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COUNTING OF INDEPENDENT SETS INCLUDING THE SET OF LEAVES IN GRAPHS WITH TWO ELEMENTARY CYCLES

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Abstract. A subset S of vertices of a graph G is called independent if no two vertices in S are adjacent. In this paper, we investigate the number $\sigma_L(G)$ of all independent sets that include all leaves in graphs with two elementary cycles having only one vertex in common. In particular, we determine the minimum and maximum possible values of $\sigma_L(G)$ among all such graphs and characterize the extremal graphs that attain these values.

***Keywords**:* independent set, Fibonacci number, graphs with two elementary cycles.
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1. INTRODUCTION AND HISTORICAL BACKGROUND

In this paper, we consider only undirected, connected, simple graphs and use the standard terminology and notation of graph theory. Let P_n, $n \geq 1$, C_n, $n \geq 3$, and S_n, $n \geq 3$ denote a path, a cycle, and a star with n vertices, respectively. We define P_0 as the empty path. The *subdivision* $sub_{xy}(G)$ of an edge $e = xy$ in G means inserting a new vertex of degree 2 into the edge e. If $xy \in E(G)$, we say that x is a *neighbor* of y. The set of all neighbors of x is called the *open neighborhood* of x and is denoted by $N(x)$. The set $N(x) \cup \{x\}$ is called the *closed neighborhood* of x and is denoted by $N[x]$. For a subset $X \subseteq V(G)$, we define $N(X) = \bigcup_{x\in X} N(x)$ and $N[X] = \bigcup_{x\in X} N[x]$. For $X \subset V(G) \cup E(G)$, $G \setminus X$ denotes the graph obtained from G by deleting the set X.

Recall that a vertex of degree 1 in a graph G is called a *leaf*. For $x \in V(G)$, denote by $L(x)$ the set of all leaves adjacent to the vertex x, and let $|L(x)| = l(x)$ and $L(G) = \bigcup_{x\in V(G)} L(x)$. A vertex $x \in V(G)$ with $L(x) \neq \emptyset$ is called a *support vertex*. If $l(x) \geq 2$, then x is called a *strong support vertex* and the set of all strong support vertices is denoted by S_s. If $l(x) = 1$, then x is called a *weak support vertex* and the set of all weak support vertices is denoted by S_w. The unique leaf adjacent to a weak support vertex is called a *single leaf*. The set of all support vertices in G is denoted by $S(G)$, so $S(G) = S_s \cup S_w$. For any subset $U \subseteq V(G)$, we define $L(U) = \bigcup_{x\in U} L(x)$. A vertex $x \in V(G)$ is called a *penultimate vertex* if it is not a leaf and is adjacent to at least $d_G(x) - 1$ leaves in G. By a unicyclic graph, we mean a graph of order n, with $n \geq 3$, obtained from an n-vertex tree by adding exactly one edge.

Let $\widetilde{G}$ be an arbitrary graph. For a graph G of order n, with $n \geq 2$, we define *$\widetilde{G}$-addition* as a local augmentation of G. This operation, denoted by $G \mapsto ad_{\widetilde{G}(x,y)}(G)$, consists of adding the graph $\widetilde{G}$ to G by identifying a vertex $x \in V(G)$ with a fixed vertex $y \in V(\widetilde{G})$.

Let $n \geq k \geq 3$ be integers. We define $P_{n,k}$ as the graph of order n obtained by attaching the path P_{n-k+1} to the cycle C_k using $\widetilde{G}$-addition, that is, $P_{n,k} = ad_{P_{n-k+1}(x,y)}(C_k)$ where $y \in V(P_{n-k+1})$ is a *leaf* and $x \in V(C_k)$. Similarly, for $n \geq k+2 \geq 5$, we define $C_{n,k}$ as the graph of order n obtained by attaching the cycle C_k to C_{n-k+1} using $\widetilde{G}$-addition, that is, $C_{n,k} = ad_{C_k(x,y)}(C_{n-k+1})$ where $x \in V(C_{n-k+1})$ and $y \in V(C_k)$. The graph $H_{n,3}$, for $n \geq 4$, is obtained from the star $K_{1,n-1}$ by adding an edge between two of its leaves.

A subset $S \subseteq V(G)$ is called an *independent set* of G if no two vertices in S are adjacent in G. The empty set and all subsets containing exactly one vertex are also independent by definition. The number of all independent sets in G is denoted by $\sigma(G)$. For $x \in V(G)$, let $\sigma_x(G)$ (respectively, $\sigma_{-x}(G)$) denote the number of independent sets S such that $x \in S$ (respectively, $x \notin S$). The basic rule for counting independent sets is given by $\sigma(G) = \sigma_x(G) + \sigma_{-x}(G)$. Moreover, for $x, y \in V(G)$, let $\sigma_{x,y}(G)$ (respectively, $\sigma_{x,-y}(G)$) denote the number of independent sets S such that $x, y \in S$ (respectively, $x \in S$ and $y \notin S$).

The topic of counting all independent sets was initiated from a purely mathematical perspective in 1982 by Prodinger and Tichy in [12], and in the next decades it has been discussed in many papers; see, for example, [2, 4–7]. This interest was further motivated by their observation that the number of independent sets in paths is given by Fibonacci numbers.

The parameter $\sigma(G)$ was called the *Fibonacci number of a graph* in [12], based on the following facts: $\sigma(P_n) = F_{n+1}$ and $\sigma(C_n) = L_n$, where the *Fibonacci numbers* F_n are defined recursively by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, and the *Lucas numbers* L_n are defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. It is well known and also used for our future considerations, that Fibonacci numbers are also defined for negative indices by $F_{-1} = 0$ and $F_{-n} = (-1)^n F_{n-2}$ for $n \geq 2$.

The paper by Prodinger and Tichy gave impetus to the study of counting independent sets. Additionally, in [8], Merrifield and Simmons introduced the parameter $\sigma(G)$ in the combinatorial chemistry. The parameter $\sigma(G)$ of a molecular graph is used in molecular design and belongs to the class of so-called topological indices, which serve as quantitative descriptors of the structure of a molecular graph. For historical reasons, the parameter $\sigma(G)$ is named as the Merrifield–Simmons index.

