



ON THE NUMBER OF SOLUTIONS TO THE INEQUALITY WITH PRIMES

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Abstract. Let p_1 and p_2 be the prime numbers. For sufficiently large values of the natural number N we prove that the number of solutions to the inequality $|N - p_1 - p_2| \leq H$ is greater than $\mathcal{O}\left(\frac{HN^{0.525+o(1)}}{(\ln(N))^2}\right)$ provided that $H \geq N^{0.07+o(1)}$.

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1. INTRODUCTION

In 2001, R. Baker, G. Harman, and J. Pintz [6] obtained for the number of solutions to inequality $|p - N| \leq H$ in prime numbers p the correct lower bound in order provided that $H \geq N^{0.525+o(1)}$. If we assume the Riemann hypothesis is true then a given positive number N can be approached by a prime number p at a distance of $\mathcal{O}(N^{0.5+o(1)})$.

One would then ask about approximating a given positive number N by the sum of two prime numbers. In 1975, H. Montgomery and R. Vaughan showed in their paper [4] that inequality

$$|p_1 + p_2 - N| \leq H \tag{1}$$

is solvable for $H \geq N^{7/72+o(1)}$. S. Gritsenko and D. Goryashin [2] proved that the number of solutions to the inequality (1) in primes p_1 and p_2 is greater than

$$\mathcal{O}\left(\frac{HN^{0.525+o(1)}}{(\ln(N))^2}\right)$$

provided that $H \geq N^{7/80+o(1)}$. In this work, we improve the result of S. Gritsenko and D. Goryashin by obtaining the lower bound of the number of solutions to the inequality (1) for smaller H .

THEOREM 1. Let N be the sufficiently large natural number and let $J(N, H)$ denote the number of solutions to the inequality $|N - p_1 - p_2| < H$ in the prime numbers p_1 and p_2 . Then for all $N^{7/100+o(1)} \leq H \leq 0.1N$ we have

$$J(N, H) \gg \frac{N^{0.525+o(1)}H}{(\ln(N))^2}.$$

Before moving on to the proof of Theorem 1, we note that our proof is based on the results of R. Baker, G. Harman and J. Pintz [6] and L. Guth and J. Maynard's new density theorem [3].

In the following, we use Vinogradov's notation $\ll$ and the big $\mathcal{O}$ notation equivalently. The meaning of these notations is that

$$f(x) \ll g(x) \Rightarrow |f(x)| \leq C|g(x)| \text{ for some constant } C \text{ and } x > x_0.$$

However, when $g(x)$ is complicated we prefer to use Vinogradov's notation than the big $\mathcal{O}$ notation.

2. SOME PRELIMINARY LEMMAS

Let $\Lambda(n)$ denote the von Mangoldt function, and $\psi(x) = \sum_{n \leq x} \Lambda(n)$ denote the Chebyshev function.

LEMMA 1. *Let $2 \leq T \leq x$. Then,*

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + \mathcal{O}\left(\frac{x(\ln(x))^2}{T}\right),$$

where $\rho = \beta + i\gamma$ runs through the zeros of the Riemann zeta function in the critical strip.

Proof. See [5], p.52.

Let $\pi(x) = \sum_{p \leq x} 1$ denote number of primes $p \leq x$.

LEMMA 2 (R. Baker, G. Harman, J. Pintz [6]). *For all $x > x_0$, the interval $[x - x^{0.525}, x]$ contains prime number.*

Note that in [6] a lower bound on the number of primes in the interval $[x - x^{0.525}, x]$ is obtained:

$$\pi(x) - \pi(x - x^{0.525}) \geq 0.09 \frac{x^{0.525}}{\ln(x)}. \quad (2)$$

LEMMA 3 (Zero-free region for $\zeta(s)$). *There exist an absolute constant $c > 0$ such that the Riemann zeta function $\zeta(s)$ has no zero in the following region of the s -plane*

$$t \geq 10, \quad \sigma \geq 1 - \frac{c}{(\ln(\ln(t)))^{1/3} (\ln(t))^{2/3}}.$$

Proof. See [5], p.119.

LEMMA 4. *Let $S(t)$ be a complex-valued function, continuously differentiable function on the interval $[t_0, t_k]$, and let*

$$t_0 < t_1 < \cdots < t_k.$$

Then setting $\delta = \min_{0 \leq r < k} (t_{r+1} - t_r)$, we have

$$\sum_{r=1}^k |S(t_r)|^2 \leq \frac{1}{\delta} \int_{t_0}^{t_k} |S(t)|^2 dt + 2 \left(\int_{t_0}^{t_k} |S(t)|^2 dt \right)^{1/2} \left(\int_{t_0}^{t_k} |S'(t)|^2 dt \right)^{1/2}.$$

Proof. See [7], p. 94.

