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CLASSICAL SOLUTIONS FOR SOME CLASSES OF NONLINEAR PARABOLIC SPDES WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We investigate the existence and uniqueness of the solution in the strong sense to a class of non-linear SPDE associated with a measure-valued branching process by the stochastic characteristics method developed by Kunita. We also use the Itô-Wentzell formula proved by Tubaro and results of Barbu and Beznea.

***Keywords**:* Neumann problem, strong solution for non-linear SPDE, stochastic characteristics system.
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1. INTRODUCTION

The need to describe important phenomena appearing in control theory, chemistry, physics and biology strongly are motivated the study of stochastic partial differential equations (SPDEs) in the last decades. In order to illustrate the range of applications we give the following examples: reaction diffusion equation (see [19]), Zakai's equation which arises in filtering theory (see [31]), simulation of a random motion of a string cf. [18], changes in structure of population cf. [17], Krylov equation or backward diffusion equation (see [32] and the references therein).

SPDEs are also used to model a free field connected with the relativistic quantum theory (see [21]) or an electrical potential of nerve cells utilized in neurophysiology (see [20], [29]).

Other important examples of SPDEs are the Markovian lifting equation which arises in the study of stochastic delay equations (see [15]), the Helmholtz parabolic equation which models the diffraction in a random nonuniform medium (see [33]), the equation of the number of particles related to continuous branching models with geographical structure used in chemistry and the population biology or the equation of stochastic quantization (see [13]).

Concerning the existence, uniqueness and regularity of solutions to SPDEs, important contributions are due to Krylov and Rozovskii [25], Krylov [23], [24], Rozovsky and Lozotsky [32], Pardoux [31], Kunita [26], Da Prato and Zabzyck [16], Nualart and Zakai [30], Tubaro [36], Vinter and Kwong [37].

The aim of this paper is to extend the results related to the existence of the solution to some classes of nonlinear parabolic SPDEs for non-linear PDEs with Neumann boundary conditions introduced in [3] where the measure-valued branching processes introduced in [4] and [11] are used. Our method is based on the stochastic characteristics method in the generalized sense cf. [34, 35]. A numerical solution to the Neumann problem in a Lipschitz domain was developed in [27] and a numerical solution for the non-linear Dirichlet problem of a branching process was developed in [28].

The objective of the paper is to prove the existence of the classical solutions for some classes of nonlinear parabolic SPDEs with Neumann boundary conditions in connection with the results of [3].

The paper is organized as follows. In Section 2 we set a nonlinear parabolic SPDE with Neumann boundary conditions, related to the problem studied in [3]. In Section 3 we introduce the stochastic characteristics system (cf. [34,35]) and a second nonlinear problem in relation with the problem from [3]. Further, we obtain the main results concerning the existence of the classical solutions of the problems considered.

2. SETTING OF THE PROBLEMS

Preliminaries. We present bellow a nonlinear parabolic problem with Neumann boundary condition introduced in [3] in order to construct solutions in the strong sense.

Let O be a bounded, open subset of R^d, $d \geq 1$, with smooth boundary Γ (for instance, of class C^2).

We consider the following nonlinear parabolic problem from [3] :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{2}\Delta u + \alpha u = 0 \quad \text{in } (0,\infty) \times 0, \\ \dfrac{\partial u}{\partial \nu} + \beta(u) = g \quad \text{on } \Gamma, \\ u(0,\cdot) = f \quad \text{in } 0, \end{cases} \tag{1}$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative to the boundary Γ of O, g is a positive continuously differentiable function on Γ, $f \in C(\bar{O})$, $\alpha \in \mathbb{R}_+^\star$ and $\beta : \mathbb{R} \to \mathbb{R}_-$ is a continuous mapping defined as

$$\beta(u) = \begin{cases} \displaystyle\int_0^\infty (e^{-su} - 1)\eta(ds) - bu, & \text{if } u \geq 0, \\ 0, & if u < 0. \end{cases} \tag{2}$$

where η is a positive measure on $\mathbb{R}_+$ satisfying $\int_{\mathbb{R}_+} (s \wedge 1)\, \eta(ds) < \infty$ and $b \in \mathbb{R}_+$.

We suppose that

$$\int_{\mathbb{R}_+} (s \wedge 1)\, \eta(ds) + b \leq \gamma := \inf_{v \in H^1(O)} \frac{||\nabla v||^2_{L^2(O)} + \alpha ||v||^2_{L^2(O)}}{||v||^2_{L^2(\Gamma)}}. \tag{3}$$

Note that the condition (3) is equivalent with the property of the function $u \longrightarrow \beta(u) + \gamma(u)$ to be nondecreasing.

