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ON THE NUMBER OF SOLUTIONS TO THE INEQUALITY WITH PRIMES

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Abstract. Let p_1 and p_2 be the prime numbers. For sufficiently large values of the natural number N we prove that the number of solutions to the inequality $|N-p_1-p_2| \leq H$ is greater than $\mathscr{O}\left(\frac{HN^{0.525+o(1)}}{(\ln(N))^2}\right)$ provided that $H \geq N^{0.07+o(1)}$.

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1. INTRODUCTION

In 2001, R. Baker, G. Harman, and J. Pintz [6] obtained for the number of solutions to inequality $|p-N| \leq H$ in prime numbers p the correct lower bound in order provided that $H \geq N^{0.525+o(1)}$. If we assume the Riemann hypothesis is true then a given positive number N can be approached by a prime number p at a distance of $\mathscr{O}\left(N^{0.5+o(1)}\right)$.

One would then ask about approximating a given positive number N by the sum of two prime numbers. In 1975, H. Montgomery and R. Vaughan showed in their paper [4] that inequality

$$|p_1+p_2-N| \leq H \tag{1}$$

is solvable for $H \geq N^{7/72+o(1)}$. S. Gritsenko and D. Goryashin [2] proved that the number of solutions to the inequality (1) in primes p_1 and p_2 is greater than

$$\mathscr{O}\left(\frac{HN^{0.525+o(1)}}{(\ln(N))^2}\right)$$

provided that $H \geq N^{7/80+o(1)}$. In this work, we improve the result of S. Gritsenko and D. Goryashin by obtaining the lower bound of the number of solutions to the inequality (1) for smaller H.

THEOREM 1. *Let N be the sufficiently large natural number and let $J(N,H)$ denote the number of solutions to the inequality $|N-p_1-p_2| < H$ in the prime numbers p_1 and p_2. Then for all $N^{7/100+o(1)} \leq H \leq 0.1N$ we have*

$$J(N,H) \gg \frac{N^{0.525+o(1)}H}{(\ln(N))^2}.$$

Before moving on to the proof of Theorem 1, we note that our proof is based on the results of R. Baker, G. Harman and J.Pintz [6] and L. Guth and J. Maynard's new density theorem [3].

In the following, we use Vinogradov's notation $\ll$ and the big $\mathcal{O}$ notation equivalently. The meaning of these notations is that

$$f(x) \ll g(x) \Rightarrow |f(x)| \leq C|g(x)| \text{ for some constant } C \text{ and } x > x_0.$$

However, when $g(x)$ is complicated we prefer to use Vinogradov's notation than the big $\mathcal{O}$ notation.

2. SOME PRELIMINARY LEMMAS

Let $\Lambda(n)$ denote the von Mangoldt function, and $\psi(x) = \sum_{n \leq x} \Lambda(n)$ denote the Chebyshev function.

LEMMA 1. *Let $2 \leq T \leq x$. Then,*

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + \mathcal{O}\left(\frac{x(\ln(x))^2}{T}\right),$$

where $\rho = \beta + i\gamma$ runs through the zeros of the Riemann zeta function in the critical strip.

Proof. See [5], p.52.
Let $\pi(x) = \sum_{p \leq x} 1$ denote number of primes $p \leq x$.

LEMMA 2 (R. Baker, G. Harman, J. Pintz [6]). *For all $x > x_0$, the interval $[x - x^{0.525}, x]$ contains prime number.*

Note that in [6] a lower bound on the number of primes in the interval $[x - x^{0.525}, x]$ is obtained:

$$\pi(x) - \pi(x - x^{0.525}) \geq 0.09 \frac{x^{0.525}}{\ln(x)}. \tag{2}$$

LEMMA 3 (Zero-free region for $\zeta(s)$). *There exist an absolute constant $c > 0$ such that the Riemann zeta function $\zeta(s)$ has no zero in the following region of the s-plane*

$$t \geq 10, \quad \sigma \geq 1 - \frac{c}{(\ln(\ln(t)))^{1/3}(\ln(t))^{2/3}}.$$

Proof. See [5], p.119.

