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INFINITE FAMILIES OF CONGRUENCES FOR 4- AND 6-REGULAR PARTITIONS

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Abstract. Recently, Ballantine and Merca proved some congruences modulo powers of 2 for $b_4(n)$ and congruences modulo 3 for $b_6(n)$, where $b_t(n)$ denotes the number of t-regular partitions of n. Motivated by Ballantine and Merca's works on congruences of $b_t(n)$, we present a characterization of congruences modulo 8 for $b_4(n)$, from which, we obtain infinite families of congruences modulo 8 for $b_4(n)$. Furthermore, we also prove infinite families of congruences modulo 3 for $b_6(n)$ based on Newman's identities. Those congruences involve primes which are congruent to 1 modulo 24.

***Keywords**:* partition, congruences, regular partition.
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1. INTRODUCTION

Recall that a partition of n is a non-increasing sequence of positive integers, called parts, whose sum is n. If $t \geq 2$ is an integer, then a partition is called a t-regular partition if there is no part divisible by t. As usual, let $b_t(n)$ denote the number of t-regular partitions of n and set $b_t(0) = 1$. The generating function of $b_t(n)$ is

$$\sum_{n=0}^{\infty} b_t(n)q^n = \frac{(q^t;q^t)_\infty}{(q;q)_\infty}, \tag{1}$$

where here and throughout this paper, $(q;q)_\infty := \prod_{n=1}^{\infty}(1-q^n)$.

In recent years, congruence properties for $b_t(n)$ are investigated in many interesting papers by Andrews, Hirschhorn and Sellers [1], Ballantine and Merca [2], Chen [4], Cui and Gu [5,6], Keith [7], Keith and Zanello [8], Lin and Wang [9], Merca [10,11], Xia [14] and Yao [15,16]. For example, Andrews, Hirschhorn and Sellers [1] proved that for $n \geq 0$,

$$\begin{aligned} b_4(9n+4) &\equiv 0 \pmod{4}, \\ b_4(9n+7) &\equiv 0 \pmod{12}. \end{aligned}$$

They also proved the following infinite families of congruences modulo 2 for $b_4(n)$: for $n,\alpha \geq 0$,

$$b_4\left(3^{2\alpha+2}n+\frac{j\times 3^{2\alpha+1}-1}{8}\right)\equiv 0 \pmod{2},$$

where $j\in\{11,17,19\}$. Merca [10] proved that $b_4(n)$ is odd if and only if n is a triangular number. He also established some relations between $b_4(n)$ and the number of partitions into parts not congruent to 2 modulo 4. Chen [4] proved that for $n,\alpha\geq 1$,

$$b_4\left(5^{2\alpha}n+\frac{r\cdot 5^{2\alpha-1}-1}{8}\right)\equiv 0 \pmod{4},$$

where $r\in\{13,21,29,37\}$. Xia [13] proved that for $n,\alpha\geq 1$,

$$b_4\left(3^{4\alpha}n+\frac{j\cdot 3^{4\alpha-1}-1}{8}\right)\equiv 0 \pmod{8},$$

where $j\in\{11,19\}$. In [2], Ballantine and Merca proved that for $n\geq 0$,

$$b_4(25n+8)\equiv b_4(25n+13)\equiv b_4(25n+18)\equiv b_4(25n+23)\equiv 0 \pmod{16}.$$

Very recently, Ballantine and Merca [3] proved infinite families of congruences modulo 3 for $b_6(n)$. More precisely, they proved the following theorem.

THEOREM 1 [3]. *Let α be a nonnegative integer and let p_i $(1\leq i\leq \alpha+1)$ be primes. If $p_{\alpha+1}\equiv 3 \pmod{4}$ and $j\not\equiv 0 \pmod{p_{\alpha+1}}$, then for all $n\geq 0$,*

$$b_6\left(p_1^2\cdots p_{\alpha+1}^2 n+\frac{p_1^2\cdots p_\alpha^2 p_{\alpha+1}(24j+5p_{\alpha+1})-5}{24}\right)\equiv 0 \pmod{3}.$$

Motivated by Ballantine and Merca's works on congruences of $b_4(n)$ and $b_6(n)$, we investigate congruences modulo 8 for $b_4(n)$ and congruences modulo 3 for $b_6(n)$ in this paper.