The function $-\beta$ appearing in 1 is called a branching mechanism. Now, we give an example of branching mechanism verifying the condition 3.

We set

$$\beta_N(u) = \frac{m}{\Gamma(1-m)} \int_0^N \frac{e^{-su} - 1}{s^{m+1}} ds,$$

with $N > 0$ and $0 < m < 1$. For a convenient number $c < 0$, the limit case is cu^m, since

$$-u^m = \frac{m}{\Gamma(1-m)} \int_0^\infty \frac{1 - e^{-su}}{s^{m+1}} ds = \lim_{N \to \infty} \beta_N(u).$$

Remark 1. It is stated in the proof of Proposition 2.2 from [3] that $V_t f = \lim_{n\to\infty} (I + \frac{t}{n} A)^{-1} f$ in $L^2(O)$ for $f \in L^2(O)$.

Problem 1. Let O and Γ as before and set $d=3$. Let $(\Omega, \mathscr{K}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space and $(W_t)_{t\geq 0}$ be a 3-dimensional Wiener process on this space.

Considering the following non-linear parabolic SPDEs with Neumann boundary condition

$$\begin{cases} d\mathsf{v} = [\mathscr{L}\mathsf{v}(t,x) - \alpha \mathsf{v}(t,x)]dt + \sum_{i=1}^{3} B_i \mathsf{v}(t,x) dW_t^i, t \geq 0, x \in O, \\ \dfrac{\partial \mathsf{v}}{\partial \nu} + \beta(\mathsf{v}) = g \text{ on } \Gamma, \\ \mathsf{v}(0,\cdot) = f \text{ in } 0, \end{cases} \tag{4}$$

where

$$\mathscr{L}\mathsf{v}(t,x) = \frac{1}{2}\sum_{i=1}^{3} a_{ij}(t,x)\frac{\partial^2 \mathsf{v}(t,x)}{\partial x_i x_j} - \sum_{i=1}^{3} b_i(t,x)\frac{\partial \mathsf{v}(t,x)}{\partial x_i},$$

$$B_i \mathsf{v}(t,x) = \sum_{i=j}^{3} \alpha_{ij}(t,x)\frac{\partial \mathsf{v}(t,x)}{\partial x_j}, \ i = 1,\cdots,n,$$

with a_{ij}, b_i, α_{ij} smooth enough and α_{ij} with compact support in O.

Let us denote $\sigma_1 = (\alpha_{ij})^T$ the 3×3 matrix given by the coefficients of the white noise term and for every $i = 1,\cdots,n$, $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3})^T$.

Let σ_2 be a matrix 3×3 such that $\sigma_2\sigma_2^T = (a_{ij}) - \sigma_1\sigma_1^T$; we put $\sigma = (\sigma_1, \sigma_2)$. Notice that $\sigma\sigma^T = (a_{ij})$. We denote $\nabla\sigma_1 \cdot \sigma_1(t,x)$ the 3-dimensional vector with i-th component

$$\sum_{j=1}^{3}\sum_{h=1}^{3} \frac{\partial \alpha_{ij}(t,x)}{\partial x_h} \cdot \alpha_{hj}(t,x).$$

Also let g and f be two enough smooth functions.

3. THE EXISTENCE RESULTS USING THE STOCHASTIC CHARACTERISTICS SYSTEM

We consider the random field $\xi_j(t;s,x)$ of stochastic characteristics introduced by Kunita (see [26]). We define $\xi_j(t;s,x)$ the solution of the stochastic differential equation

$$\begin{cases} d\xi(t) = \dfrac{1}{2}\nabla\sigma_1\sigma_1(t,\xi(t))dt - \sigma_1^T(t,\xi(t))dW_t, \\ \xi(s) = x, \ x \in \mathbb{R}^3. \end{cases} \tag{5}$$

Problem (5) has for every $s \geq 0$, a unique solution (see Theorem 6.1.2 from [26]); moreover the first derivative $\dfrac{\partial \xi(s;t,x)}{\partial x_l}$ verifies a linear stochastic differential equation that by the regularity of the $\alpha_{ij}\,'s$ has a bounded solution. When the initial time is $s=0$ we will write $\xi(t,x) = \xi(t;0,x)$.

Taking into account the assumptions on the coefficients of (5) by Theorem 6.1.5 from [26] we have that the solution $\xi(t;s,x)$ has a modification that is a C^2-diffeomorfism for all $t \geq s$.

Let us denote by $\eta(s;t,x)$ the inverse mapping of ξ, *i.e.*, the unique process such that

$$\eta(s;t,\xi(t;s,x)) = \xi(t;s,\eta(s;t,x)) = x \ \ a.s.$$

The process $\eta(s;t,x)$ is a solution to the Itô backward differential equation

$$d\eta(s) = \frac{1}{2}\nabla\sigma_1\sigma_1(s,\eta(s))ds + \sigma_1^T(s,\eta(s))dW_s$$

with $\eta(t;t,x) = x$. We define the process $\eta(t,x) := \eta(0;t,x)$.