A review of the literature shows that the topic is current, see, for example, [1, 3, 9, 11]. Many papers dealing with the theory of counting independent sets in graphs have appeared. For instance, the survey by Gutman and Wagner [4] collects and classifies results related to $\sigma(G)$, see this paper and its references for further details.

We recall that for trees and unicyclic graphs, the following results have been proved:

THEOREM 1 [12]. *If T_n is a tree of order $n, n \geq 1$, then $F_{n+1} \leq \sigma(T_n) \leq 2^{n-1} + 1$. Moreover $\sigma(T_n) = F_{n+1}$ if and only if $T_n \cong P_n$ and $\sigma(T_n) = 2^{n-1} + 1$ if and only if $T_n \cong S_n$.*

THEOREM 2 [10]. *If G is a unicyclic graph of order n, then $L_n \leq \sigma(G) \leq 3 \cdot 2^{n-3} + 1$. Moreover, $\sigma(G) = L_n$ if and only if $G \cong C_n$ or $G \cong P_{n,3}$ and $\sigma(G) = 3 \cdot 2^{n-3} + 1$ if and only if $G \cong C_4$ or $G \cong H_{n,3}$.*

Let $R_{n,3,3}$ denote the graph of order n, where $n \geq 6$, obtained by attaching the cycle C_3 to $P_{n-2,3}$ using $\widetilde{G}$-addition, that is, $R_{n,3,3} = ad_{C_3(x,y)}(P_{n-2,3})$, where $x \in V(P_{n-2,3})$ is the unique leaf of $P_{n-2,3}$, and $y \in V(C_3)$. The graph $H_{n,3,3}$, for $n \geq 5$, is obtained from the star $K_{1,n-1}$ by adding two edges in such a way that the resulting graph contains two edge-disjoint triangles.

For graphs with exactly two elementary cycles, the following result was proved:

THEOREM 3 [13]. *Let G be an arbitrary graph of order n, $n \geq 5$, with exactly two elementary cycles. Then $5F_{n-3} \leq \sigma(G) \leq 1 + 9 \cdot 2^{n-2}$. Moreover $\sigma(G) = 5F_{n-3}$ for $G \cong C_{5,3}$ and $G \cong R_{n,3,3}$ for $n \geq 6$ and $\sigma(G) = 1 + 9 \cdot 2^{n-2}$ for $G \cong H_{n,3,3}$.*

In this paper, we determine the number of independent sets that include the set $L(G)$ as a subset. Independent sets containing $L(G)$ as a subset in trees and unicyclic graphs were previously studied in [14], [16].

This subfamily of independent sets is of particular interest, as sets that include the set of leaves $L(G)$ may have applications in localization problems related to security. For example, a graph can model a system of communication routes in a border area. To ensure border protection, outposts must be placed so that each endpoint is monitored by a guard post. At the same time, to optimize placement, the posts should be located at intersections, but no two should be adjacent. In terms of graph theory, such a configuration corresponds to an independent set that includes the set of leaves. Depending on the number of available outposts, one may select an independent set of the desired cardinality. In particular, it is important to determine the number of independent sets that include the set of leaves.

Every independent set that includes the set $L(G)$ as a subset can be extended by locally augmenting the graph for example, by attaching new leaves to existing support vertices. Such local augmentations do not affect the number of independent sets that include $L(G)$ as a subset.

Let $\sigma_L(G)$ denote the total number of independent sets in G including the set $L(G)$. If $L(G) = \{w\}$, then for simplicity we also put $\sigma_w(G)$ instead of $\sigma_{\{w\}}(G)$ or $\sigma_L(G)$. For trees and unicyclic graphs, the following results was proved:

THEOREM 4 [14]. *Let T be an arbitrary n-vertex tree, $n \geq 3$. Then $1 \leq \sigma_L(T) \leq F_{n-3}$. Moreover $\sigma_L(T) = F_{n-3}$ if $T \cong P_n$.*

THEOREM 5 [15]. *Let G be a unicyclic graph of order n, $n \geq 5$, and $G \neq C_n$. Then $\sigma_L(G) \leq F_{n-1}$. Moreover, $\sigma_L(G) = F_{n-1}$ if $G \cong P_{n,n-1}$.*

Let $\mathscr{C}$ be a family of graphs with two elementary cycles having only one vertex in common. In this paper we study the parameter $\sigma_L(G)$, where $G \in \mathscr{C}$.

2. TRANSFORMATIONS THAT PRESERVE OR INCREASE $\sigma_L(G)$

Let $S \subseteq V(G)$ be an independent set. Then $\sigma_L(G) = \sigma(G \setminus N[L])$. If $\delta(G) \geq 2$, that is, $L(G) = \emptyset$, then $\sigma_L(G) = \sigma(G)$. Consequently, for all graphs with minimum degree $\delta(G) \geq 2$, the parameter $\sigma_L(G)$ coincides with the Merrifield–Simmons index. Let $\mathscr{F}_L$ be the family of all independent sets of G that include $L(G)$ as a subset. Then $|\mathscr{F}_L| = |\mathscr{F}'|$ where $\mathscr{F}'$ is the family of all independent sets in $G \setminus N[L]$. Let x be an arbitrary vertex of $V(G)$. Let $\sigma_{L,x}(G)$ (respectively, $\sigma_{L,-x}(G)$) denote the number of independent sets that include $L(G)$ as a subset and contain x (respectively, do not contain x). Then $\sigma_L(G) = \sigma_{L,x}(G) + \sigma_{L,-x}(G)$ is the basic rule for counting independent sets that include $L(G)$ in a graph G.

Since $C_{n,k}$ has no leaves, that is, $L(C_{n,k}) = \emptyset$, it follows that $\sigma_L(C_{n,k}) = \sigma(C_{n,k})$. In the subsequent considerations, we assume that $G \neq C_{n,k}$.

The following lemma is immediate.

LEMMA 1. *Let G be a graph of order n, where $n \geq 6$ and $G \in \mathscr{C}$. Then $\sigma_L(G) = 1$ if and only if $V(G) = L(G) \cup S(G)$.*

Next, we determine the maximum number of independent sets that include $L(G)$ in graphs belonging to $\mathscr{C}$, we also characterize graphs that attain this maximum. We present a sequence of results that describe how to gradually transform a graph into another graph of the same order such that $\sigma_L(G)$ increases or remains unchanged at each step. To determine the maximum value of the parameter $\sigma_L(G)$ and to characterize the extremal graphs that achieve it, we first prove several necessary results.