The fist goal of this paper is to present a characterization of congruences modulo 8 for $b_4(n)$. To state the main results on congruences modulo 8 for $b_4(n)$, define

$$\mu_1(n):=\begin{cases}1, & \text{if } n=k(k-1)/2 \text{ for some positive integer } k,\\ 0, & \text{otherwise,}\end{cases} \tag{2}$$

$$V_{1,2}(n):=\sum_{\substack{m,k\geq 1,\\ 2m^2+k(k-1)/2=n}}(-1)^m, \tag{3}$$

$$V_{1,4}(n):=\sum_{\substack{m,k\geq 1,\\ 4m^2+k(k-1)/2=n}}1. \tag{4}$$

The main results on congruences modulo 8 for $b_4(n)$ can be stated as follows.

THEOREM 2. *For $n\geq 1$,*

$$b_4(n)\equiv \mu_1(n)-2V_{1,2}(n)+4V_{1,4}(n) \pmod{8}. \tag{5}$$

For example, setting $n=200$ in (5), we deduce that $\mu_1(200)=0$, $V_{1,2}=1$, $V_{1,4}=1$ and

$$b_4(200)\equiv 0-2\times 1+4\times 1\equiv 2 \pmod{8}.$$

In fact, $b_4(200)=122\,730\,022\,082$.

Based on Theorem 2, we obtain the following corollary.

COROLLARY 1. *Let p be a prime with $p \equiv 7 \pmod 8$. If n, α are nonnegative integers with $p \nmid n$, then*

$$b_4\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod 8. \tag{6}$$

The second goal of this paper is to establish infinite families of congruences modulo 3 for $b_6(n)$ involving other choices of primes.

THEOREM 3. *Let p be a prime with $p \equiv 1 \pmod{24}$. If $b_6(5(p-1)/24) \equiv 0 \pmod 3$, then for $n, \alpha \geq 0$ with $p \nmid (24n+5)$, then*

$$b_6\left(p^{2\alpha+1}n + \frac{5(p^{2\alpha+1}-1)}{24}\right) \equiv 0 \pmod 3. \tag{7}$$

If $b_6(5(p-1)/24) \not\equiv 0 \pmod 3$, then for $n, \alpha \geq 0$ with $p \nmid (24n+5)$, then

$$b_6\left(p^{3\alpha+2}n + \frac{5(p^{3\alpha+2}-1)}{24}\right) \equiv 0 \pmod 3. \tag{8}$$

For example, setting $p = 73$ in Theorem 3 and using the fact that $b_6(15) = 143$, we deduce that for $\alpha \geq 0$,

$$b_6\left(73^{3k+2}n + \frac{5(73^{3k+2}-1)}{24}\right) \equiv 0 \pmod 3,$$

where $73 \nmid (24n+5)$.

2. PROOFS OF THEOREM 2 AND COROLLARY 1

It is easy to check that

$$\begin{aligned}\sum_{m,n=1}^{\infty}(-1)^{m+n}q^{m^2+n^2} &= \sum_{\substack{m,n=1,\\ m>n}}^{\infty}(-1)^{m+n}q^{m^2+n^2} + \sum_{\substack{m,n=1,\\ n>m}}^{\infty}(-1)^{m+n}q^{m^2+n^2} + \sum_{n=1}^{\infty}q^{2n^2}\\ &= 2\sum_{\substack{m,n=1,\\ m>n}}^{\infty}(-1)^{m+n}q^{m^2+n^2} + \sum_{n=1}^{\infty}q^{2n^2}.\end{aligned} \tag{9}$$

To prove the main results of this paper, we require the following two identities due to Gauss:

$$1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} \tag{10}$$

and

$$\sum_{k=1}^{\infty} q^{k(k-1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{11}$$

