiv \left(1-2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}+4\sum_{n=1}^{\infty} q^{4n^2}\right) \sum_{k=1}^{\infty} q^{k(k-1)/2} \pmod 8 \qquad \text{(by (9))} \\
&= \sum_{n=0}^{\infty} (\mu_1(n)-2V_{1,2}(n)+4V_{1,4}(n)) q^n, \qquad (12)
\end{aligned}
$$

which yields (5) after comparing the coefficients of q^n on both sides of (12). The proof of Theorem 2 is complete.

Now, we turn to prove Corollary 1.

It follows from (2) that if $p \nmid n$, then

$$\mu_1\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{8}\right)=0. \qquad (13)$$

In addition, from (3) and (4), we can rewrite $V_{1,2}(n)$ and $V_{1,4}(n)$ as

$$V_{1,2}(n) = \sum_{\substack{m,k\geq 1,\\ (4m)^2+(2k-1)^2=8n+1}} (-1)^m, \qquad (14)$$

$$V_{1,4}(n) = \sum_{\substack{m,k\geq 1,\\ 2(4m)^2+(2k-1)^2=8n+1}} 1. \qquad (15)$$

From (14), we know that if $8n+1$ is not of the form x^2+y^2, then $V_{1,2}(n)=0$. Note that if N is of the form x^2+y^2, then $\nu_p(N)$ is even since p is a prime with $p\equiv 7 \pmod 8$ and $\left(\frac{-1}{p}\right)=-1$. Here $\nu_p(N)$ denotes the highest power of p dividing N and $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. It is easy to check that if $p\nmid n$, then

$$\nu_p\left(8\left(p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{8}\right)+1\right)=\nu_p(8p^{2\alpha+1}n+p^{2\alpha+2})=2\alpha+1$$

is odd. Therefore, $8\left(p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{8}\right)+1$ is not of the form x^2+y^2 and

$$V_{1,2}\left(p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{8}\right)=0. \qquad (16)$$

It follows from (15) that if $8n+1$ is not of the form x^2+2y^2, then $V_{1,4}(n)=0$. The facts that $\nu_p\left(8\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{8}\right)+1\right)$ is odd and $\left(\frac{-2}{p}\right)=-1$ imply that $8\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{8}\right)+1$ is not of the form x^2+2y^2 and

$$V_{1,4}\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{8}\right)=0. \qquad (17)$$

Congruence (6) follows from (5), (13), (16) and (17). This completes the proof of Corollary 1. □

3. PROOF OF THEOREM 3

Define

$$\sum_{n=0}^{\infty} a(n)q^n := \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty}. \tag{18}$$

Newman [12] proved that if p is a prime with $p \equiv 1 \pmod{24}$, then

$$a\left(pn + \frac{5(p-1)}{24}\right) = a(5(p-1)/24)a(n) - a\left(\frac{n - \frac{5(p-1)}{24}}{p}\right). \tag{19}$$

If $3|a(5(p-1)/24)$, then

$$a\left(pn + \frac{5(p-1)}{24}\right) \equiv -a\left(\frac{n - \frac{5(p-1)}{24}}{p}\right) \pmod{3}. \tag{20}$$

If $p \nmid (24n+5)$, then $\frac{n-\frac{5(p-1)}{24}}{p}$ is not an integer and

$$a\left(\frac{n - \frac{5(p-1)}{24}}{p}\right) = 0. \tag{21}$$

It follows from (20) and (21) that if $3|a(5(p-1)/24)$ and $p \nmid (24n+5)$, then

$$a\left(pn + \frac{5(p-1)}{24}\right) \equiv 0 \pmod{3}. \tag{22}$$

Replacing n by $pn + \frac{5(p-1)}{24}$ in (20) yields

$$a\left(p^2 n + \frac{5(p^2-1)}{24}\right) \equiv -a(n) \pmod{3}. \tag{23}$$

By (23) and mathematical induction, we deduce that for $n, \alpha \geq 0$,

$$a\left(p^{2\alpha} n + \frac{5(p^{2\alpha}-1)}{24}\right) \equiv (-1)^{\alpha} a(n) \pmod{3}. \tag{24}$$