The next two results are immediate.

LEMMA 2. *Let G be a graph of order n, where $n \geq 7$ such that $G \in \mathscr{C}$ and let $x \in S_s(G)$. For any subset $L'(x) \subset L(x)$, we have $\sigma_L(G) = \sigma_L(G \setminus L'(x))$.*

LEMMA 3. *Let G be a graph of order n, where $n \geq 7$ such that $G \in \mathscr{C}$ and let $x \in S_w(G)$ with $L(x) = \{z\}$. If x is not a penultimate vertex, then $\sigma_L(G) \leq \sigma_L(G \setminus \{z\})$.*

LEMMA 4. *Let G be a graph of order n, where $n \geq 7$ and $G \in \mathscr{C}$. Let $x \in S_s(G)$ with $L(x) = \{z_1, \ldots, z_k\}$, where $k \geq 2$. Then for any $1 \leq i \leq k$, either*

(i) $\sigma_L(G) < \sigma_L(\mathrm{sub}_{uv}(G \setminus \{z_i\}))$ *if u is a penultimate vertex and $v \in N(u) \setminus L(u)$, or*

(ii) $\sigma_L(G) < \sigma_L(\mathrm{sub}_{xq}(G \setminus \{z_i\}))$ *if G has no penultimate vertex and q is a vertex belonging to a cycle.*

Proof. Let $z_i \in L(x)$, $1 \leq i \leq k$, be a fixed vertex. Then by Lemma 2 we have that $\sigma_L(G) = \sigma_L(G \setminus \{z_i\})$, let us consider the following cases.

(i) Let u be a penultimate vertex of G and $v \in N(u) \setminus L(u)$. Denote $G' = sub_{uv}(G \setminus \{z_i\})$ and let $\mathscr{F}_L$ and $\mathscr{F}'_L$ be families of all independent sets including the set of leaves in G and in G', respectively. Clearly $\sigma_L(G') = \sigma_{L,u}(G') + \sigma_{L,-u}(G')$. Since u is the penultimate vertex in G hence u is the penultimate vertex in G', too. Let $S \in \mathscr{F}'_L$. Then it is obvious that $u \notin S$. This gives that $\sigma_L(G') = |\mathscr{F}'_{L,-u}|$. Let z be the vertex inserted into the edge uv. Of course $|\mathscr{F}'_{L,-u}| = |\mathscr{F}'_{L,z}| + |\mathscr{F}'_{L,-z}| = |\mathscr{F}'_{L,z}| + \sigma_L(G)$. Since $|\mathscr{F}'_{L,z}| \geq 1$ so $\sigma_L(G') > \sigma_L(G)$.

(ii) Since a graph G has no penultimate vertex so every support vertex belongs to the subgraph $C_{p,l}$ in G. Let q be a neighbor of x and q belongs to a cycle in G. Let z be a vertex inserted into edge xq. Denote $G' = sub_{xq}(G \setminus \{z_i\})$. Then $\sigma_L(G') = \sigma_{L,z}(G') + \sigma_{L,-z}(G')$. Since $\sigma_{L,-z}(G') = \sigma_L(G)$ hence $\sigma_L(G') = \sigma_{L,z}(G') + \sigma_L(G) > \sigma_L(G)$, which ends the proof.

□

In the similar way we can prove following results:

THEOREM 6. *Let G be a graph of order n, where $n \geq 6$ and $G \in \mathscr{C}$. Let $x \in S_w(G)$ with $L(x) = \{z\}$. Let u be a penultimate vertex of G and $v \in N(u) \setminus L(u)$. Then $\sigma_L(G) \leq \sigma_L(sub_{uv}(G \setminus \{z\}))$.*

THEOREM 7. *Let G be a graph of order n, where $n \geq 7$ such that $G \in \mathscr{C}$ and let G has at least two leaves. Let $x \in S_w(G)$ with $L(x) = \{z\}$. If G has no penultimate vertex then $\sigma_L(G) < \sigma_L(sub_{xq}(G \setminus \{z\}))$, where q is a vertex belonging to a cycle.*

THEOREM 8. *Let G be a graph of order n, with $n \geq 9$ and $G \in \mathscr{C}$. Suppose that G contains two subtrees isomorphic to P_t and P_m, with $t, m \geq 3$, such that:*

(i) *P_t and P_m have only one vertex x in common,*

(ii) *the endpoints of P_t and P_m distinct from x are leaves in G, and*

(iii) *all internal vertices of P_t and P_m have degree two in G.*

Then $\sigma_L(G) < \sigma_L(ad_{P_t(u,x)}(G \setminus (P_t \setminus \{x\})))$ where u is the end vertex of P_m distinct from x, which is identified with the initial vertex x of P_t.

Proof. Denote $G' = ad_{P_t(u,x)}(G \setminus (P_t \setminus \{x\}))$. Clearly, $\sigma_L(G) = \sigma_{L,x}(G) + \sigma_{L,-x}(G)$. Let S be an arbitrary independent set in G that includes the set of leaves. Then $u \in S$. Define $S^* = S \cap (V(G) \setminus (V(P_t) \cup V(P_m))$, $S_1 = S \cap V(P_m)$, $S_2 = S \cap V(P_t)$. We consider two cases:

(1). $x \in S$.

Identifying the initial vertex x of P_t with the end vertex u of P_m, we observe that the set $S^* \cup S_1 \cup S_2$ is an independent set in the graph G' including the set L. Moreover, the set $S^* \cup S_1 \cup S_2 \setminus \{u\}$ is an additional independent set in the graph G' including the set L. Hence, $\sigma_{L,x}(G) < \sigma_{L,x}(G')$.

(2). $x \notin S$.

Let $v \in N(x) \cap V(P_t)$. If $v \notin S$, then $S^* \cup S_1 \cup S_2$ is an independent set in the graph G' including the set L. If $v \in S$, then $S^* \cup S_1 \cup S_2 \setminus \{u\}$ is an independent set in the graph G' including the set L.

Consequently, in both subcases, $\sigma_{L,-x}(G) < \sigma_{L,-x}(G')$.