In light of (1), (10) and (11),

$$\sum_{n=0}^{\infty} b_4(n)q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2}\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}$$

$$
\begin{aligned}
&=\frac{1}{1+2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}}\sum_{k=1}^{\infty}q^{k(k-1)/2}\\
&=\left(1+\sum_{j=1}^{\infty}(-2)^j\left(\sum_{t=1}^{\infty}(-1)^t q^{2t^2}\right)^j\right)\sum_{k=1}^{\infty}q^{k(k-1)/2}\\
&\equiv\left(1-2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}+4\sum_{m,n=1}^{\infty}(-1)^{m+n}q^{2m^2+2n^2}\right)\sum_{k=1}^{\infty}q^{k(k-1)/2}\\
&\equiv\left(1-2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}+4\sum_{n=1}^{\infty}q^{4n^2}\right)\sum_{k=1}^{\infty}q^{k(k-1)/2}\pmod 8\qquad\text{(by (9))}\\
&=\sum_{n=0}^{\infty}(\mu_1(n)-2V_{1,2}(n)+4V_{1,4}(n))q^n, \qquad (12)
\end{aligned}
$$

which yields (5) after comparing the coefficients of q^n on both sides of (12). The proof of Theorem 2 is complete.

Now, we turn to prove Corollary 1.

It follows from (2) that if $p\nmid n$, then

$$\mu_1\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{8}\right)=0. \qquad (13)$$

In addition, from (3) and (4), we can rewrite $V_{1,2}(n)$ and $V_{1,4}(n)$ as

$$V_{1,2}(n)=\sum_{\substack{m,k\geq 1,\\ (4m)^2+(2k-1)^2=8n+1}}(-1)^m, \qquad (14)$$

$$V_{1,4}(n)=\sum_{\substack{m,k\geq 1,\\ 2(4m)^2+(2k-1)^2=8n+1}}1. \qquad (15)$$

From (14), we know that if $8n+1$ is not of the form x^2+y^2, then $V_{1,2}(n)=0$. Note that if N is of the form x^2+y^2, then $\nu_p(N)$ is even since p is a prime with $p\equiv 7 \pmod 8$ and $\left(\frac{-1}{p}\right)=-1$. Here $\nu_p(N)$ denotes the highest power of p dividing N and $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. It is easy to check that if $p\nmid n$, then

$$\nu_p\left(8\left(p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{8}\right)+1\right)=\nu_p(8p^{2\alpha+1}n+p^{2\alpha+2})=2\alpha+1$$

is odd. Therefore, $8\left(p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{8}\right)+1$ is not of the form x^2+y^2 and

$$V_{1,2}\left(p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{8}\right)=0. \qquad (16)$$

It follows from (15) that if $8n+1$ is not of the form x^2+2y^2, then $V_{1,4}(n)=0$. The facts that $\nu_p\left(8\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{8}\right)+1\right)$ is odd and $\left(\frac{-2}{p}\right)=-1$ imply that $8\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{8}\right)+1$ is not of the form x^2+2y^2 and

$$V_{1,4}\left(p^{2\alpha-1}n+\frac{p^{2\alpha}-1}{8}\right)=0. \qquad (17)$$

Congruence (6) follows from (5), (13), (16) and (17). This completes the proof of Corollary 1. □

3. PROOF OF THEOREM 3

Define

$$\sum_{n=0}^{\infty} a(n)q^n := \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty}. \tag{18}$$

Newman [12] proved that if p is a prime with $p \equiv 1 \pmod{24}$, then

$$a\left(pn + \frac{5(p-1)}{24}\right) = a(5(p-1)/24)a(n) - a\left(\frac{n - \frac{5(p-1)}{24}}{p}\right). \tag{19}$$

If $3|a(5(p-1)/24)$, then

$$a\left(pn + \frac{5(p-1)}{24}\right) \equiv -a\left(\frac{n - \frac{5(p-1)}{24}}{p}\right) \pmod{3}. \tag{20}$$

