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ANALYTICAL SOLUTIONS TO A MODEL OF INVISCID LIQUID-GAS TWO-PHASE FLOW WITH CORIOLIS FORCE AND FREE BOUNDARY

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Abstract. In this paper, we consider the free boundary problem of a model of inviscid liquid-gas two-phase flow with Coriolis force. In the three-dimensional cylindrical symmetry case, we construct two classes of global analytical solutions with the free boundary pulsating between two positive constants, and prove that the solutions are periodic when the adiabatic exponent $\gamma = 2$. From the analytical solutions constructed in this paper, we find that the Coriolis force can prevent the free boundary from spreading out infinitely.

***Keywords**:* two-phase flow, free boundary, analytical solutions, cylindrical symmetry, Coriolis force.

1. INTRODUCTION AND MAIN RESULTS

In [6], Huang, Wang and Yuan considered the following model of inviscid liquid-gas two-phase flow:

$$\begin{cases} m_t + \nabla\cdot(m\vec{u}_g) = 0, \\ n_t + \nabla\cdot(n\vec{u}_l) = 0, \\ (m\vec{u}_g + n\vec{u}_l)_t + \nabla\cdot(m\vec{u}_g\otimes\vec{u}_g + n\vec{u}_l\otimes\vec{u}_l) + \nabla p(m,n) = \vec{0}, \end{cases} \tag{1.1}$$

where $m = \alpha_g\rho_g$ and $n = \alpha_l\rho_l$ are the masses of gas and liquid, respectively; α_g and $\alpha_l \in [0,1]$ represent the gas and liquid volume fractions satisfying $\alpha_g + \alpha_l = 1$; ρ_g and ρ_l denote the densities of the gas and liquid, respectively; $\vec{u}_g$ and $\vec{u}_l$ represent the velocities of the gas and liquid, respectively; and $p(m,n)$ is the pressure term of two phases, which was taken as

$$p(m,n) = (\gamma-1)(m+n)^{\gamma} \tag{1.2}$$

in [6], where $\gamma > 1$ is the adiabatic index. For simplicity, the authors in [6] assumed that $\vec{u}_g = \vec{u}_l = \vec{u}$ and neglected the momentum of the gas phase in the mixture momentum equation, then, (1.1) was reduced to

$$\begin{cases} m_t + \nabla\cdot(m\vec{u}) = 0, \\ n_t + \nabla\cdot(n\vec{u}) = 0, \\ n[\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u}] + \nabla p(m,n) = \vec{0}. \end{cases} \tag{1.2}$$

The liquid-gas two-phase flows are very important in the industry applications because they can be used to model boilers, condensers, pipelines for oil and natural gas, etc. We may refer to [1,11] for more physical background of the liquid-gas two-phase flows. For the system (1.3), the nonlinear stability and existence of vortex sheet solutions were obtained in [6] and some instant results were given in [12]. Dong and Yuen [4] constructed some special self-similar solutions for the system (1.3) in some symmetric cases. Recently, Dong [3] investigated the free boundary value problem for the system (1.3) with radial symmetry, two classes of global analytical solutions were constructed by using a self-similar ansatz and the free boundary was shown to spread outward at least linearly in time, see [5] for the three-dimensional cylindrical symmetry case.

On one hand, from [3, 5] we can see that the free boundary for (1.3) expands out infinitely as time grows up, on the other hand, it was shown that the Coriolis force can suppress the blowup of smooth solutions for the shallow water system, see [2, 9]. So a natural interesting problem arises: Can the Coriolis force prevent the free boundary from spreading out infinitely for the system (1.3) ? In this paper, we will investigate this problem by constructing some analytical solutions in the case of three-dimensional cylindrical symmetry. To this end, we add the Coriolis force to the third equation of (1.3), then we have

$$\begin{cases} m_t + \nabla \cdot (m\vec{u}) = 0, \\ n_t + \nabla \cdot (n\vec{u}) = 0, \\ n[\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u}] + \nabla p(m,n) + kn\vec{f} \times \vec{u} = \vec{0}, \end{cases} \tag{1.4}$$

where $k > 0$ is a constant and $\vec{f} = (0,0,1)$. As in [10], we can use the cylindrical symmetric transformation