LEMMA 4. *Let $S(t)$ be a complex-valued function, continuously differentiable function on the interval $[t_0, t_k]$, and let*

$$t_0 < t_1 < \cdots < t_k.$$

Then setting $\delta = \min_{0 \leq r < k} (t_{r+1} - t_r)$, we have

$$\sum_{r=1}^{k} |S(t_r)|^2 \leq \frac{1}{\delta} \int_{t_0}^{t_k} |S(t)|^2 dt + 2\left(\int_{t_0}^{t_k} |S(t)|^2 dt\right)^{1/2} \left(\int_{t_0}^{t_k} |S'(t)|^2 dt\right)^{1/2}.$$

Proof. See [7], p. 94.

LEMMA 5. *The number of zeros ρ_n of the zeta function that satisfy $T \leq \Im(\rho_n) \leq T + 1$ does not exceed $\mathcal{O}(\log(T))$.*

Proof. See [7], p. 58.

LEMMA 6 (L. Guth, J. Maynard [3]). *Let $N(\sigma,T)$ denote the number of zeros ρ of $\zeta(s)$ with $\Re(s)\geq\sigma$ and $|\Im(s)|\leq T$. Then we have*

$$N(\sigma,T)\ll T^{15(1-\sigma)/(3+5\sigma)+o(1)}.$$

LEMMA 7. *Let $Y\geq 2$ and $T\geq 2$. Then we have*

$$\int_{0.5Y}^{2Y}\left|\sum_{0<|\gamma|\leq T}x^{\rho}\right|^2dx\ll Y^2\ln(Y)(\ln(T))^2\max_{0.5\leq\alpha<1}Y^{2\alpha-1}N(\alpha,T),$$

$$\int_{0.5Y}^{2Y}\left|\sum_{0<|\gamma|\leq T}\rho x^{\rho}\right|^2dx\ll Y^2T^2\ln(Y)(\ln(T))^2\max_{0.5\leq\alpha<1}Y^{2\alpha-1}N(\alpha,T),$$

where $\rho=\beta+i\gamma$ runs through the zeros of the Riemann zeta function in the critical strip.

Proof. 1) Proof of the first inequality. We write

$$\left|\sum_{0<|\gamma|\leq T}x^{\rho}\right|^2=\left(\sum_{0<|\gamma|\leq T}x^{\rho}\right)\overline{\left(\sum_{0<|\gamma|\leq T}x^{\rho}\right)}=\left(\sum_{0<|\gamma_1|\leq T}x^{\beta_1+i\gamma_1}\right)\left(\sum_{0<|\gamma_2|\leq T}\overline{x^{\beta_2+i\gamma_2}}\right)$$

$$=\left(\sum_{0<|\gamma_1|\leq T}x^{\beta_1+i\gamma_1}\right)\left(\sum_{0<|\gamma_2|\leq T}x^{\beta_2-i\gamma_2}\right)=\sum_{0<|\gamma_1|\leq T}\sum_{0<|\gamma_2|\leq T}x^{\beta_1+\beta_2+i(\gamma_1-\gamma_2)}.$$

Thus, we have

$$\int_{0.5Y}^{2Y}\left|\sum_{0<|\gamma|\leq T}x^{\rho}\right|^2dx=\sum_{0<|\gamma_1|\leq T}\sum_{0<|\gamma_2|\leq T}\int_{0.5Y}^{2Y}x^{\beta_1+\beta_2+i(\gamma_1-\gamma_2)}dx$$

$$=\sum_{0<|\gamma_1|\leq T}\sum_{0<|\gamma_2|\leq T}\frac{(2Y)^{\beta_1+\beta_2+1+i(\gamma_1-\gamma_2)}-(0.5Y)^{\beta_1+\beta_2+1+i(\gamma_1-\gamma_2)}}{\beta_1+\beta_2+1+i(\gamma_1-\gamma_2)}$$

$$\ll Y\sum_{0<|\gamma_1|\leq T}\sum_{0<|\gamma_2|\leq T}\frac{Y^{\beta_1+\beta_2}}{1+|\gamma_1-\gamma_2|}$$