Finally from the above cases we obtain that $\sigma_L(G) < \sigma_L(G')$, which ends the proof. □

Using the same method, we can prove the following:

THEOREM 9. *Let G be a graph of order n, with $n \geq 9$ and $G \in \mathscr{C}$. Suppose that G contains two subtrees isomorphic to P_t and P_m with $t, m \geq 3$ such that*

(i) *P_t has exactly one vertex x in common with a cycle, and P_m has exactly one vertex y in common with a cycle, where $x \neq y$,*

(ii) *the endpoints of P_t and P_m distinct from x and y are leaves in G,*

(iii) *all internal vertices of P_t and P_m have degree two in G.*

Then $\sigma_L(G) < \sigma_L(ad_{P_t(u,x)}(G \setminus (P_t \setminus \{x\})))$, where u is the end vertex of P_m distinct from x, which is identified with the initial vertex x of P_t.

Let $k \geq 3$, $l \geq 3$, $0 \leq r \leq [l/2]$, and $n \geq k+l$. By $G_n^{k,l,r}$ we denote the graph of order n consisting of two cycles C_k and C_l that have exactly one common vertex, say x, and a path $P_{n-k-l+1}$ attached to a vertex $y \in V(C_l)$ (not necessarily distinct from x), such that $d_{G_n^{k,l,r}}(x,y) = r$, where $r \geq 0$. Clearly, $|L(G_n^{k,l,r})| = 1$. Let $L(G_n^{k,l,r}) = \{w\}$.

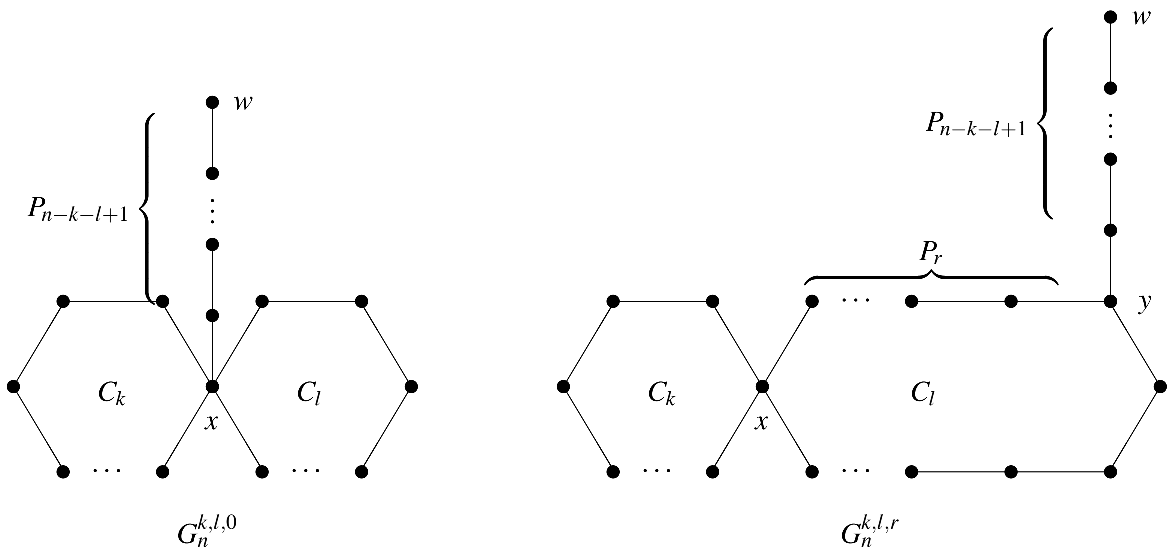

Fig.1 – Graphs $G_n^{k,l,r}$ for arbitrary $r \geq 0$.

From previous theorems, we can conclude:

THEOREM 10. *If $G \in \mathscr{C}$ is a graph of order n, for which the value of $\sigma_L(G)$ attains its maximum possible value among all graphs of order n belonging to $\mathscr{C}$ and $G \ncong C_{n,k}$, then $G \cong G_n^{k,l,r}$ for some integers n,k,l, and r.*

It follows from Theorem 10 that to determine the maximum value of $\sigma_L(G)$, it suffices to study the behavior of the parameter $\sigma_L(G_n^{k,l,r})$.

3. THE BEHAVIOR OF THE PARAMETER $\sigma_L(G_N^{K,L,R})$

In future calculations we use the following well-known identities for Fibonacci numbers

$$F_m F_n + F_{m-1} F_{n-1} = F_{m+n}, \tag{1}$$

$$F_m F_n - F_{m+1} F_{n-1} = (-1)^n F_{m-n}. \tag{2}$$

From (1) we get $F_{m-1}F_{n-1} + F_{m-2}F_{n-2} = F_{m+n-2}$. Hence and by (1) we get

$$F_m F_n - F_{m-2} F_{n-2} = F_{m+n-1}. \tag{3}$$

From (2) we get $F_{m+1}F_{n-1} - F_{m+2}F_{n-2} = (-1)^{n-1} F_{m-n+2}$. Hence and by (2) we get

$$F_m F_n - F_{m+2} F_{n-2} = (-1)^{n-1} F_{m-n+1}. \tag{4}$$

LEMMA 5. *Let* $k \geq 3$, $l \geq 3$, $0 \leq r \leq [l/2]$, $n \geq k+l$. *Then*

(i) $\sigma_w(G_n^{k,l,0}) = F_{k-2}F_{l-2}F_{n-k-l-1} + F_kF_lF_{n-k-l}$,

(ii) $\sigma_w(G_n^{k,l,1}) = F_{k-2}F_{l-2}F_{n-k-l} + F_kF_{n-k-1}$,

(iii) $\sigma_w(G_n^{k,l,2}) = F_{k-2}F_{n-k-3} + F_k(F_{l-2}F_{n-k-l} + F_{n-k-2})$, $l \geq 4$,

(iv) $\sigma_w(G_n^{k,l,r}) = F_{k-2}(F_{l-r-2}F_{r-2}F_{n-k-l-1} + F_{l-r-1}F_{r-1}F_{n-k-l}) + F_k(F_{l-r-1}F_{r-1}F_{n-k-l-1} + F_{l-r}F_rF_{n-k-l})$, $l \geq 6$, $r \geq 3$.

