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TWO NEW q- CONGRUENCES FROM GASPER'S KARLSSON-MINTON TYPE SUMMATION

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Abstract. We give two new q-congruences by using the method of "creative microscoping" and Gasper's Karlsson-Minton type summation. In particular, we present a q-analogue of a congruence of Barman and Saikia.

***Keywords**:* supercongruence, p-adic Gamma function, cyclotomic polynomials, creative microscoping.
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1. INTRODUCTION

Rodriguez-Villegas [12] studied hypergeometric families of Calabi–Yau manifolds, and found a number of possible supercongruences. For instance, he observed that, for any prime $p > 2$,

$$\sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^2}{k!^2} \equiv (-1)^{(p-1)/2} \pmod{p^2}, \tag{1}$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ $(n \geqslant 1)$ is the *rising factorial*. Mortenson [11] first confirmed the congruence (1). Later, the first author and Zeng [4] obtained a q-analogue of (1):

$$\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2}{(q^2;q^2)_k^2} q^{2k} \equiv (-1)^{(p-1)/2} q^{(p^2-1)/4} \pmod{[p]^2} \quad \text{for any odd prime } p.$$

Here and throughout the paper, $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ $(n \geqslant 1)$ is the q-*shifted factorial*, and $[n] = 1 + q + \cdots + q^{n-1}$ is the q-*integer*. For convenience, we will also adopt the condensed notation $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n$.

In 2020, Barman and Saikia [1] gave a generalization of (1) as follows: for $d \geqslant 1$ and any prime p satisfying $p \equiv 1 \pmod{d^2+d}$,

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv (-1)^{d+1} \Gamma_p(\tfrac{1}{d})^d \Gamma_p(\tfrac{d}{d+1})^{d+1} \pmod{p^2}, \tag{2}$$

where $\Gamma_p(x)$ denotes the p-adic Gamma function (see [8]).

Let $\Phi_n(q)$ be the n-th *cyclotomic polynomial* in q, which can be written as

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(k,n)=1}} (q - \zeta^k),$$

where ζ is an n-th primitive root of unity. The first aim of this note is to give the following q-analogue of Barman and Saikia's congruence (2).

THEOREM 1. *Let d and n be positive integers with $n\equiv 1 \pmod{d^2+d}$. Then, modulo $\Phi_n(q)^2$,*

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^d;q^{d^2+d})_k^{d+1} q^{(d^2+d)k}}{(q^{d+1};q^{d^2+d})_k^d (q^{d^2+d};q^{d^2+d})_k} \equiv \frac{(-1)^{(n-1)/(d+1)}(q^{d^2+d};q^{d^2+d})_{(n-1)/(d+1)} q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(q^{d+1};q^{d^2+d})^d_{(n-1)/(d^2+d)}}. \tag{3}$$

For n prime, letting $q\to 1$ in Theorem 1, we arrive at the following congruence: for $d\geqslant 1$ and any prime $p\equiv 1 \pmod{d^2+d}$,

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv \frac{(-1)^{(p-1)/(d+1)}(\frac{p-1}{d+1})!}{(\frac{1}{d})^d_{(p-1)/(d^2+d)}} \pmod{p^2}. \tag{4}$$

In view of properties of the p-adic Gamma function (see [10, Section 2]), it is not hard to show that

$$\frac{(-1)^{(p-1)/(d+1)}(\frac{p-1}{d+1})!}{(\frac{1}{d})^d_{(p-1)/(d^2+d)}} \equiv (-1)^{d+1}\Gamma_p(\tfrac{1}{d})^d\Gamma_p(\tfrac{d}{d+1})^{d+1} \pmod{p^2}. \tag{5}$$

Hence, the congruence (4) is equivalent to (2).

We shall also establish the following congruence similar to (2).