$$=Y\sum_{0<|\gamma_1|\leq T}\left(\sum_{\substack{0<|\gamma_2|\leq T\\ \beta_1\leq\beta_2}}\frac{Y^{\beta_1+\beta_2}}{1+|\gamma_1-\gamma_2|}+\sum_{\substack{0<|\gamma_2|\leq T\\ \beta_1>\beta_2}}\frac{Y^{\beta_1+\beta_2}}{1+|\gamma_1-\gamma_2|}\right)$$

$$\ll Y\sum_{0<|\gamma_1|\leq T}\left(\sum_{\substack{0<|\gamma_2|\leq T\\ 0.5\leq\beta_2<1}}\frac{Y^{2\beta_2}}{1+|\gamma_1-\gamma_2|}+\sum_{\substack{0<|\gamma_2|\leq T\\ 0.5\leq\beta_1<1}}\frac{Y^{2\beta_1}}{1+|\gamma_1-\gamma_2|}\right)$$

$$=2Y\sum_{0<|\gamma_1|\leq T}\sum_{\substack{0<|\gamma_2|\leq T\\ 0.5\leq\beta<1}}\frac{Y^{2\beta}}{1+|\gamma_1-\gamma_2|}$$

$$\ll Y\sum_{\substack{0<|\gamma_1|\leq T\\ 0.5\leq\beta<1}}Y^{2\beta}\left(\sum_{\substack{0<|\gamma_2|\leq T\\ |\gamma_1-\gamma_2|\leq 1}}1+\sum_{\substack{0<|\gamma_2|\leq T\\ |\gamma_1-\gamma_2|>1}}\frac{1}{|\gamma_1-\gamma_2|}\right).$$

By lemma 5 we have

$$\sum_{\substack{0<|\gamma_2|\le T\\ |\gamma_1-\gamma_2|\le 1}} 1 \ll \ln(\gamma_1)$$

and

$$\begin{aligned}\sum_{\substack{0<|\gamma_2|\le T\\ |\gamma_1-\gamma_2|>1}} \frac{1}{|\gamma_1-\gamma_2|} &= \sum_{1\le n\le T}\sum_{n<|\gamma_1-\gamma_2|\le n+1}\frac{1}{|\gamma_1-\gamma_2|}\\ &\le \sum_{1\le n\le T}\frac{1}{n}\sum_{n<|\gamma_1-\gamma_2|\le n+1} 1\\ &\ll (\ln(T))^2.\end{aligned}$$

Thus

$$\int_{0.5Y}^{2Y}\left|\sum_{0<|\gamma|\le T} x^{\rho}\right|^2 dx \ll Y(\ln(T))^2 \sum_{\substack{0<|\gamma_1|\le T\\ 0.5\le\beta<1}} Y^{2\beta}. \tag{3}$$

Inserting the identity

$$Y^{2\beta} = Y + 2\ln(Y)\int_{0.5}^{\beta} Y^{2u}du.$$

into (3) we obtain

$$\begin{aligned}\int_{0.5Y}^{2Y}\left|\sum_{0<|\gamma|\le T} x^{\rho}\right|^2 dx &\ll Y(\ln(T))^2 \sum_{\substack{0<|\gamma_1|\le T\\ 0.5\le\beta<1}}\left(Y+2\ln(Y)\int_{0.5}^{\beta}Y^{2u}du\right)\\ &\ll Y^2(\ln(T))^2N(T) + Y\ln(Y)(\ln(T))^2\int_{0.5}^{1}Y^{2u}N(u,T)du\\ &\ll Y^2\ln(Y)(\ln(T))^2\max_{0.5\le\alpha<1}Y^{2\alpha-1}N(\alpha,T).\end{aligned}$$