LEMMA 5. *The number of zeros ρ_n of the zeta function that satisfy $T \leq \Im(\rho_n) \leq T + 1$ does not exceed $\mathcal{O}(\log(T))$.*

Proof. See [7], p. 58.

LEMMA 6 (L. Guth, J. Maynard [3]). Let $N(\sigma, T)$ denote the number of zeros ρ of $\zeta(s)$ with $\Re(s) \geq \sigma$ and $|\Im(s)| \leq T$. Then we have

$$N(\sigma, T) \ll T^{15(1-\sigma)/(3+5\sigma)+o(1)}.$$

LEMMA 7. Let $Y \geq 2$ and $T \geq 2$. Then we have

$$\int_{0.5Y}^{2Y} \left| \sum_{0 < |\gamma| \leq T} x^\rho \right|^2 dx \ll Y^2 \ln(Y) (\ln(T))^2 \max_{0.5 \leq \alpha < 1} Y^{2\alpha-1} N(\alpha, T),$$

$$\int_{0.5Y}^{2Y} \left| \sum_{0 < |\gamma| \leq T} \rho x^\rho \right|^2 dx \ll Y^2 T^2 \ln(Y) (\ln(T))^2 \max_{0.5 \leq \alpha < 1} Y^{2\alpha-1} N(\alpha, T),$$

where $\rho = \beta + i\gamma$ runs through the zeros of the Riemann zeta function in the critical strip.

Proof. 1) Proof of the first inequality. We write

$$\begin{aligned} \left| \sum_{0 < |\gamma| \leq T} x^\rho \right|^2 &= \left(\sum_{0 < |\gamma| \leq T} x^\rho \right) \overline{\left(\sum_{0 < |\gamma| \leq T} x^\rho \right)} = \left(\sum_{0 < |\gamma_1| \leq T} x^{\beta_1 + i\gamma_1} \right) \left(\sum_{0 < |\gamma_2| \leq T} \overline{x^{\beta_2 + i\gamma_2}} \right) \\ &= \left(\sum_{0 < |\gamma_1| \leq T} x^{\beta_1 + i\gamma_1} \right) \left(\sum_{0 < |\gamma_2| \leq T} x^{\beta_2 - i\gamma_2} \right) = \sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} x^{\beta_1 + \beta_2 + i(\gamma_1 - \gamma_2)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \int_{0.5Y}^{2Y} \left| \sum_{0 < |\gamma| \leq T} x^\rho \right|^2 dx &= \sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} \int_{0.5Y}^{2Y} x^{\beta_1 + \beta_2 + i(\gamma_1 - \gamma_2)} dx \\ &= \sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} \frac{(2Y)^{\beta_1 + \beta_2 + 1 + i(\gamma_1 - \gamma_2)} - (0.5Y)^{\beta_1 + \beta_2 + 1 + i(\gamma_1 - \gamma_2)}}{\beta_1 + \beta_2 + 1 + i(\gamma_1 - \gamma_2)} \\ &\ll Y \sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} \frac{Y^{\beta_1 + \beta_2}}{1 + |\gamma_1 - \gamma_2|} \\ &= Y \sum_{0 < |\gamma_1| \leq T} \left(\sum_{\substack{0 < |\gamma_2| \leq T \\ \beta_1 \leq \beta_2}} \frac{Y^{\beta_1 + \beta_2}}{1 + |\gamma_1 - \gamma_2|} + \sum_{\substack{0 < |\gamma_2| \leq T \\ \beta_1 > \beta_2}} \frac{Y^{\beta_1 + \beta_2}}{1 + |\gamma_1 - \gamma_2|} \right) \\ &\ll Y \sum_{0 < |\gamma_1| \leq T} \left(\sum_{\substack{0 < |\gamma_2| \leq T \\ 0.5 \leq \beta_2 < 1}} \frac{Y^{2\beta_2}}{1 + |\gamma_1 - \gamma_2|} + \sum_{\substack{0 < |\gamma_2| \leq T \\ 0.5 \leq \beta_1 < 1}} \frac{Y^{2\beta_1}}{1 + |\gamma_1 - \gamma_2|} \right) \\ &= 2Y \sum_{0 < |\gamma_1| \leq T} \sum_{\substack{0 < |\gamma_2| \leq T \\ 0.5 \leq \beta < 1}} \frac{Y^{2\beta}}{1 + |\gamma_1 - \gamma_2|} \\ &\ll Y \sum_{\substack{0 < |\gamma_1| \leq T \\ 0.5 \leq \beta < 1}} Y^{2\beta} \left(\sum_{\substack{0 < |\gamma_2| \leq T \\ |\gamma_1 - \gamma_2| \leq 1}} 1 + \sum_{\substack{0 < |\gamma_2| \leq T \\ |\gamma_1 - \gamma_2| > 1}} \frac{1}{|\gamma_1 - \gamma_2|} \right). \end{aligned}$$