THEOREM 2. *Let $d\geqslant 1$ and let p be a prime with $p\equiv 1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(p-1)/(d+1)} \frac{k(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv \frac{(-1)^{d+2}}{2(d^2+d)}\Gamma_p(\tfrac{1}{d})^d\Gamma_p(\tfrac{d}{d+1})^{d+1} \pmod{p^2}. \tag{6}$$

Since $\Gamma_p(1)=-1$ and $\Gamma_p(\frac12)^2=(-1)^{(p+1)/2}$, for $d=1$, the congruence (6) reduces to

$$\sum_{k=0}^{p-1}\frac{k(\frac12)_k^2}{k!^2} \equiv \frac{(-1)^{(p+1)/2}}{4} \pmod{p^2},$$

of which a generalization modulo p^3 for $p>3$ has already been given by Sun [13, Theorem 1.2, (1.8) and (1.10)].

It is easy to see that, for any prime $p>2$,

$$\sum_{k=0}^{(p+1)/2}\frac{(-\frac12)_k^2}{k!^2} = \frac{2p+3}{4^{p+1}}\binom{p+1}{(p+1)/2}^2 \equiv 0 \pmod{p^2}. \tag{7}$$

The last aim of this note is to give the following generalization of (7).

THEOREM 3. *Let d and n be positive integers with $n\equiv 2d+1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(n+1)/(d+1)} \frac{(q^{-d};q^{d^2+d})_k^{d+1} q^{(d^2+d)k}}{(q^{d+1};q^{d^2+d})_k^d (q^{d^2+d};q^{d^2+d})_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{8}$$

In particular, letting n be a prime and taking $q\to 1$ in Theorem 3, we are led to the conclusion.

COROLLARY 1. *Let $d \geqslant 1$ and let p be a prime with $p \equiv 2d+1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(p+1)/(d+1)} \frac{(-\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv 0 \pmod{p^2}.$$

Like the proof of Theorem 2, we can also deduce the following congruence from Theorem 3.

COROLLARY 2. *Let $d \geqslant 1$ and let p be a prime with $p \equiv 2d+1 \pmod{d^2+d}$. Then*

$$\sum_{k=0}^{(p+1)/(d+1)} \frac{k(-\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!} \equiv 0 \pmod{p^2}.$$

2. PROOF OF THEOREM 1

We will make use of Gasper's Karlsson-Minton type summation (see [2, (1.9.9)]; and see [3, (5.13)] for a generalization): for all non-negative integers $n_1,\dots,n_m$,

$$\sum_{k=0}^{N} \frac{(q^{-N}, b_1q^{n_1},\dots,b_mq^{n_m};q)_k}{(q,b_1,\dots,b_m;q)_k} q^k = (-1)^N \frac{(q;q)_N b_1^{n_1}\cdots b_m^{n_m}}{(b_1;q)_{n_1}\cdots(b_m;q)_{n_m}} q^{\binom{n_1}{2}+\cdots+\binom{n_m}{2}}, \tag{9}$$

where $N = n_1+\cdots+n_m$. For some recent congruences and q-congruences related to (9), see [5,7,9].

We first build the following generalization of Theorem 1 with an extra parameter a by employing the "creative microscoping" method devised in [6].

THEOREM 4. *Let $d,n>1$ be integers with $n \equiv 1 \pmod{d^2+d}$. Let a be an indeterminate. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/(d+1)} \frac{(a^dq^d, a^{d-2}q^d,\dots,aq^d;q^{d^2+d})_k}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,a^2q^{d+1},q^{d+1};q^{d^2+d})_k} \\
&\quad\times \frac{(a^{-d}q^d,a^{2-d}q^d,\dots,a^{-1}q^d;q^{d^2+d})_k q^{(d^2+d)k}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\dots,a^{-2}q^{d+1};q^{d^2+d})_k (q^{d^2+d};q^{d^2+d})_k} \\
&\equiv \frac{(-1)^{(n-1)/(d+1)}(q^{d^2+d};q^{d^2+d})_{(n-1)/(d+1)}}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,a^2q^{d+1},q^{d+1};q^{d^2+d})_{(n-1)/(d^2+d)}} \\
&\quad\times \frac{q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\dots,a^{-2}q^{d+1};q^{d^2+d})_{(n-1)/(d^2+d)}}
\end{aligned} \tag{10}$$