If $p \nmid (24n+5)$, then $\frac{n - \frac{5(p-1)}{24}}{p}$ is not an integer and

$$a\left(\frac{n - \frac{5(p-1)}{24}}{p}\right) = 0. \tag{21}$$

It follows from (20) and (21) that if $3|a(5(p-1)/24)$ and $p \nmid (24n+5)$, then

$$a\left(pn + \frac{5(p-1)}{24}\right) \equiv 0 \pmod{3}. \tag{22}$$

Replacing n by $pn + \frac{5(p-1)}{24}$ in (20) yields

$$a\left(p^2 n + \frac{5(p^2-1)}{24}\right) \equiv -a(n) \pmod{3}. \tag{23}$$

By (23) and mathematical induction, we deduce that for $n, \alpha \geq 0$,

$$a\left(p^{2\alpha} n + \frac{5(p^{2\alpha}-1)}{24}\right) \equiv (-1)^\alpha a(n) \pmod{3}. \tag{24}$$

Replacing n by $pn + \frac{5(p-1)}{24}$ in (24) and utilizing (22), we find that if $3|a(5(p-1)/24)$ and $p \nmid (24n+5)$, then for $n, \alpha \geq 0$,

$$a\left(p^{2\alpha+1} n + \frac{5(p^{2\alpha+1}-1)}{24}\right) \equiv 0 \pmod{3}. \tag{25}$$

It follows from (19) that if $a(5(p-1)/24) \equiv 1 \pmod{3}$, then

$$a\left(pn + \frac{5(p-1)}{24}\right) \equiv a(n) - a\left(\frac{n - \frac{5(p-1)}{24}}{p}\right) \pmod{3}. \tag{26}$$

Replacing n by $pn + \frac{5(p-1)}{24}$ in (26) yields

$$a\left(p^2 n + \frac{5(p^2-1)}{24}\right) \equiv a\left(pn + \frac{5(p-1)}{24}\right) - a(n) \pmod{3},$$

from which with (26), we arrive at

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv -a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right) \pmod 3. \tag{27}$$

By (27), we see that if $a(5(p-1)/24)\equiv 1 \pmod 3$ and $p\nmid(24n+5)$, then

$$a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right)=0$$

and

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv 0 \pmod 3. \tag{28}$$

Replacing n by $pn+\frac{5(p-1)}{24}$ in (27) yields

$$a\left(p^3n+\frac{5(p^3-1)}{24}\right)\equiv -a(n) \pmod 3. \tag{29}$$

By (29) and mathematical induction, we deduce that for $n,\alpha\geq 0$,

$$a\left(p^{3\alpha}n+\frac{5(p^{3\alpha}-1)}{24}\right)\equiv (-1)^{\alpha}a(n) \pmod 3. \tag{30}$$

Replacing n by $p^2n+\frac{5(p^2-1)}{24}$ in (30) and using (28), we see that if $a(5(p-1)/24)\equiv 1 \pmod 3$, then for $n,\alpha\geq 0$ with $p\nmid(24n+5)$,

$$a\left(p^{3\alpha+2}n+\frac{5(p^{3\alpha+2}-1)}{24}\right)\equiv 0 \pmod 3. \tag{31}$$

Identity (19) implies that if $a(5(p-1)/24)\equiv 2 \pmod 3$, then

$$a\left(pn+\frac{5(p-1)}{24}\right)\equiv 2a(n)-a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right) \pmod 3. \tag{32}$$

Replacing n by $pn+\frac{5(p-1)}{24}$ in (32) yields

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv 2a\left(pn+\frac{5(p-1)}{24}\right)-a(n) \pmod 3. \tag{33}$$

Substituting (32) into (33) yields

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right) \pmod 3, \tag{34}$$

which implies that if $a(5(p-1)/24)\equiv 2 \pmod 3$ and $p\nmid(24n+5)$, then

$$a\left(\frac{n-\frac{5(p-1)}{24}}{p}\right)=0$$

and

$$a\left(p^2n+\frac{5(p^2-1)}{24}\right)\equiv 0 \pmod 3. \tag{35}$$

If we replace n by $pn+\frac{5(p-1)}{24}$ in , we arrive at

$$a\left(p^3n+\frac{5(p^3-1)}{24}\right)\equiv a(n) \pmod 3. \tag{36}$$

By (36) and mathematical induction, we deduce that for $n,\alpha\geq 0$,

$$a\left(p^{3\alpha}n+\frac{5(p^{3\alpha}-1)}{24}\right)\equiv a(n) \pmod 3. \tag{37}$$