Proof. (i) For $n \geq k+l+2$ we get

$$\begin{aligned}
\sigma_w(G_n^{k,l,0}) &= \sigma_{w,x}(G_n^{k,l,0}) + \sigma_{w,-x}(G_n^{k,l,0}) \\
&= \sigma(P_{k-3})\sigma(P_{l-3})\sigma(P_{n-k-l-2}) + \sigma(P_{k-1})\sigma(P_{l-1})\sigma(P_{n-k-l-1}) \\
&= F_{k-2}F_{l-2}F_{n-k-l-1} + F_kF_lF_{n-k-l}.
\end{aligned}$$

For $n = k+l+1$ we get

$$\begin{aligned}
\sigma_w(G_n^{k,l,0}) &= \sigma_{w,x}(G_n^{k,l,0}) + \sigma_{w,-x}(G_n^{k,l,0}) \\
&= \sigma(P_{k-3})\sigma(P_{l-3}) + \sigma(P_{k-1})\sigma(P_{l-1}) = F_{k-2}F_{l-2} + F_kF_l \\
&= F_{k-2}F_{l-2}F_{n-k-l-1} + F_kF_lF_{n-k-l}.
\end{aligned}$$

For $n = k+l$ we get

$$\sigma_w(G_n^{k,l,0}) = \sigma_{w,-x}(G_n^{k,l,0}) = \sigma(P_{k-1})\sigma(P_{l-1}) = F_kF_l = F_{k-2}F_{l-2}F_{n-k-l-1} + F_kF_lF_{n-k-l}.$$

(ii) For $n \geq k+l+1$ we get

$$\begin{aligned}
\sigma_w(G_n^{k,l,1}) &= \sigma_{w,x}(G_n^{k,l,1}) + \sigma_{w,-x}(G_n^{k,l,1}) \\
&= \sigma(P_{k-3})\sigma(P_{l-3})\sigma(P_{n-k-l-1}) + \sigma(P_{k-1})\sigma(P_{n-k-2}) \\
&= F_{k-2}F_{l-2}F_{n-k-l} + F_kF_{n-k-1}.
\end{aligned}$$

For $n = k+l$ we get

$$\begin{aligned}
\sigma_w(G_n^{k,l,1}) &= \sigma_{w,x}(G_n^{k,l,1}) + \sigma_{w,-x}(G_n^{k,l,1}) \\
&= \sigma(P_{k-3})\sigma(P_{l-3}) + \sigma(P_{k-1})\sigma(P_{l-2}) = F_{k-2}F_{l-2} + F_kF_{l-1} \\
&= F_{k-2}F_{l-2}F_{n-k-l} + F_kF_{n-k-1}.
\end{aligned}$$

(iii) For $n \geq k+l+1$ we get

$$\begin{aligned}
\sigma_w(G_n^{k,l,2}) &= \sigma_{w,x}(G_n^{k,l,2}) + \sigma_{w,-x}(G_n^{k,l,2}) \\
&= \sigma(P_{k-3})\sigma(P_{n-k-4}) + \sigma(P_{k-1})\Big(\sigma(P_{l-3})\sigma(P_{n-k-l-1}) + \sigma(P_{n-k-3})\Big) \\
&= F_{k-2}F_{n-k-3} + F_k(F_{l-2}F_{n-k-l} + F_{n-k-2}).
\end{aligned}$$

For $n = k+l$ we get

$$\begin{aligned}
\sigma_w(G_n^{k,l,2}) &= \sigma_{w,x}(G_n^{k,l,2}) + \sigma_{w,-x}(G_n^{k,l,2}) \\
&= \sigma(P_{k-3})\sigma(P_{l-4}) + \sigma(P_{k-1})\sigma(P_{l-3}) \cdot 2 = F_{k-2}F_{l-3} + 2F_kF_{l-2} \\
&= F_{k-2}F_{n-k-3} + F_k(F_{l-2}F_{n-k-l} + F_{n-k-2}).
\end{aligned}$$

(iv) For $n \geq k+l+2$ and $r \geq 3$ we get

$$\begin{aligned}
\sigma_w(G_n^{k,l,r}) &= \sigma_{w,x}(G_n^{k,l,r}) + \sigma_{w,-x}(G_n^{k,l,r}) \\
&= \sigma(P_{k-3})\Big(\sigma(P_{l-r-3})\sigma(P_{r-3})\sigma(P_{n-k-l-2}) \\
&\quad + \sigma(P_{l-r-2})\sigma(P_{r-2})\sigma(P_{n-k-l-1})\Big) \\
&\quad + \sigma(P_{k-1})\Big(\sigma(P_{l-r-2})\sigma(P_{r-2})\sigma(P_{n-k-l-2}) \\
&\quad + \sigma(P_{l-r-1})\sigma(P_{r-1})\sigma(P_{n-k-l-1})\Big) \\
&= F_{k-2}(F_{l-r-2}F_{r-2}F_{n-k-l-1} + F_{l-r-1}F_{r-1}F_{n-k-l}) \\
&\quad + F_k(F_{l-r-1}F_{r-1}F_{n-k-l-1} + F_{l-r}F_rF_{n-k-l}).
\end{aligned}$$

For $n = k+l+1$ we get by (1)

$$\begin{aligned}
\sigma_w(G_n^{k,l,r}) &= \sigma_{w,x}(G_n^{k,l,r}) + \sigma_{w,-x}(G_n^{k,l,r}) \\
&= \sigma(P_{k-3})\sigma(P_{l-3}) + \sigma(P_{k-1})\sigma(P_{l-1}) = F_{k-2}F_{l-2} + F_kF_l \\
&= F_{k-2}(F_{l-r-2}F_{r-2}F_{n-k-l-1} + F_{l-r-1}F_{r-1}F_{n-k-l}) \\
&\quad + F_k(F_{l-r-1}F_{r-1}F_{n-k-l-1} + F_{l-r}F_rF_{n-k-l}).
\end{aligned}$$

For $n = k+l$ and $r \geq 2$ we get

$$\begin{aligned}
\sigma_w(G_n^{k,l,r}) &= \sigma_{w,x}(G_n^{k,l,r}) + \sigma_{w,-x}(G_n^{k,l,r}) \\
&= \sigma(P_{k-3})\sigma(P_{l-r-2})\sigma(P_{r-2}) + \sigma(P_{k-1})\sigma(P_{l-r-1})\sigma(P_{r-1}) \\
&= F_{k-2}F_{l-r-1}F_{r-1} + F_kF_{l-r}F_r \\
&= F_{k-2}(F_{l-r-2}F_{r-2}F_{n-k-l-1} + F_{l-r-1}F_{r-1}F_{n-k-l}) \\
&\quad + F_k(F_{l-r-1}F_{r-1}F_{n-k-l-1} + F_{l-r}F_rF_{n-k-l}).
\end{aligned}$$

This ends the proof of Lemma 5. □

If we substitute in (iv) r by 0, 1 or 2 then we get (i), (ii) or (iii) respectively. We distinguish these cases because of the manner of proof and the later use.