if d is odd, and

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/(d+1)} \frac{(a^dq^d, a^{d-2}q^d,\dots,a^2q^d,q^d;q^{d^2+d})_k}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,aq^{d+1};q^{d^2+d})_k} \\
&\quad\times \frac{(a^{-d}q^d,a^{2-d}q^d,\dots,a^{-2}q^d;q^{d^2+d})q^{(d^2+d)k}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\dots,a^{-1}q^{d+1};q^{d^2+d})_k (q^{d^2+d};q^{d^2+d})_k} \\
&\equiv \frac{(-1)^{(n-1)/(d+1)}(q^{d^2+d};q^{d^2+d})_{(n-1)/(d+1)}}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,aq^{d+1};q^{d^2+d})_{(n-1)/(d^2+d)}} \\
&\quad\times \frac{q^{\frac{(n-1)(n+1+d-d^2)}{2(d+1)}}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\dots,a^{-1}q^{d+1};q^{d^2+d})_{(n-1)/(d^2+d)}}
\end{aligned} \tag{11}$$

if d is even.

Proof. It is obvious that $\gcd(d,n)=1$, and therefore none of the numbers $d,2d,\dots(n-1)d$ are divisible by n. This indicates that the denominators on the left-hand side of (10) do not have the factor $1-aq^n$ nor $1-a^{-1}q^n$. Thus, for $a=q^{-n}$ or $a=q^n$, the left-hand side of (10) may be written as

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^{-(n-1)d},q^{-(n-1)d+2n},\dots,q^{-n+d};q^{d^2+d})_k}{(q^{-(d-1)n+d+1},q^{-(d-3)n+d+1},\dots,q^{-2n+d+1},q^{d+1};q^{d^2+d})_k} \times \frac{(q^{(n+1)d},q^{(n+1)d-2n},\dots,q^{n+d};q^{d^2+d})_k q^{(d^2+d)k}}{(q^{(d-1)n+d+1},q^{(d-3)n+d+1},\dots,q^{2n+d+1};q^{d^2+d})_k(q^{d^2+d};q^{d^2+d})_k}. \tag{12}$$

Letting $q\mapsto q^{d^2+d}$, $N=(n-1)/(d+1)$, $m=d$, $b_j=q^{-(d-1)n+d+1+(2j-2)n}$ and $n_j=(n-1)/(d^2+d)$ $(1\leqslant j\leqslant d)$ in (9), we conclude that (12) is equal to

$$\frac{(-1)^{(n-1)/(d+1)}(q^{d^2+d};q^{d^2+d})_{(n-1)/(d+1)}}{(q^{-(d-1)n+d+1},q^{-(d-3)n+d+1},\dots,q^{-2n+d+1},q^{d+1};q^{d^2+d})_{(n-1)/(d^2+d)}} \times \frac{q^{(n-1)+d(d^2+d)\binom{(n-1)/(d^2+d)}{2}}}{(q^{(d-1)n+d+1},q^{(d-3)n+d+1},\dots,q^{2n+d+1};q^{d^2+d})_{(n-1)/(d^2+d)}},$$

which is just the $a=q^{-n}$ or $a=q^n$ case of the right-hand side (10). This proves the q-congruence (10).