Replacing n by $pn + \frac{5(p-1)}{24}$ in (24) and utilizing (22), we find that if $3|a(5(p-1)/24)$ and $p \nmid (24n+5)$, then for $n, \alpha \geq 0$,

$$a\left(p^{2\alpha+1} n + \frac{5(p^{2\alpha+1}-1)}{24}\right) \equiv 0 \pmod{3}. \tag{25}$$

It follows from (19) that if $a(5(p-1)/24) \equiv 1 \pmod{3}$, then

$$a\left(pn + \frac{5(p-1)}{24}\right) \equiv a(n) - a\left(\frac{n - \frac{5(p-1)}{24}}{p}\right) \pmod{3}. \tag{26}$$

Replacing n by $pn+\frac{5(p-1)}{24}$ in (26) yields

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv a\left(pn+\frac{5(p-1)}{24}\right)-a(n) \pmod 3,$$

from which with (26), we arrive at

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv -a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right) \pmod 3. \tag{27}$$

By (27), we see that if $a(5(p-1)/24)\equiv 1 \pmod 3$ and $p\nmid(24n+5)$, then

$$a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right)=0$$

and

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv 0 \pmod 3. \tag{28}$$

Replacing n by $pn+\frac{5(p-1)}{24}$ in (27) yields

$$a\left(p^3n+\frac{5(p^3-1)}{24}\right)\equiv -a(n) \pmod 3. \tag{29}$$

By (29) and mathematical induction, we deduce that for $n,\alpha\geq 0$,

$$a\left(p^{3\alpha}n+\frac{5(p^{3\alpha}-1)}{24}\right)\equiv (-1)^{\alpha}a(n) \pmod 3. \tag{30}$$

Replacing n by $p^2n+\frac{5(p^2-1)}{24}$ in (30) and using (28), we see that if $a(5(p-1)/24)\equiv 1 \pmod 3$, then for $n,\alpha\geq 0$ with $p\nmid(24n+5)$,

$$a\left(p^{3\alpha+2}n+\frac{5(p^{3\alpha+2}-1)}{24}\right)\equiv 0 \pmod 3. \tag{31}$$

Identity (19) implies that if $a(5(p-1)/24)\equiv 2 \pmod 3$, then

$$a\left(pn+\frac{5(p-1)}{24}\right)\equiv 2a(n)-a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right) \pmod 3. \tag{32}$$

Replacing n by $pn+\frac{5(p-1)}{24}$ in (32) yields

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv 2a\left(pn+\frac{5(p-1)}{24}\right)-a(n) \pmod 3. \tag{33}$$

Substituting (32) into (33) yields

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right) \pmod 3, \tag{34}$$

which implies that if $a(5(p-1)/24) \equiv 2 \pmod 3$ and $p \nmid (24n+5)$, then

$$a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right) = 0$$

and

$$a\left(p^2 n + \frac{5(p^2-1)}{24}\right) \equiv 0 \pmod 3. \tag{35}$$

If we replace n by $pn + \frac{5(p-1)}{24}$ in , we arrive at

$$a\left(p^3 n + \frac{5(p^3-1)}{24}\right) \equiv a(n) \pmod 3. \tag{36}$$

By (36) and mathematical induction, we deduce that for $n, \alpha \geq 0$,

$$a\left(p^{3\alpha} n + \frac{5(p^{3\alpha}-1)}{24}\right) \equiv a(n) \pmod 3. \tag{37}$$