$$\vec{u} = \left(u\frac{x_1}{r} - v\frac{x_2}{r}, \quad u\frac{x_2}{r} + v\frac{x_1}{r}, \quad w \right), \quad r = \sqrt{x_1^2 + x_2^2}\,, \quad u = u(r,t)\,, \quad v = v(r,t)\,, \quad w = w(r,t) \tag{1.5}$$

to reformulate the system (1.4) as

$$\begin{cases} m_t + (mu)_r + \dfrac{mu}{r} = 0, \\ n_t + (nu)_r + \dfrac{nu}{r} = 0, \\ n\left(u_t + uu_r - \dfrac{v^2}{r} - kv \right) + [\,p(m,n)]_r = 0, \\ v_t + uv_t + \dfrac{uv}{r} + ku = 0, \\ w_t + uw_r = 0, \end{cases} \tag{1.6}$$

where the scalar functions u, v and w represent the radial component, the angular component, and the axial component of the velocity $\vec{u}$. We consider the system (1.6) in the cylinder

$$\mathbf{C}(t) = \left\{ r \,|\, 0 \le r \le a(t) \right\} \subset \mathbf{R}^3$$

and supplement (1.6) with the following initial and boundary data:

$$(m,n,u,v,w)|_{t=0} = (m_0(r), n_0(r), u_0(r), v_0(r), w_0(r))\,, \quad 0 \le r \le a_0\,, \tag{1.7}$$

$$m(a(t),t) = n(a(t),t) = 0\,, \tag{1.8}$$

where $a(t)$ is the free boundary separating the fluid from vacuum, $a_0 > 0$ is the initial location of the free boundary. The free boundary problem (1.6)–(1.8) describes the dynamic evolution of a rotational two-phase flow in a cylinder surrounded by vacuum.

Our main results are stated as follows.

THEOREM 1. *For the problem (1.6)–(1.8), there exists a class of global analytical solutions*

$$m(r,t) = \frac{K\left[\dfrac{\lambda}{2\gamma(1+K)^\gamma} \left(1 - \dfrac{r^2}{a(t)^2} \right) \right]^{\frac{1}{\gamma-1}}}{a(t)^2}, \tag{1.9}$$

$$n(r,t) = \frac{\left[\dfrac{\lambda}{2\gamma(1+K)^\gamma} \left(1 - \dfrac{r^2}{a(t)^2} \right) \right]^{\frac{1}{\gamma-1}}}{a(t)^2}, \tag{1.10}$$

$$u(r,t)=\frac{a'(t)}{a(t)}r\,,\quad v(r,t)=\left[\frac{a_0{}^2(b_0+\frac{k}{2})}{a(t)^2}-\frac{k}{2}\right]r\,,\quad w(r,t)=\frac{a_0c_0}{a(t)}r\,, \tag{1.11}$$

where K , λ are two arbitrary positive constants, $b_0\neq 0$ and c_0 satisfy $v_0(r)=b_0r$ and $w_0(r)=c_0r$, respectively, and $a(t)$ satisfies the following ordinary differential equation:

$$a''(t)=\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2}{a(t)^3}-\frac{k^2}{4}a(t)+\frac{\lambda}{a(t)^{2\gamma-1}}\,,\quad a(0)=a_0\,,\quad a'(0)=a_1 \tag{1.12}$$

with a_0 and a_1 being the initial location and slope of the free boundary. Moreover, there exist two positive constants M_1, M_2 such that

$$M_1\le a(t)\le M_2 \tag{1.13}$$

THEOREM 2. *For the problem (1.6)–(1.8), there exists a class of global analytical solutions*

$$m(r,t)=\frac{\left[\frac{\lambda(\beta-1)}{2\beta(\gamma-1)}\left(1-\frac{r^2}{a(t)^2}\right)\right]^{\frac{\beta}{\gamma(\beta-1)}}-\left[\frac{\lambda(\beta-1)}{2\beta(\gamma-1)}\left(1-\frac{r^2}{a(t)^2}\right)\right]^{\frac{1}{\beta-1}}}{a(t)^2}\,, \tag{1.14}$$

$$n(r,t)=\frac{\left[\frac{\lambda(\beta-1)}{2\beta(\gamma-1)}\left(1-\frac{r^2}{a(t)^2}\right)\right]^{\frac{1}{\beta-1}}}{a(t)^2}\,, \tag{1.15}$$

$$u(r,t)=\frac{a'(t)}{a(t)}r\,,\quad v(r,t)=\left[\frac{a_0{}^2\left(b_0+\frac{k}{2}\right)}{a(t)^2}-\frac{k}{2}\right]r\,,\quad w(r,t)=\frac{a_0c_0}{a(t)}r\,, \tag{1.16}$$

where $\lambda>0$ and $\beta>1$ are two arbitrary constants, $b_0\neq 0$ and c_0 satisfy $v_0(r)=b_0r$ and $w_0(r)=c_0r$, respectively, and $a(t)$ satisfies (1.12) and (1.13).