We recall that the coefficients $\alpha_{ij}(t,x)$ have compact support in O. From this and equation (5) it follows that for any $x \notin O$, $\xi(t,x) = x$ for all t. Thus, we find that O is η- invariant in the sense of [1].

In the sequel, we impose that the following assumptions hold, taking into account the existence result from [3] that the solution u of the problem (1) exists and belongs to $C^1([0,\infty];H^2(O))$.

Assumption 1 (compatibility conditions).

$$\langle(\sigma_2\sigma_2^T)\nabla\eta_i(t,\xi(t,x)),\nabla\eta_j(t,\xi(t,x))\rangle = 0, \quad \text{for all } i \neq j;$$

$$\langle(\sigma_2\sigma_2^T)\nabla\eta_i(t,\xi(t,x)),\nabla\eta_i(t,\xi(t,x))\rangle = 1 \quad \text{for all } i;$$

$$\frac{1}{2}Tr[(\sigma_2\sigma_2^T)\nabla^2\eta_i(t,\xi(t,x))] + \langle b(t,\xi(t,x)) - \frac{1}{2}\nabla\sigma_1\sigma_1(t,\xi(t,x)),\nabla\eta_i(t,\xi(t,x))\rangle = 0 \quad \text{for all } j.$$

Assumption 2 (regularity of u). We assume that $u \in C^1([0,\infty];H^4(O))$.

THEOREM 1. *According to the assumptions from the beginning of Section 2, Assumption 1 and Assumption 2 the problem* (4) *admits a unique classical solution* $\mathsf{v} \in L^2[(\Omega,\mathscr{K},\mathbb{P});C([0,\infty);C^2(\bar{O}))]$ *which is adapted to the filtration* $(\mathscr{F}_t)_{t\geq 0}$.

Proof. We consider $\mathsf{v}(t,x) := u(t,\xi(t,x))$. It is known that $H^4(O) \hookrightarrow C^2(\bar{O})$. Applying the Itô formula (see Theorem 3.3.1 from [26] or Proposition 2 from [36]) to $u(t,\xi(t,x))$ it follows that $\mathsf{v}(t,x)$ is the unique classical solution of the problem (4) which belongs to $L^2[(\Omega,\mathscr{K},\mathbb{P});C([0,\infty);C^2(\bar{O}))]$. The fact v is adapted to $(\mathscr{F}_t)_{t\geq 0}$ follows from the adaptivity of $\xi(t,x)$ and by Remark 1. □

Conversely, if problem (4) admits a classical solution $\mathsf{v}(t,x)$ for *a.e.* $\omega \in \Omega$ applying Itô formula to $\mathsf{v}(t,\eta(t,x))$ we get that problem (1) has a solution.

Problem 2. Let O and Γ be as before and $d = 3$. Let $(\Omega,\mathscr{K},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$ be a complete probability space and $(W_t)_{t\geq 0}$ be a 3-dimensional Wiener process on this space.

Considering the following nonlinear parabolic SPDEs with Neumann boundary condition

$$\begin{cases} d\mathsf{v} = [\mathscr{L}\mathsf{v}(t,x) + h(t,x)]dt + [\theta^T(t,x)v(t,x) + \gamma^T(t,x)]dW_t,\ t > 0, x \in O, \\ \dfrac{\partial \mathsf{v}}{\partial \nu} + \beta(\mathsf{v}) = g \text{ on } \Gamma, \\ \mathsf{v}(0,\cdot) = f \text{ in } 0, \end{cases} \tag{6}$$

where

$$\mathscr{L}\mathsf{v}(t,x) = \frac{1}{2}\Delta\mathsf{v}(t,x) + b^T(t,x)\nabla\mathsf{v}(t,x) + c(t,x)\mathsf{v}(t,x),$$

$b(t,x)$ is a 3-dimensional column-vector composed of the coefficients $b^i(t,x)$, $c(t,x)$ and $h(t,x)$ are scalar functions, $\theta(t,x)$ and $\gamma(t,x)$ are 3-dimensional column-vectors composed of the coefficients $\theta^i(t,x)$ and $\gamma^i(t,x)$ respectively.