COROLLARY 1. *The following inequalities are true:*

(i) $\sigma_w(G_n^{4,n-4,0}) > \sigma_w(G_n^{3,3,0})$ *for* $n \geq 7$.

(ii) $\sigma_w(G_n^{4,n-4,0}) \geq \sigma_w(G_n^{3,4,0})$ *for* $n \geq 7$ *and equality holds iff* $n = 7$.

(iii) $\sigma_w(G_n^{4,n-4,0}) \geq \sigma_w(G_n^{4,4,0})$ *for* $n \geq 8$ *and equality holds iff* $n = 8$.

Proof. Above inequalities are easy to verify by using the Lemma 5 (i). □

From the Lemma 5 it follows the following relations between the numbers of independent sets of $G_n^{k,l,r}$ for special r.

LEMMA 6. *Let* $k \geq 3$, $l \geq 3$, $0 \leq r \leq [l/2]$, $n \geq k+l$. *Then*

(i) $\sigma_w(G_n^{k,l,0}) = \sigma_w(G_n^{k,l,1}) + F_{k-1}F_{l-2}F_{n-k-l-2} \geq \sigma_w(G_n^{k,l,1})$,

(ii) $\sigma_w(G_n^{k,l,0}) = \sigma_w(G_n^{k,l,2}) + F_{k-1}F_{l-3}F_{n-k-l-2} \geq \sigma_w(G_n^{k,l,2})$, $l \geq 4$,

(iii) $\sigma_w(G_n^{k,l,0}) = \sigma_w(G_n^{k,l,r}) + F_{k-1}F_{l-r-1}F_{r-1}F_{n-k-l-2} \geq \sigma_w(G_n^{k,l,r})$, $l \geq 6$, $r \geq 3$.

The equations hold if and only if $n = k+l+1$.

Proof. (i) By the Lemma 5 (i), (ii) and (3) we have

$$\begin{aligned}\sigma_w(G_n^{k,l,0})-\sigma_w(G_n^{k,l,1}) &= F_{k-2}F_{l-2}(F_{n-k-l-1}-F_{n-k-l})+F_k(F_lF_{n-k-l}-F_{n-k-1})\\ &= -F_{k-2}F_{l-2}F_{n-k-l-2}+F_kF_{l-2}F_{n-k-l-2}\\ &= F_{k-1}F_{l-2}F_{n-k-l-2}.\end{aligned}$$

(ii) By the Lemma 5 (iii), (ii) and (3) for $l\geq 4$ we get

$$\begin{aligned}\sigma_w(G_n^{k,l,2})-\sigma_w(G_n^{k,l,1}) &= F_{k-2}(F_{n-k-3}-F_{l-2}F_{n-k-l})+F_k(F_{l-2}F_{n-k-l}+F_{n-k-2}-F_{n-k-1})\\ &= -F_{k-2}F_{l-4}F_{n-k-l-2}+F_k(F_{l-2}F_{n-k-l}-F_{n-k-3})\\ &= -F_{k-2}F_{l-4}F_{n-k-l-2}+F_kF_{l-4}F_{n-k-l-2}\\ &= F_{k-1}F_{l-4}F_{n-k-l-2}.\end{aligned}$$

Hence and by virtue of (i) we get for $l\geq 4$

$$\sigma_w(G_n^{k,l,0})-\sigma_w(G_n^{k,l,2})=F_{k-1}F_{l-2}F_{n-k-l-2}-F_{k-1}F_{l-4}F_{n-k-l-2}=F_{k-1}F_{l-3}F_{n-k-l-2}.$$

(iii) By the Lemma 5 (i), (iv) and (1) for $l\geq 6$, $r\geq 3$

$$\begin{aligned}\sigma_w(G_n^{k,l,0})-\sigma_w(G_n^{k,l,r}) &= F_{k-2}(F_{l-2}F_{n-k-l-1}-F_{l-r-2}F_{r-2}F_{n-k-l-1}-F_{l-r-1}F_{r-1}F_{n-k-l})\\ &\quad +F_k(F_lF_{n-k-l}-F_{l-r-1}F_{r-1}F_{n-k-l-1}-F_{l-r}F_rF_{n-k-l})\\ &= F_{k-2}(F_{l-r-1}F_{r-1}F_{n-k-l-1}-F_{l-r-1}F_{r-1}F_{n-k-l})\\ &\quad +F_k(F_{l-r-1}F_{r-1}F_{n-k-l}-F_{l-r-1}F_{r-1}F_{n-k-l-1})\\ &= -F_{k-2}F_{l-r-1}F_{r-1}F_{n-k-l-2}+F_kF_{l-r-1}F_{r-1}F_{n-k-l-2}\\ &= F_{k-1}F_{l-r-1}F_{r-1}F_{n-k-l-2}.\end{aligned}$$

Because $F_p=0$ iff $p=-1$ then $F_{n-k-l-2}=0$ iff $n=k+l+1$. The remaining subscripts are positive. Hence it follows that the equalities in (i)–(iii) hold if and only if $n=k+l+1$. □

LEMMA 7. *Let $n\geq k+l$, $l\geq 3$. Then*

(i) $\sigma_w(G_n^{k,l,0})=\sigma_w(G_n^{k-1,l+1,0})+(-1)^kF_{l-k}F_{n-k-l+1}$ *for* $k\geq 4$.

(ii) $\sigma_w(G_n^{k,l,0})=\sigma_w(G_n^{k-2,l+2,0})+(-1)^{k-1}F_{l-k+1}F_{n-k-l+1}$ *for* $k\geq 5$.