THEOREM 3. *When $\gamma=2$, if $b_0\le -k$ or $b_0>0$, then the solutions constructed in Theorems 1 and 2 are non-trivially periodic; if $-k<b_0<0$, then the solutions constructed in Theorems 1 and 2 are non-trivially periodic except for the case with $a_0=\sqrt[4]{\frac{-\lambda}{b_0^2+b_0k}}$ and $a_1=0$, if $a_0=\sqrt[4]{\frac{-\lambda}{b_0^2+b_0k}}$ and $a_1=0$, then the solutions constructed in Theorems 1 and 2 are stabl*e.

Remark 1. Our results indicates that the Coriolis force can prevent the free boundary from spreading out infinitely.

Remark 2. System (1.4) models the large-scale two-phase flow motions in a thin layer of fluids under the influence of the Coriolis rotational force. In recent years, Coriolis flowmeters were used to measure two-phase flows, see [13] for instance. Consequently, studying the solutions to two-phase models with Coriolis force is very important. Unfortunately, there are few results in this regard. To our knowledge, the asymptotic limits of dissipative turbulent solutions to a compressible two-fluid model with Coriolis force were investigated in [7], where the Coriolis force was taken as $k(m+n)\vec{f}\times\vec{u}$. But in this paper, we omit the term $km\vec{f}\times\vec{u}$ because the liquid phase is much heavier than the gas phase.

2. PROOF OF THE RESULTS

Proof of Theorem 1. By Lemma 3 of [14], we know that the equations $(1.6)_1$ and $(1.6)_2$ have solutions with the form

$$m=\frac{f_1(s)}{a(t)^2}, \quad n=\frac{f_2(s)}{a(t)^2}, \quad u=\frac{a'(t)}{a(t)}r, \tag{2.1}$$

where $s=\frac{r}{a(t)}$, $f_1, f_2 \geq 0 \in C^1$ and $0<a(t)\in C^1$. Let

$$v(r,t)=b(t)r, \tag{2.2}$$

where $b(t)\in C^1$ will be determined later. We substitute $u=\frac{a'(t)}{a(t)}r$ and $v=b(t)r$ into $(1.6)_4$ to obtain

$$b'(t)+[2b(t)+k]\frac{a'(t)}{a(t)}=0, \tag{2.3}$$

which can be solved out as

$$b(t)=\frac{a_0{}^2\left(b_0+\frac{k}{2}\right)}{a(t)^2}-\frac{k}{2}, \tag{2.4}$$

where $b_0\neq 0$ satisfies $v_0(r)=b_0 r$, so we obtain $v(r,t)=\left[\frac{a_0{}^2\left(b_0+\frac{k}{2}\right)}{a(t)^2}-\frac{k}{2}\right]r$. Similarly, if we let $w=c(t)r$ and substitute it and $u=\frac{a'(t)}{a(t)}r$ into $(1.6)_5$, we can obtain $w(r,t)=\frac{a_0 c_0}{a(t)}r$.

We plug (1.11) and (2.1) into $(1.5)_3$ and use (1.2) to have

$$\frac{f_2(s)}{a(t)^2}\left\{\frac{a''(t)}{a(t)}r-\left[\frac{a_0{}^2\left(b_0+\frac{k}{2}\right)}{a(t)^2}-\frac{k}{2}\right]r-k\left[\frac{a_0{}^2\left(b_0+\frac{k}{2}\right)}{a(t)^2}-\frac{k}{2}\right]r\right\}+\gamma(\gamma-1)\cdot\frac{[f_1(s)+f_2(s)]^{\gamma-1}}{a(t)^{2(\gamma-1)}}\cdot\frac{f_1'(s)+f_2'(s)}{a(t)^3}=0 \tag{2.5}$$

that is

$$sf_2(s)\left[a''(t)-\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2}{a(t)^3}+\frac{k^2}{4}a(t)\right]+(\gamma-1)\cdot\frac{\left\{[f_1(s)+f_2(s)]^{\gamma}\right\}'}{a(t)^{2\gamma-1}}=0. \tag{2.6}$$