Definition 1. *We say that* v *is a strong classical solution to the problem* (6) *if it is* $(\mathscr{F}_t)_{t\geq 0}$ *adapted and the following relations hold almost surely*

$$\begin{cases} \mathsf{v}(t,x) = \mathsf{v}(0,x) + \int_0^t [\mathscr{L}\mathsf{v}(s.x) + h(x)]d\xi + \int_0^t [\theta^T(s,x)\mathsf{v}(s,x) + \gamma^T(s,x)]dW_s, \ t > 0, \ x \in O \\ \dfrac{\partial \mathsf{v}}{\partial \nu} + \beta(\mathsf{v}) = g \ on \ \Gamma, \\ \mathsf{v}(0,\cdot) = f \ in \ 0. \end{cases}$$

Further, we assume the following conditions on the problem (6), taking into account the existence result from [3] that the solution u of the problem (1) exists and belongs to $C^1([0,\infty];H^2(O))$.

Assumption 1. (smoothness of the coefficients). We suppose that the coefficients in (6) are sufficiently smooth to apply the Itô formula and $\theta^i(t,x)$ and $\gamma^i(t,x)$ have compact support in O for any i.

Assumption 2. (compatibility conditions). We assume that

$$b(t,x) + a(t,x)\nabla\eta(t,x) = 0,$$

$$c(t,x) - \frac{1}{2}|\theta(t,x)|^2 + b^T(t,x)\nabla\eta(t,x) + \frac{1}{2}\nabla^T\eta(t,x)\nabla\eta(t,x) + \sum_{i,j=1}^{3} \frac{\partial^2\eta(t,x)}{\partial x_i \partial x_j} = \alpha,$$

$$e^{-\eta(t,x)}h(t,x) - \frac{1}{2}|\theta(t,x)|^2\xi(t,x) - e^{-\eta(t,x)}\gamma^T(t,x)\theta(t,x) + \mathscr{L}\xi(t,x) + \nabla^T\xi(t,x)\nabla\eta(t,x) +$$

$$\frac{1}{2}\xi(t,x)\nabla^T\eta(t,x)\nabla\eta(t,x) + \frac{1}{2}\xi(t,x)\sum_{i,j=1}^{3} \frac{\partial^2\eta(t,x)}{\partial x_i \partial x_j} = 0,$$

where

$$d\eta = \theta^T(t,x)dW_t, \ \eta(0,x) = 0,$$

$$d\xi = \gamma^T(t,x)e^{-\eta(t,x)}dW_t, \ \xi(0,x) = 0.$$

Assumption 3. (regularity of u). We assume that $u \in C^1([0,\infty);H^4(O))$.

THEOREM 2. *Under the assumptions from above the problem* (6) *has a unique classical solution in the strong sense.*

Proof. We consider $\mathsf{v}(t,x) = e^{\eta(t,x)}u(t,x) + e^{\eta(t,x)}\xi(t,x)$. It is known that $H^4(0) \hookrightarrow C^2(\bar{O})$. Applying Itô formula (see Theorem 3.3.1 from [26] or [32]) to $e^{\eta(t,x)}u(t,x) + e^{\eta(t,x)}\xi(t,x)$ it follows that $\mathsf{v}(t,x)$ is the unique solution of the problem (6) which belongs to $L^2[(\Omega,\mathscr{K},\mathbb{P});C([0,\infty);C^2(\bar{O}))]$. Using similar technique as in the case of problem (4), we obtain the adaptivity of v. □

Conversely, if the problem (6) admits a classical solution $\mathsf{v}(t,x)$ for *a.e.* $\omega \in \Omega$ applying Itô formula to $e^{-\eta(t,x)}\mathsf{v}(t,x) - \xi(t,x)e^{-\eta(t,x)}$ we get that problem (1) has a solution.

Remark 2. As in the case of problem (4) Theorem 2 was proved under rather strong assumptions, in particular that $u \in C^1([0,\infty);H^4(0))$ which allow us to obtain a classical solution v for problem (6). If we consider $u \in C^1([0,\infty);H^2(O)])$ then it can be obtained applying Itô formula proved in [22] a generalized solution v of problem (6) in the sense of [32].

Final remark. Our aim was to prove the existence of the classical solutions for some classes of nonlinear parabolic SPDEs with Neumann boundary conditions considered in [3]. In the first step, we set the problem (4) related to a class of nonlinear parabolic SPDEs with Neumann boundary conditions. Then, we get the existence of a classical solution for this problem, by using some results concerning the stochastic characteristics system cf. [34, 35]. Further, we consider another class of SPDE defined in (6) and we obtained the main result concerning the existence of the classical solution. It is a challenge to investigate numerical methods for SDEs with jumps, in connection with the fragmentation and avalanche phenomena studied in [6–9], taking into account the associate measure-valued branching processes from [10] and [12], an analogue SPDE similar to [4], and the sochastic solutions to the evolution equations of non-local branching processes from [12] and [5].

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