Proof. (i) By the Lemma 5 (i) and (2) we have for $k\geq 4$

$$\begin{aligned}\sigma_w(G_n^{k,l,0})-\sigma_w(G_n^{k-1,l+1,0}) &= F_{k-2}F_{l-2}F_{n-k-l-1}+F_kF_lF_{n-k-l}\\ &\quad -F_{k-3}F_{l-1}F_{n-k-l-1}-F_{k-1}F_{l+1}F_{n-k-l}\\ &= (F_{k-2}F_{l-2}-F_{k-3}F_{l-1})F_{n-k-l-1}+(F_kF_l-F_{k-1}F_{l+1})F_{n-k-l}\\ &= (-1)^{k-2}F_{l-k}F_{n-k-l-1}+(-1)^kF_{l-k}F_{n-k-l}\\ &= (-1)^kF_{l-k}(F_{n-k-l-1}+F_{n-k-l})\\ &= (-1)^kF_{l-k}F_{n-k-l+1}.\end{aligned}$$

(ii) By the Lemma 5 (i) and (4) we have for $k\geq 5$

$$\begin{aligned}
\sigma_w(G_n^{k,l,0}) - \sigma_w(G_n^{k-2,l+2,0}) &= F_{k-2}F_{l-2}F_{n-k-l-1} + F_kF_lF_{n-k-l} \\
&\quad - F_{k-4}F_lF_{n-k-l-1} - F_{k-2}F_{l+2}F_{n-k-l} \\
&= (F_{k-2}F_{l-2} - F_{k-4}F_l)F_{n-k-l-1} + (F_kF_l - F_{k-2}F_{l+2})F_{n-k-l} \\
&= (-1)^{k-3}F_{l-k+1}F_{n-k-l-1} + (-1)^{k-1}F_{l-k+1}F_{n-k-l} \\
&= (-1)^{k-1}F_{l-k+1}(F_{n-k-l-1} + F_{n-k-l}) \\
&= (-1)^{k-1}F_{l-k+1}F_{n-k-l+1}.
\end{aligned}$$

This ends the proof. □

LEMMA 8. *Let* $n \geq k+l$, $k \geq 3$. *Then*

(i) $\sigma_w(G_n^{k,l,0}) = \sigma_w(G_n^{k,l-1,0}) + (-1)^l A$, $l \geq 4$, *where* $A > 0$ *for* $n \geq k+2l-2$.

(ii) $\sigma_w(G_n^{k,l,0}) = \sigma_w(G_n^{k,l-1,0}) + (-1)^{n-k-l} B$, $l \geq 4$, *where* $B > 0$ *for* $n \leq k+2l-2$.

(iii) $\sigma_w(G_n^{k,l,0}) = \sigma_w(G_n^{k,l-2,0}) + (-1)^{l-1} C$, $l \geq 5$, *where* $C > 0$ *for* $n \geq k+2l-3$.

(iv) $\sigma_w(G_n^{k,l,0}) = \sigma_w(G_n^{k,l-2,0}) + (-1)^{n-k-l} D$, $l \geq 5$, *where* $D > 0$ *for* $n \leq k+2l-3$.

Proof. Because Fibonacci numbers are also defined for negative indices, so we prove that A,B,C,D are positive numbers.

(i) By the Lemma 5 (i) and (2) we get for $l \geq 4$

$$\begin{aligned}
\sigma_w(G_n^{k,l,0}) - \sigma_w(G_n^{k,l-1,0}) &= F_{k-2}F_{l-2}F_{n-k-l-1} + F_kF_lF_{n-k-l} \\
&\quad - F_{k-2}F_{l-3}F_{n-k-l} - F_kF_{l-1}F_{n-k-l+1} \\
&= F_{k-2}(F_{l-2}F_{n-k-l-1} - F_{l-3}F_{n-k-l}) + F_k(F_lF_{n-k-l} - F_{l-1}F_{n-k-l+1}) \\
&= F_{k-2}(-1)^{l-2}F_{n-k-2l+1} + F_k(-1)^lF_{n-k-2l} \\
&= (-1)^l(F_{k-2}F_{n-k-2l+1} + F_kF_{n-k-2l}).
\end{aligned}$$

If $n-k-2l \geq -2$ then $A = F_{k-2}F_{n-k-2l+1} + F_kF_{n-k-2l} > 0$.

(ii) We have

$$\begin{aligned}
F_{k-2}F_{n-k-2l+1} + F_kF_{n-k-2l} &= F_{k-2}(-1)^{-n+k+2l-1}F_{-n+k+2l-3} + F_k(-1)^{-n+k+2l}F_{-n+k+2l-2} \\
&= (-1)^{-n+k+2l-1}(F_{k-2}F_{-n+k+2l-3} - F_kF_{-n+k+2l-2}) \\
&= (-1)^{-n+k-1}(F_{k-2}F_{-n+k+2l-3} - F_{k-2}F_{-n+k+2l-2} - F_{k-1}F_{-n+k+2l-2}) \\
&= (-1)^{-n+k-1}(-F_{k-2}F_{-n+k+2l-4} - F_{k-1}F_{-n+k+2l-2}) \\
&= (-1)^{-n+k}(F_{k-2}F_{-n+k+2l-4} + F_{k-1}F_{-n+k+2l-2}).
\end{aligned}$$

Thus we get by (i)

$$\begin{aligned}
\sigma_w(G_n^{k,l,0}) - \sigma_w(G_n^{k,l-1,0}) &= (-1)^{-n+k+l}(F_{k-2}F_{-n+k+2l-4} + F_{k-1}F_{-n+k+2l-2}) \\
&= (-1)^{n-k-l}(F_{k-2}F_{-n+k+2l-4} + F_{k-1}F_{-n+k+2l-2}),
\end{aligned}$$

where $B = F_{k-2}F_{-n+k+2l-4} + F_{k-1}F_{-n+k+2l-2} > 0$ for $-n+k+2l-4 \geq -2$.