Similarly, for $a=q^{-n}$ or $a=q^n$, the left-hand side of (11) may be expressed as

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^{-(n-1)d},q^{-(n-1)d+2n},\dots,q^{-2n+d},q^{d};q^{d^2+d})_k}{(q^{-(d-1)n+d+1},q^{-(d-3)n+d+1},\dots,q^{-n+d+1};q^{d^2+d})_k} \times \frac{(q^{(n+1)d},q^{(n+1)d-2n},\dots,q^{2n+d};q^{d^2+d})_k q^{(d^2+d)k}}{(q^{(d-1)n+d+1},q^{(d-3)n+d+1},\dots,q^{n+d+1};q^{d^2+d})_k(q^{d^2+d};q^{d^2+d})_k}. \tag{13}$$

Letting $q\mapsto q^{d^2+d}$, $N=(n-1)/(d+1)$, $m=d$, $b_j=q^{-(d-1)n+d+1+(2j-2)n}$ and $n_j=(n-1)/(d^2+d)$ $(1\leqslant j\leqslant d)$ in (9), we deduce that (13) is equal to the $a=q^{-n}$ or $a=q^n$ case of the right-hand side (11). This establishes (11). □

Proof of Theorem 1. Note that $\Phi_n(q)$ is a factor of $1-q^m$ if and only if m is divisible by n. Hence, when $a=1$ the denominators of (10) are all coprime with $\Phi_n(q)$. Meanwhile, when $a=1$ the polynomial $(1-aq^n)(a-q^n)=(1-q^n)^2$ incorporates the factor $\Phi_n(q)^2$. The proof of (3) then follows immediately from the $a=1$ case of (10) and (11). □

3. PROOF OF THEOREM 2

Let $n>1$ be an integer with $n\equiv 1 \pmod{d^2+d}$. Performing the substitution $q\mapsto q^{-1}$ in (3), we get its dual form: modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^d;q^{d^2+d})_k^{d+1}}{(q^{d+1};q^{d^2+d})_k^d(q^{d^2+d};q^{d^2+d})_k} \equiv \frac{(-1)^{(n-1)/(d+1)}(q^{d^2+d};q^{d^2+d})_{(n-1)/(d+1)}q^{\frac{(1-n)(nd+d^2)}{2(d+1)}}}{(q^{d+1};q^{d^2+d})^d_{(n-1)/(d^2+d)}}. \tag{14}$$

Subtracting (3) from (14) and dividing both sides by $1-q$, we are led to

$$\sum_{k=0}^{(n-1)/(d+1)} \frac{(q^d;q^{d^2+d})_k^{d+1}(1-q^{(d^2+d)k})}{(q^{d+1};q^{d^2+d})_k^d(q^{d^2+d};q^{d^2+d})_k(1-q)} \equiv \frac{(-1)^{(n-1)/(d+1)}(q^{d^2+d};q^{d^2+d})_{(n-1)/(d+1)}q^{\frac{(1-n)(nd+d^2)}{2(d+1)}}(1-q^{\frac{n^2-1}{2}})}{(q^{d+1};q^{d^2+d})^d_{(n-1)/(d^2+d)}(1-q)} \pmod{\Phi_n(q)^2}.$$

Letting $n=p$ be a prime and taking the limit as $q\to 1$ in the above q-supercongruence, we obtain the following result: for any positive integer d and prime $p\equiv 1 \pmod{d^2+d}$,

$$\sum_{k=0}^{(p-1)/(d+1)}\frac{k(\frac{1}{d+1})_k^{d+1}}{(\frac{1}{d})_k^d k!}\equiv\frac{(-1)^{(p-1)/(d+1)}(\frac{p-1}{d+1})!(p^2-1)}{2(\frac{1}{d})^d_{(p-1)/(d^2+d)}(d^2+d)}$$
$$\equiv\frac{(-1)^{(p+d)/(d+1)}(\frac{p-1}{d+1})!}{2(\frac{1}{d})^d_{(p-1)/(d^2+d)}(d^2+d)}\pmod{p^2}.$$

The proof then follows from the congruence (5).