Replacing n by $p^2 n + \frac{5(p^2-1)}{24}$ in (37) and using (35), we see that if $a(5(p-1)/24) \equiv 2 \pmod 3$, then for $n, \alpha \geq 0$ with $p \nmid (24n+5)$,

$$a\left(p^{3\alpha+2} n + \frac{5(p^{3\alpha+2}-1)}{24}\right) \equiv 0 \pmod 3. \tag{38}$$

Setting $t = 6$ in (1), we get

$$\sum_{n=0}^{\infty} b_6(n) q^n = \frac{(q^6;q^6)_\infty}{(q;q)_\infty}. \tag{39}$$

By (39) and the fact that

$$(1-q^{6n}) \equiv (1-q^{2n})^3 \pmod 3,$$

we arrive at

$$\sum_{n=0}^{\infty} b_6(n) q^n \equiv \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty} \pmod 3. \tag{40}$$

Combining (18) and (40) yields

$$b_6(n) \equiv a(n) \pmod 3. \tag{41}$$

Theorem 3 follows from (25), (31), (38) and (41). This completes the proof. □

4. CONCLUDING REMARKS

As seen in Introduction, congruence properties for t-regular partition functions have received a lot of attention in recent years. In this study, we give a characterization of congruences modulo 8 for $b_4(n)$ and prove infinite families of congruences modulo 3 for $b_6(n)$. A natural question is to extend the congruences in this paper to modulo 9, 32, 64, etc. However, it will likely require a different approach since the methods used in this paper run into serious limitations beyond the modulus of 9.

ACKNOWLEDGEMENTS

This work was supported by the Natural Science Foundation of Jiangsu Province of China (BK20200267) and the research project of Jiangsu Agri-animal Husbandry Vocational College (NSF2025CB22).

REFERENCES

[1] Andrews GE, Hirschhorn MD, Sellers JA. Arithmetic properties of partitions with even parts distinct. Ramanujan J. 2010; 23: 169–181.

[2] Ballantine C, Merca M. 4-Regular partitions and the pod function. Quaest. Math. 2023; 46: 2027–2051.

[3] Ballantine C, Merca M. 6-regular partitions: new combinatorial properties, congruences, and linear inequalities. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 2024; 159: 117.

[4] Chen SC. On the number of partitions with distinct even parts. Discrete Math. 2011; 311: 940–943.

[5] Cui SP, Gu NSS. Arithmetic properties of l-regular partitions. Adv. Appl. Math. 2013; 51: 507–523.

[6] Cui SP, Gu NSS. Congruences for 9-regular partitions modulo 3. Ramanujan J. 2015; 38: 503–512.

[7] Keith WJ. Congruences for 9-regular partitions modulo 3. Ramanujan J. 2014; 35: 157–164.

[8] Keith WJ, Zanello F. Parity of the coefficients of certain eta-quotients. J. Number Theory. 2022; 235: 275–304.

[9] Lin BLS, Wang AYZ. Generalization of Keith's conjecture on 9-regular partitions and 3-cores. Bull. Austral. Math. Soc. 2014; 90: 204–212.

[10] Merca M. New relations for the number of partitions with distinct even parts. J. Number Theory. 2017; 176: 1–12.

[11] Merca M. Ramanujan-type congruences modulo 4 for partitions into distinct parts. Analele Stiintifice ale Universitatii Ovidius Constanta-Seria Matematica. 2022; 30: 185–199.

[12] Newman M. Modular forms whose coefficients possess multiplicative properties. Ann. Math. 1959 70: 478–489.

[13] Xia EXW. New infinite families of congruences modulo 8 for partitions with even parts distinct. Electron. J. Combin. 2014; 21: #P4.8.

[14] Xia EXW. Congruences for some l-regular partitions modulo l. J. Number Theory. 2015; 152: 105–117.

[15] Yao OXM. New congruences modulo powers of 2 and 3 for 9-regular partitions. J. Number Theory. 2014; 142: 89–101.

[16] Yao OXM. New parity results for 3-regular partitions, Quaest. Math. 2023; 46: 465–471.

Received January 21, 2025