2) Proof of the second inequality. We write

$$\begin{aligned}\int_{0.5Y}^{2Y}\left|\sum_{0<|\gamma|\le T}\rho x^{\rho}\right|^2 dx &= \sum_{0<|\gamma_1|\le T}\sum_{0<|\gamma_2|\le T}(\beta_1+i\gamma_1)(\beta_2-i\gamma_2)\int_{0.5Y}^{2Y}x^{\beta_1+\beta_2+i(\gamma_1-\gamma_2)}dx\\ &\ll T^2\sum_{0<|\gamma_1|\le T}\sum_{0<|\gamma_2|\le T}\left|\int_{0.5Y}^{2Y}x^{\beta_1+\beta_2+i(\gamma_1-\gamma_2)}dx\right|\\ &= T^2\sum_{0<|\gamma_1|\le T}\sum_{0<|\gamma_2|\le T}\left|\frac{(2Y)^{\beta_1+\beta_2+1+i(\gamma_1-\gamma_2)}-(0.5Y)^{\beta_1+\beta_2+1+i(\gamma_1-\gamma_2)}}{\beta_1+\beta_2+1+i(\gamma_1-\gamma_2)}\right|\\ &\ll YT^2\sum_{0<|\gamma_1|\le T}\sum_{0<|\gamma_2|\le T}\frac{Y^{\beta_1+\beta_2}}{1+|\gamma_1-\gamma_2|}\end{aligned}$$

In the above argument we proved the following estimate

$$\sum_{0<|\gamma_1|\le T}\sum_{0<|\gamma_2|\le T}\frac{Y^{\beta_1+\beta_2}}{1+|\gamma_1-\gamma_2|} \ll Y(\ln(Y))(\ln(T))^2\max_{0.5\le\alpha<1}Y^{2\alpha-1}N(\alpha,T),$$

and the proof of the second inequality follows. □

LEMMA 8. *Let $0 < \varepsilon < 0.1$ be arbitrary small number and let $N(\sigma,T)$ denote the number of zeros ρ of $\zeta(s)$ with $\Re(s) \geq \sigma$ and $|\Im(s)| \leq T$. Then we have*

$$N(\sigma,T) \ll T^{(30/13+\varepsilon)(1-\sigma)}(\ln(T))^{14}.$$

Proof. It follows from Lemma 6 that

$$N(\sigma,T) \ll T^{15(1-\sigma)/(3+5\sigma)+\varepsilon_1}.$$

Here $0 < \varepsilon_1 < 0.01$ is arbitrary small number. Put $\varepsilon_2 = 169/(40/\varepsilon_1 - 140) > 0$. Let us consider 3 cases:

Case 1: If $0.7+\varepsilon_2 < \sigma < 13/15$ then we have

$$\frac{15}{3+5\sigma} < \frac{15}{3+5(0.7+\varepsilon_2)} = \frac{30}{13} - \frac{300\varepsilon_2}{169+130\varepsilon_2}.$$

Thus

$$\begin{aligned} N(\sigma,T) &\ll T^{15(1-\sigma)/(3+5\sigma)+\varepsilon_1} \\ &< T^{30(1-\sigma)/13+\varepsilon_1-40\varepsilon_2/(169+130\varepsilon_2)} \\ &= T^{30(1-\sigma)/13}. \end{aligned} \tag{4}$$

Case 2: If $\sigma \leq 0.7+\varepsilon_2$ then we use the Ingham's density estimate [1]

$$N(\sigma,T) \ll T^{3(1-\sigma)/(2-\sigma)}(\ln(T))^5. \tag{5}$$

We have

$$\frac{3}{2-\sigma} \leq \frac{3}{2-(0.7+\varepsilon_2)} = \frac{30}{13} + \frac{300\varepsilon_2}{40-260\varepsilon_2} \leq \frac{30}{13} + \varepsilon.$$

Here we take $\varepsilon = 10\varepsilon_2$. Then, it follows from (5) that

$$N(\sigma,T) \ll T^{(30/13+\varepsilon)(1-\sigma)}(\ln(T))^5. \tag{6}$$

Case 3: If $\sigma \geq 13/15$ then we use density estimate (see Theorem 12.1, [8]) as following

$$N(\sigma,T) \ll T^{2(1-\sigma)/\sigma}(\ln(T))^{14}.$$

Since $\sigma \geq 13/15$ then

$$N(\sigma,T) \ll T^{30(1-\sigma)/13}(\ln(T))^{14}. \tag{7}$$

We finish the proof by combining (4), (6) and (7). □

3. PROOF OF THEOREM 1

Let $\varepsilon > 0$ be arbitrary small number and put $N_1 = N^{0.525+3.75\varepsilon}$. For the number $J(N,H)$ of solutions to the inequality $0 \leq N - p_1 - p_2 < H$ we have

$$J(N,H) = \sum_{\substack{2 \leq p_1 \leq N \\ N-p_1-H < p_2 \leq N-p_1}} 1 \geq \sum_{\substack{N-2N_1 < p_1 \leq N-N_1 \\ N-p_1-H < p_2 \leq N-p_1}} 1. \tag{8}$$