(iii) By the Lemma 5 (i) and (4) we get for $l \geq 5$

$$\begin{aligned}
\sigma_w(G_n^{k,l,0}) - \sigma_w(G_n^{k,l-2,0}) &= F_{k-2}F_{l-2}F_{n-k-l-1} + F_kF_lF_{n-k-l} \\
&\quad - F_{k-2}F_{l-4}F_{n-k-l+1} - F_kF_{l-2}F_{n-k-l+2} \\
&= F_{k-2}(F_{l-2}F_{n-k-l-1} - F_{l-4}F_{n-k-l+1}) \\
&\quad + F_k(F_lF_{n-k-l} - F_{l-2}F_{n-k-l+2}) \\
&= F_{k-2}(-1)^{l-3}F_{n-k-2l+2} + F_k(-1)^{l-1}F_{n-k-2l+1} \\
&= (-1)^{l-1}(F_{k-2}F_{n-k-2l+2} + F_kF_{n-k-2l+1}).
\end{aligned}$$

If $n-k-2l+1 \geq -2$ then $C = F_{k-2}F_{n-k-2l+2} + F_kF_{n-k-2l+1} > 0$.
(iv) Similarly as in (ii) we get by (iii)

$$\sigma_w(G_n^{k,l,0}) - \sigma_w(G_n^{k,l-2,0}) = (-1)^{n-k-l}(F_{k-2}F_{-n+k+2l-5} + F_{k-1}F_{-n+k+2l-3}),$$

where $D = F_{k-2}F_{-n+k+2l-5} + F_{k-1}F_{-n+k+2l-3} > 0$ for $-n+k+2l-5 \geq -2$. □

4. EXTREMAL BOUNDS FOR PARAMETER $\sigma_L(G)$

In this section we give the upper bound for the parameter $\sigma_L(G)$ in $(n,n+1)$-graphs.

THEOREM 11 *Let* $k \geq 3$, $l \geq 3$, $0 \leq r \leq [l/2]$, $n \geq k+l \geq 7$. *Then* $\sigma_w(G_n^{k,l,r}) \leq \sigma_w(G_n^{4,n-4,0}) = 5F_{n-4}$. *Moreover,* $\sigma_w(G_n^{k,l,r}) = 5F_{n-4}$ *if and only if* $G_n^{k,l,r} \cong G_n^{4,n-4,0}$.

Proof. Suppose that k,l are fixed. By Lemma 6 it follows that for $0 < r \leq [l/2]$

$$\sigma_w(G_n^{k,l,r}) \leq \sigma_w(G_n^{k,l,0}) \tag{5}$$

and equality holds if and only if $n = k+l+1$. Therefore, it suffices to consider graphs of the form $G_n^{k,l,0}$.

Now suppose that the sum $k+l$ is fixed and $k+l \geq 8$. Without loss of generality, assume $k \leq l$ because $\sigma_w(G_n^{k,l,0}) = \sigma_w(G_n^{l,k,0})$. Let $k > 4$. If k is odd then, by Lemma 7 (i), $\sigma_w(G_n^{k,l,0}) < \sigma_w(G_n^{k-1,l+1,0})$ and $k-1$ is even. If k is even then, by Lemma 7 (ii), $\sigma_w(G_n^{k,l,0}) < \sigma_w(G_n^{k-2,l+2,0})$ and $k-2$ is even. For $k=3$ we have, by Lemma 7 (i), $\sigma_w(G_n^{3,l,0}) < \sigma_w(G_n^{4,l-1,0})$. Hence it follows that for $k \neq 4$, $k+l \geq 8$

$$\sigma_w(G_n^{k,l,0}) < \sigma_w(G_n^{4,k+l-4,0}). \tag{6}$$

Thus it is suffices to consider the graphs $G_n^{4,l,0}$, $l \geq 4$.

Consider any graph $G_n^{4,l,0}$ and suppose that $4 < l < n-4$. We have either $n \geq 2l+2$ or $n \leq 2l+1$. Let $n \geq 2l+2$. If l is odd then, by Lemma 8 (i), $\sigma_w(G_n^{4,l,0}) < \sigma_w(G_n^{4,l-1,0})$. If l is even then, by Lemma 8 (iii), $\sigma_w(G_n^{4,l,0}) < \sigma_w(G_n^{4,l-2,0})$ and $l-2 \geq 4$. Now suppose that $n \leq 2l+1$. If $n-l$ is odd then, by Lemma 8 (ii), $\sigma_w(G_n^{4,l,0}) < \sigma_w(G_n^{4,l-1,0})$. If $n-l$ is even then $n \geq l+6$ (because $n > 4+l$), $n-(l+2)$ is even and $n \leq 2l+5$. Hence and by Lemma 8 (iv) $\sigma_w(G_n^{4,l,0}) < \sigma_w(G_n^{4,l+2,0})$ and $l+2 \leq n-4$.

Thus for every graph $G_n^{4,l,0}$ with $4 < l < n-4$ there exist a graph $G_n^{4,l',0}$ such that $\sigma_w(G_n^{4,l',0}) > \sigma_w(G_n^{4,l,0})$ and $4 \leq l' \leq n-4$. For any n the number of different graphs $G_n^{4,l,0}$ is finite. Thus the graph with maximal independent sets among the graphs $G_n^{4,l,0}$, $4 \leq l \leq n-4$, is a graph isomorphic with $G_n^{4,4,0}$ or $G_n^{4,n-4,0}$. Hence, by (6) and by assumption $k+l \geq 8$ it follows that in the set of graphs $G_n^{k,l,0}$, $k \geq 3$, $l \geq 3$, the maximal number of independent sets is realized in the graph which is isomorphic with one of the graphs: $G_n^{3,3,0}$, $G_n^{3,4,0}$, $G_n^{4,4,0}$, $G_n^{4,n-4,0}$. By virtue of Corollary 1 we see that for $n \geq 7$ the extremal graph is only the graph isomorphic to $G_n^{4,n-4,0}$.

Thus by (5) for $n \geq 7$

$$\sigma_w(G_n^{k,l,r}) \leq \sigma_w(G_n^{4,n-4,0})$$

and the equality holds if and only if $k=4$, $l=n-4$, $r=0$ or $k=n-4$, $l=4$, $r=0$. Moreover, by Lemma 5 (i), $\sigma_w(G_n^{4,n-4,0}) = F_4F_{n-6}F_{-1} + F_4F_{n-4}F_0 = 5F_{n-4}$, which ends the proof. $\square$

From Theorems 10 and 11 we immediately obtain the main result of this paper.

THEOREM 12 *Let G be a graph of order n, $n \geq 7$ and $G \in \mathscr{C}$. Then $\sigma_L(G) \leq 5F_{n-4}$. Moreover the equality holds only for $G \cong G_n^{4,n-4,0}$.*

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