4. PROOF OF THEOREM 3

We will utilize another Karlsson-Minton type summation due to Gasper (see [2, (1.9.11)]): for all non-negative integers $n_1,\dots,n_m$,

$$\sum_{k=0}^{N}\frac{(q^{-N},b_1q^{n_1},\dots,b_mq^{n_m};q)_k}{(q,b_1,\dots,b_m;q)_k}q^k=0, \tag{15}$$

where $N>n_1+\cdots+n_m$.

We first establish the following parametric generalization of Theorem 3.

THEOREM 5. *Let $d,n>1$ be integers with $n\equiv 2d+1 \pmod{d^2+d}$. Let a be an indeterminate. Then, modulo $(1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{(n+1)/(d+1)}\frac{(a^dq^{-d},a^{d-2}q^{-d},\dots,aq^{-d};q^{d^2+d})_k}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,a^2q^{d+1},q^{d+1};q^{d^2+d})_k}$$
$$\times\frac{(a^{-d}q^{-d},a^{2-d}q^{-d},\dots,a^{-1}q^{-d};q^{d^2+d})_kq^{(d^2+d)k}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\dots,a^{-2}q^{d+1};q^{d^2+d})_k(q^{d^2+d};q^{d^2+d})_k}\equiv 0 \tag{16}$$

if d is odd, and

$$\sum_{k=0}^{(n+1)/(d+1)}\frac{(a^dq^{-d},a^{d-2}q^{-d},\dots,a^2q^{-d},q^{-d};q^{d^2+d})_k}{(a^{d-1}q^{d+1},a^{d-3}q^{d+1},\dots,aq^{d+1};q^{d^2+d})_k}$$
$$\times\frac{(a^{-d}q^{-d},a^{2-d}q^{-d},\dots,a^{-2}q^{-d};q^{d^2+d})q^{(d^2+d)k}}{(a^{1-d}q^{d+1},a^{3-d}q^{d+1},\dots,a^{-1}q^{d+1};q^{d^2+d})_k(q^{d^2+d};q^{d^2+d})_k}\equiv 0 \tag{17}$$

if d is even.

Proof. It is easy to see that $\gcd(d,n)=1$ and so none of the numbers $d,2d,\dots(n-1)d$ are multiples of n. This implies that the denominators of the left-hand sides of (16) have no factors $1-aq^n$ and $1-a^{-1}q^n$. Therefore, for $a=q^{-n}$ or $a=q^n$, the left-hand side of (16) can be expressed as

$$\sum_{k=0}^{(n+1)/(d+1)}\frac{(q^{-(n+1)d},q^{-(n+1)d+2n},\dots,q^{-n+d};q^{d^2+d})_k}{(q^{-(d-1)n+d+1},q^{-(d-3)n+d+1},\dots,q^{-2n+d+1},q^{d+1};q^{d^2+d})_k}$$
$$\times\frac{(q^{(n-1)d},q^{(n-1)d-2n},\dots,q^{n-d};q^{d^2+d})_kq^{(d^2+d)k}}{(q^{(d-1)n+d+1},q^{(d-3)n+d+1},\dots,q^{2n+d+1};q^{d^2+d})_k(q^{d^2+d};q^{d^2+d})_k}. \tag{18}$$

Letting $q\mapsto q^{d^2+d}$, $N=(n+1)/(d+1)$, $m=d$, $b_j=q^{-(d-1)n+d+1+(2j-2)n}$ and $n_j=(n-2d-1)/(d^2+d)$ $(1\leqslant j\leqslant d)$ in (15), we conclude that (18) is equal to 0, which is just the $a=q^{-n}$ or $a=q^n$ case of the right-hand side of (16). Namely, the congruence (16) holds. Exactly in the same way, we can prove the q-congruence (17). □

Proof of Theorem 3. When $a = 1$, the polynomial $(1 - aq^n)(a - q^n)$ contains the factor $\Phi_n(q)^2$, which is coprime with the denominators of the left-hand sides of (16) and (17). Hence, the congruence (8) immediately follows from the $a = 1$ case of (16) and (17). □

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