We have

$$\sum_{N-p_1-H<p_2\le N-p_1} 1 = \sum_{N-p_1-H<n\le N-p_1} \frac{\Lambda(n)}{\ln(n)} - \sum_{\substack{N-p_1-H<n=q^k\le N-p_1\\ k\ge 2\\ q\text{ - prime}}} \frac{\Lambda(n)}{\ln(n)}$$

$$\gg \frac{1}{\ln(N)} \sum_{N-p_1-H<n\le N-p_1} \Lambda(n) - \sum_{\substack{N-p_1-H<n=q^k\le N-p_1\\ k\ge 2\\ q\text{ - prime}}} \frac{\Lambda(n)}{\ln(n)}. \tag{9}$$

We establish that

$$\begin{aligned}\sum_{\substack{N-p_1-H<n=q^k\le N-p_1\\ k\ge 2\\ q\text{ - prime}}} \frac{\Lambda(n)}{\ln(n)} &= \sum_{\substack{2\le k\le \log_2(N)\\ \sqrt[k]{N-p_1-H}<q\le \sqrt[k]{N-p_1}}} \frac{1}{k}\\ &\ll \sum_{2\le k\le \log_2(N)} \frac{\sqrt[k]{N-p_1}-\sqrt[k]{N-p_1-H}}{k}\\ &\ll 1+\sqrt{N-p_1}-\sqrt{N-p_1-H}\\ &\ln(N)\left(1+\sqrt[3]{N-p_1}-\sqrt[3]{N-p_1-H}\right)\\ &\ll \frac{H}{\sqrt{N_1}}+\ln(N).\end{aligned} \tag{10}$$

It follows form (8), (9) and (10) that

$$J(N,H) \gg \frac{1}{\ln(N)} \sum_{\substack{-2N_1<p_1\le N-N_1\\ N-p_1-H<n\le N-p_1}} \Lambda(n) + \mathscr{O}\left(\frac{H\sqrt{N_1}}{\ln(N)}+N_1\right). \tag{11}$$

Put $T = N_1(\ln(N))^3/H$. By Lemma 1 we have

$$\sum_{\substack{N-2N_1<p_1\le N-N_1\\ N-p_1-H<n\le N-p_1}} \Lambda(n) = H\left(\pi(N-N_1)-\pi(N-2N_1)\right) - W + \mathscr{O}\left(\frac{H}{\ln(N)}\right) \tag{12}$$

where

$$W = \sum_{N-2N_1<p_1\le N-N_1} \sum_{0<|\gamma|\le T} \frac{(N-p_1)^\rho-(N-p_1-H)^\rho}{\rho}.$$

Next, Lemma 2 yields

$$\pi(N-N_1)-\pi(N-2N_1) \gg \frac{N_1}{\ln(N)}. \tag{13}$$

By combining (11), (12) and (13) we obtain

$$J(N,H) \gg \frac{HN_1}{\ln(N)} - W + \mathscr{O}\left(\frac{H\sqrt{N_1}}{\ln(N)}+N_1\right) + \mathscr{O}\left(\frac{H}{\ln(N)}\right). \tag{14}$$

To prove Theorem 1 it suffices to show that

$$W \ll HN_1 \exp\left(-\sqrt[10]{\ln(N)}\right).$$

First we write

$$\sum_{0<|\gamma|\le T}\frac{(N-p_1)^\rho-(N-p_1-H)^\rho}{\rho}=\sum_{0<|\gamma|\le T}\int_{N-p_1-H}^{N-p_1}x^{\rho-1}dx$$
$$=\int_0^H\sum_{0<|\gamma|\le T}(x_1+N-p_1-H)^{\rho-1}dx$$
$$\ll\frac{H}{N_1}\left|\sum_{0<|\gamma|\le T}(x_1+N-p_1-H)^\rho\right|$$

here $x_1\in[0,H]$ which maximize the modulus of the sum over zeros ρ of the Riemann's zeta function. Thus, we have

$$W\ll\frac{H}{N_1}\sum_{N-2N_1<p_1\le N-N_1}\left|\sum_{0<|\gamma|\le T}(x+N-p_1-H)^\rho\right|.$$