Let $f_1(s)=Kf_2(s)=Kf(s)$ and $a(t)$ satisfy (1.12), then by (2.6), we know that

$$(\gamma-1)(1+K)^{\gamma}[f(s)^{\gamma}]'+\lambda f(s)s=0, \tag{2.7}$$

where K, λ are two arbitrary positive constants. In view of (1.8), one has $f(1)=0$, which together with (2.7) implies that

$$f(s)=\left[\frac{\lambda\left(1-s^2\right)}{2\gamma(1+K)^\gamma}\right]^{\frac{1}{\gamma-1}}. \tag{2.8}$$

So we obtain the solutions (1.9)–(1.11). By using the fixed-point theorem, we can prove that the ordinary differential equation (1.12) has a local-in-time C^2 solution, here we omit the details.

We multiply (1.12) by $a'(t)$ and integrate it on $[0,t]$ to have

$$\frac{1}{2}a'(t)^2+\frac{a_0^{\,4}\left(b_0+\frac{k}{2}\right)^2}{2a(t)^2}+\frac{k^2}{8}a(t)^2+\frac{\lambda}{(2\gamma-2)a(t)^{2\gamma-2}}=M_0, \tag{2.9}$$

where

$$M_0=\frac{1}{2}a_1^{\,2}+\frac{a_0^{\,2}(b_0+\frac{k}{2})^2}{2}+\frac{k^2}{8}a_0^{\,2}+\frac{\lambda}{(2\gamma-2)a_0^{\,2\gamma-2}}. \tag{2.10}$$

With the aid of (2.9), we obtain

$$\max\left\{\frac{a_0^{\,2}\left|b_0+\frac{k}{2}\right|}{\sqrt{2M_0}},\left[\frac{\lambda}{(2\gamma-2)G_0}\right]^{\frac{1}{2\gamma-2}}\right\}\le a(t)\le\frac{2\sqrt{2M_0}}{k}, \tag{2.11}$$

so (1.13) holds. By using the standard continuity argument, we know that the analytical solutions (1.9)–(1.11) exist globally-in-time.

Proof of Theorem 2. In the proof of Theorem 1, if we take

$$[f_1(s)+f_2(s)]^\gamma=f_2(s)^\beta=g(s)^\beta, \tag{2.12}$$

where $\beta>1$ is an arbitrary constant, we will obtain the solutions (1.14)–(1.16).

Proof of Theorem 3. We only need to prove the periodic property of $a(t)$. When $\gamma=2$, (1.12) becomes

$$a''(t)-\frac{a_0^{\,4}\left(b_0+\frac{k}{2}\right)^2+\lambda}{a(t)^3}+\frac{k^2}{4}a(t)=0,\quad a(0)=a_0,\quad a'(0)=a_1. \tag{2.13}$$

We multiply (2.13) by $a'(t)$ and integrate it on $[0,t]$ to have

$$\frac{1}{2}a'(t)^2+\frac{a_0^{\,4}\left(b_0+\frac{k}{2}\right)^2+\lambda}{2a(t)^2}+\frac{k^2}{8}a(t)^2=\theta, \tag{2.14}$$

where

$$\theta=\frac{1}{2}a_1^{\,2}+\frac{a_0^{\,4}\left(b_0+\frac{k}{2}\right)^2+\lambda}{2a_0^{\,2}}+\frac{k^2}{8}a_0^{\,2}>0. \tag{2.15}$$

We define the kinetic energy and the potential energy as

$$F_{kin}:=\frac{1}{2}a'(t)^2 \tag{2.16}$$

and

$$F_{pot} := \frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{2a(t)^2}+\frac{k^2}{8}a(t)^2\,, \tag{2.17}$$

respectively. Then, by (2.14), we know that the total energy is conserved, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}(F_{kin}+F_{pot})=0\,. \tag{2.18}$$

After direct calculations, we find that the potential F_{pot} has only one global minimum at

$$a_{\min}=\sqrt{\frac{2\sqrt{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}}{k}}\,, \tag{2.19}$$

for $a(t)\in(0,+\infty)$. Using the classical energy method for conservative systems (in section 4.3 of [8]), the solution to (2.13) has a closed trajectory. In view of (2.14), we can calculate the time for travelling the closed orbit:

$$T=\int_0^t \mathrm{d}t=\int_0^t \frac{\mathrm{d}t}{\mathrm{d}a(t)}\mathrm{d}a(t)=\int_{\underline{a}}^{\bar{a}}\frac{\mathrm{d}a(t)}{a'(t)}=\int_{\underline{a}}^{\bar{a}}\frac{\mathrm{d}a(t)}{\sqrt{2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{a(t)^2}+\frac{k^2}{4}a(t)^2\right]}} \tag{2.20}$$

where $\underline{a}=\inf_{t\geq 0}\{a(t)\}$ and $\bar{a}=\sup_{t\geq 0}\{a(t)\}$.