Replacing n by $p^2n+\frac{5(p^2-1)}{24}$ in (37) and using (35), we see that if $a(5(p-1)/24)\equiv 2 \pmod 3$, then for $n,\alpha\geq 0$ with $p\nmid(24n+5)$,

$$a\left(p^{3\alpha+2}n+\frac{5(p^{3\alpha+2}-1)}{24}\right)\equiv 0 \pmod 3. \tag{38}$$

Setting $t=6$ in (1), we get

$$\sum_{n=0}^{\infty}b_6(n)q^n=\frac{(q^6;q^6)_\infty}{(q;q)_\infty}. \tag{39}$$

By (39) and the fact that

$$(1-q^{6n})\equiv(1-q^{2n})^3 \pmod 3,$$

we arrive at

$$\sum_{n=0}^{\infty}b_6(n)q^n\equiv\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty} \pmod 3. \tag{40}$$

Combining (18) and (40) yields

$$b_6(n)\equiv a(n) \pmod 3. \tag{41}$$

Theorem 3 follows from (25), (31), (38) and (41). This completes the proof. □

4. CONCLUDING REMARKS

As seen in Introduction, congruence properties for t-regular partition functions have received a lot of attention in recent years. In this study, we give a characterization of congruences modulo 8 for $b_4(n)$ and prove infinite families of congruences modulo 3 for $b_6(n)$. A natural question is to extend the congruences in this paper to modulo 9, 32, 64, etc. However, it will likely require a different approach since the methods used in this paper run into serious limitations beyond the modulus of 9.

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REFERENCES

[1] Andrews GE, Hirschhorn MD, Sellers JA. Arithmetic properties of partitions with even parts distinct. Ramanujan J. 2010; 23: 169–181.

[2] Ballantine C, Merca M. 4-regular partitions and the pod function. Quaest. Math. 2023; 46: 2027–2051.

[3] Ballantine C, Merca M. 6-regular partitions: new combinatorial properties, congruences, and linear inequalities. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 2024; 159: 117.

[4] Chen SC. On the number of partitions with distinct even parts. Discrete Math. 2011; 311: 940–943.

[5] Cui SP, Gu NSS. Arithmetic properties of l-regular partitions. Adv. Appl. Math. 2013; 51: 507–523.

[6] Cui SP, Gu NSS. Congruences for 9-regular partitions modulo 3. Ramanujan J. 2015; 38: 503–512.

[7] Keith WJ. Congruences for 9-regular partitions modulo 3. Ramanujan J. 2014; 35: 157–164.

[8] Keith WJ, Zanello F. Parity of the coefficients of certain eta-quotients. J. Number Theory. 2022; 235: 275–304.

[9] Lin BLS, Wang AYZ. Generalization of Keith's conjecture on 9-regular partitions and 3-cores. Bull. Austral. Math. Soc. 2014; 90: 204–212.

[10] Merca M. New relations for the number of partitions with distinct even parts. J. Number Theory. 2017; 176: 1–12.

[11] Merca M. Ramanujan-type congruences modulo 4 for partitions into distinct parts. Analele Stiintifice ale Universitatii Ovidius Constanta-Seria Matematica. 2022; 30: 185–199.

[12] Newman M. Modular forms whose coefficients possess multiplicative properties. Ann. Math. 1959 70: 478–489.

[13] Xia EXW. New infinite families of congruences modulo 8 for partitions with even parts distinct. Electron. J. Combin. 2014; 21: #P4.8.

[14] Xia EXW. Congruences for some l-regular partitions modulo l. J. Number Theory. 2015; 152: 105–117.

[15] Yao OXM. New congruences modulo powers of 2 and 3 for 9-regular partitions. J. Number Theory. 2014; 142: 89–101.

[16] Yao OXM. New parity results for 3-regular partitions, Quaest. Math. 2023; 46: 465–471.

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