The Cauchy's inequality give us

$$W^2\ll\frac{H^2}{N_1}\sum_{N-2N_1<p_1\le N-N_1}\left|\sum_{0<|\gamma|\le T}(x_1+N-p_1-H)^\rho\right|^2.$$

Put

$$f(t)=\sum_{0<|\gamma|\le T}(x_1+N-t-H)^\rho.$$

Lemma 4 give us

$$W^2\ll\frac{H^2}{N_1}\left(0.5\int_{N-2N_1}^{N-N_1}|f(t)|^2dt+2\left(\int_{N-2N_1}^{N-N_1}|f(t)|^2dt\right)^{0.5}\left(\int_{N-2N_1}^{N-N_1}|f'(t)|^2dt\right)^{0.5}\right) \tag{15}$$

It follows from Lemma 7 that

$$\int_{N-2N_1}^{N-N_1}|f(t)|^2dt=\int_{N_1+x_1-H}^{2N_1+x_1-H}\left|\sum_{0<|\gamma|\le T}t^\rho\right|^2dt$$
$$\le\int_{N1/2}^{2N_1}\left|\sum_{0<|\gamma|\le T}t^\rho\right|^2dt$$
$$\ll N_1^2(\ln(N))^3\max_{0.5\le\alpha<1}N_1^{2\alpha-1}N(\alpha,T). \tag{16}$$

Similarly, we have

$$\int_{N-2N_1}^{N-N_1}|f'(t)|^2dt=\int_{N_1+x_1-H}^{2N_1+x_1-H}\left|\sum_{0<|\gamma|\le T}\rho t^{\rho-1}\right|^2dt$$
$$\ll N_1^{-2}\int_{N1/2}^{2N_1}\left|\sum_{0<|\gamma|\le T}\rho t^\rho\right|^2dt$$
$$\ll T^2(\ln(N))^3\max_{0.5\le\alpha<1}N_1^{2(\alpha-1)}N(\alpha,T). \tag{17}$$

By combining (15), (16) and (17) we obtain

$$W^2 \ll H^2 N_1^2 (\ln(N))^3 \max_{0.5\le\alpha<1} N_1^{2(\alpha-1)} N(\alpha,T). \tag{18}$$

We recall that $T = N_1(\ln(N))^3/H$ and $H \ge N^{0.07+\varepsilon} = N_1^{2/15+0.5\varepsilon}$. It follows from Lemma 8 that

$$\begin{aligned} N_1^{2(\alpha-1)} N(\alpha,T) &\ll N_1^{2(\alpha-1)} T^{(30/13+\varepsilon)(1-\alpha)} (\ln(T))^{14} \\ &\ll N_1^{2(\alpha-1)} \left(\frac{N_1(\ln(N))^3}{H}\right)^{(30/13+\varepsilon)(1-\alpha)} (\ln(N))^{14} \\ &\ll \left(\frac{N_1^{2/15+13\varepsilon/30}}{H}\right)^{(30/13+\varepsilon)(1-\alpha)} (\ln(N))^{19} \\ &\ll N_1^{-\varepsilon(30/13+\varepsilon)(1-\alpha)/15} (\ln(N))^{19} \\ &\ll N_1^{-2/13\varepsilon(1-\alpha)} (\ln(N))^{19}. \end{aligned} \tag{19}$$

From (18) and (19),

$$W^2 \ll H^2 N_1^2 (\ln(N))^{22} \max_{0.5\le\alpha<1} N_1^{-2/13\varepsilon(1-\alpha)}.$$

Therefore, by Lemma 3,

$$\begin{aligned} W^2 &\ll H^2 N_1^2 \exp\left(-\frac{c_1 \varepsilon \ln(N)}{(\ln(\ln(N)))^{1/3}(\ln(N))^{2/3}} + 22\ln(\ln(N))\right) \\ &\ll H^2 N_1^2 \exp\left(-\sqrt[10]{\ln(N)}\right). \end{aligned} \tag{20}$$

We finish the proof by combining (14) and (20).

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