Let

$$G(t):=2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{a(t)^2}+\frac{k^2}{4}a(t)^2\right], \tag{2.21}$$

$$G_0:=2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{(\underline{a}+\varepsilon)^2}+\frac{k^2}{4}(\underline{a}+\varepsilon)^2\right]>0\,, \tag{2.22}$$

$$G_1:=2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{(\bar{a}-\varepsilon)^2}+\frac{k^2}{4}(\bar{a}-\varepsilon)^2\right]>0\,, \tag{2.23}$$

where $\varepsilon>0$ is small enough. Except for the case where $-k<b_0<0$, $a_0=\sqrt[4]{\frac{-\lambda}{b_0{}^2+b_0k}}$ and $a_1=0$, equation (2.20) becomes

$$T=\int_{\underline{a}}^{\underline{a}+\varepsilon}\frac{\mathrm{d}a(t)}{\sqrt{2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{a(t)^2}+\frac{k^2}{4}a(t)^2\right]}}+\int_{\underline{a}+\varepsilon}^{\bar{a}-\varepsilon}\frac{\mathrm{d}a(t)}{\sqrt{2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{a(t)^2}+\frac{k^2}{4}a(t)^2\right]}}$$

$$+\int_{\bar{a}-\varepsilon}^{\bar{a}}\frac{\mathrm{d}a(t)}{\sqrt{2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{a(t)^2}+\frac{k^2}{4}a(t)^2\right]}}$$

$$\leq\sup_{\underline{a}\leq a(t)\leq\underline{a}+\varepsilon}\left|\frac{1}{\frac{2a_0{}^4\left(b_0+\frac{k}{2}\right)^2+2\lambda}{a(t)^3}-\frac{k^2}{2}a(t)}\right|\int_0^{G_0}\frac{\mathrm{d}G(t)}{\sqrt{G(t)}}+\int_{\underline{a}+\varepsilon}^{\bar{a}-\varepsilon}\frac{\mathrm{d}a(t)}{\sqrt{2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{a(t)^2}+\frac{k^2}{4}a(t)^2\right]}}$$

$$+\sup_{\bar{a}-\varepsilon\leq a(t)\leq\bar{a}}\left|\frac{1}{\frac{2a_0{}^4\left(b_0+\frac{k}{2}\right)^2+2\lambda}{a(t)^3}-\frac{k^2}{2}a(t)}\right|\int_0^{G_1}\frac{\mathrm{d}G(t)}{\sqrt{G(t)}}$$

$$=2\sqrt{G_0}\sup_{\underline{a}\leq a(t)\leq\underline{a}+\varepsilon}\left|\frac{1}{\frac{2a_0{}^4\left(b_0+\frac{k}{2}\right)^2+2\lambda}{a(t)^3}-\frac{k^2}{2}a(t)}\right|+\int_{\underline{a}+\varepsilon}^{\bar{a}-\varepsilon}\frac{\mathrm{d}a(t)}{\sqrt{2\theta-\left[\frac{a_0{}^4\left(b_0+\frac{k}{2}\right)^2+\lambda}{a(t)^2}+\frac{k^2}{4}a(t)^2\right]}}$$

$$+2\sqrt{G_1}\sup_{\bar{a}-\varepsilon\leq a(t)\leq\bar{a}}\left|\frac{1}{\frac{2a_0{}^4\left(b_0+\frac{k}{2}\right)^2+2\lambda}{a(t)^3}-\frac{k^2}{2}a(t)}\right|$$

$$<+\infty. \tag{2.24}$$

Obviously, if $a_0 = \sqrt[4]{\frac{-\lambda}{b_0^2 + b_0 k}}$ and $a_1 = 0$, then the solution to (2.13) is stable. The proof of Theorem 3 is finished.

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