By lemma 5 we have

$$\sum_{\substack{0 < |\gamma_2| \leq T \\ |\gamma_1 - \gamma_2| \leq 1}} 1 \ll \ln(\gamma_1)$$

and

$$\begin{aligned} \sum_{\substack{0 < |\gamma_2| \leq T \\ |\gamma_1 - \gamma_2| > 1}} \frac{1}{|\gamma_1 - \gamma_2|} &= \sum_{1 \leq n \leq T} \sum_{n < |\gamma_1 - \gamma_2| \leq n+1} \frac{1}{|\gamma_1 - \gamma_2|} \\ &\leq \sum_{1 \leq n \leq T} \frac{1}{n} \sum_{n < |\gamma_1 - \gamma_2| \leq n+1} 1 \\ &\ll (\ln(T))^2. \end{aligned}$$

Thus

$$\int_{0.5Y}^{2Y} \left| \sum_{0 < |\gamma| \leq T} x^\rho \right|^2 dx \ll Y (\ln(T))^2 \sum_{\substack{0 < |\gamma_1| \leq T \\ 0.5 \leq \beta < 1}} Y^{2\beta}. \quad (3)$$

Inserting the identity

$$Y^{2\beta} = Y + 2 \ln(Y) \int_{0.5}^{\beta} Y^{2u} du.$$

into (3) we obtain

$$\begin{aligned} \int_{0.5Y}^{2Y} \left| \sum_{0 < |\gamma| \leq T} x^\rho \right|^2 dx &\ll Y (\ln(T))^2 \sum_{\substack{0 < |\gamma_1| \leq T \\ 0.5 \leq \beta < 1}} \left(Y + 2 \ln(Y) \int_{0.5}^{\beta} Y^{2u} du \right) \\ &\ll Y^2 (\ln(T))^2 N(T) + Y \ln(Y) (\ln(T))^2 \int_{0.5}^1 Y^{2u} N(u, T) du \\ &\ll Y^2 \ln(Y) (\ln(T))^2 \max_{0.5 \leq \alpha < 1} Y^{2\alpha-1} N(\alpha, T). \end{aligned}$$

2) Proof of the second inequality. We write

$$\begin{aligned} \int_{0.5Y}^{2Y} \left| \sum_{0 < |\gamma| \leq T} \rho x^\rho \right|^2 dx &= \sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} (\beta_1 + i\gamma_1)(\beta_2 - i\gamma_2) \int_{0.5Y}^{2Y} x^{\beta_1 + \beta_2 + i(\gamma_1 - \gamma_2)} dx \\ &\ll T^2 \sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} \left| \int_{0.5Y}^{2Y} x^{\beta_1 + \beta_2 + i(\gamma_1 - \gamma_2)} dx \right| \\ &= T^2 \sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} \left| \frac{(2Y)^{\beta_1 + \beta_2 + 1 + i(\gamma_1 - \gamma_2)} - (0.5Y)^{\beta_1 + \beta_2 + 1 + i(\gamma_1 - \gamma_2)}}{\beta_1 + \beta_2 + 1 + i(\gamma_1 - \gamma_2)} \right| \\ &\ll YT^2 \sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} \frac{Y^{\beta_1 + \beta_2}}{1 + |\gamma_1 - \gamma_2|}. \end{aligned}$$

In the above argument we proved the following estimate

$$\sum_{0 < |\gamma_1| \leq T} \sum_{0 < |\gamma_2| \leq T} \frac{Y^{\beta_1 + \beta_2}}{1 + |\gamma_1 - \gamma_2|} \ll Y (\ln(Y)) (\ln(T))^2 \max_{0.5 \leq \alpha < 1} Y^{2\alpha-1} N(\alpha, T),$$

and the proof of the second inequality follows. $\square$

LEMMA 8. Let $0 < \varepsilon < 0.1$ be arbitrary small number and let $N(\sigma, T)$ denote the number of zeros ρ of $\zeta(s)$ with $\Re(s) \geq \sigma$ and $|\Im(s)| \leq T$. Then we have

$$N(\sigma, T) \ll T^{(30/13+\varepsilon)(1-\sigma)} (\ln(T))^{14}.$$

Proof. It follows from Lemma 6 that

$$N(\sigma, T) \ll T^{15(1-\sigma)/(3+5\sigma)+\varepsilon_1}.$$

Here $0 < \varepsilon_1 < 0.01$ is arbitrary small number. Put $\varepsilon_2 = 169/(40/\varepsilon_1 - 140) > 0$. Let us consider 3 cases:

Case 1: If $0.7 + \varepsilon_2 < \sigma < 13/15$ then we have

$$\frac{15}{3+5\sigma} < \frac{15}{3+5(0.7+\varepsilon_2)} = \frac{30}{13} - \frac{300\varepsilon_2}{169+130\varepsilon_2}.$$

Thus

$$\begin{aligned} N(\sigma, T) &\ll T^{15(1-\sigma)/(3+5\sigma)+\varepsilon_1} \\ &< T^{30(1-\sigma)/13+\varepsilon_1-40\varepsilon_2/(169+130\varepsilon_2)} \\ &= T^{30(1-\sigma)/13}. \end{aligned} \quad (4)$$

Case 2: If $\sigma \leq 0.7 + \varepsilon_2$ then we use the Ingham's density estimate [1]

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)} (\ln(T))^5. \quad (5)$$

We have

$$\frac{3}{2-\sigma} \leq \frac{3}{2-(0.7+\varepsilon_2)} = \frac{30}{13} + \frac{300\varepsilon_2}{40-260\varepsilon_2} \leq \frac{30}{13} + \varepsilon.$$

Here we take $\varepsilon = 10\varepsilon_2$. Then, it follows from (5) that

$$N(\sigma, T) \ll T^{(30/13+\varepsilon)(1-\sigma)} (\ln(T))^5. \quad (6)$$

Case 3: If $\sigma \geq 13/15$ then we use density estimate (see Theorem 12.1, [8]) as following

$$N(\sigma, T) \ll T^{2(1-\sigma)/\sigma} (\ln(T))^{14}.$$

Since $\sigma \geq 13/15$ then

$$N(\sigma, T) \ll T^{30(1-\sigma)/13} (\ln(T))^{14}. \quad (7)$$

We finish the proof by combining (4), (6) and (7). $\square$

3. PROOF OF THEOREM 1

Let $\varepsilon > 0$ be arbitrary small number and put $N_1 = N^{0.525+3.75\varepsilon}$. For the number $J(N, H)$ of solutions to the inequality $0 \leq N - p_1 - p_2 < H$ we have

$$J(N, H) = \sum_{\substack{2 \leq p_1 \leq N \\ N-p_1-H < p_2 \leq N-p_1}} 1 \geq \sum_{\substack{N-2N_1 < p_1 \leq N-N_1 \\ N-p_1-H < p_2 \leq N-p_1}} 1. \quad (8)$$

We have

$$\begin{aligned}
 \sum_{N-p_1-H < p_2 \leq N-p_1} 1 &= \sum_{N-p_1-H < n \leq N-p_1} \frac{\Lambda(n)}{\ln(n)} - \sum_{\substack{N-p_1-H < n=q^k \leq N-p_1 \\ k \geq 2 \\ q \text{ - prime}}} \frac{\Lambda(n)}{\ln(n)} \\
 &\gg \frac{1}{\ln(N)} \sum_{N-p_1-H < n \leq N-p_1} \Lambda(n) - \sum_{\substack{N-p_1-H < n=q^k \leq N-p_1 \\ k \geq 2 \\ q \text{ - prime}}} \frac{\Lambda(n)}{\ln(n)}. \tag{9}
 \end{aligned}$$

We establish that

$$\begin{aligned}
 \sum_{\substack{N-p_1-H < n=q^k \leq N-p_1 \\ k \geq 2 \\ q \text{ - prime}}} \frac{\Lambda(n)}{\ln(n)} &= \sum_{\substack{2 \leq k \leq \log_2(N) \\ \sqrt[k]{N-p_1-H} < q \leq \sqrt[k]{N-p_1}}} \frac{1}{k} \\
 &\ll \sum_{2 \leq k \leq \log_2(N)} \frac{\sqrt[k]{N-p_1} - \sqrt[k]{N-p_1-H}}{k} \\
 &\ll 1 + \sqrt{N-p_1} - \sqrt{N-p_1-H} \\
 &\quad \ln(N) \left(1 + \sqrt[3]{N-p_1} - \sqrt[3]{N-p_1-H} \right) \\
 &\ll \frac{H}{\sqrt{N_1}} + \ln(N). \tag{10}
 \end{aligned}$$

It follows from (8), (9) and (10) that

$$J(N, H) \gg \frac{1}{\ln(N)} \sum_{\substack{-2N_1 < p_1 \leq N-N_1 \\ N-p_1-H < n \leq N-p_1}} \Lambda(n) + \mathcal{O} \left(\frac{H\sqrt{N_1}}{\ln(N)} + N_1 \right). \tag{11}$$

Put $T = N_1(\ln(N))^3/H$. By Lemma 1 we have

$$\sum_{\substack{N-2N_1 < p_1 \leq N-N_1 \\ N-p_1-H < n \leq N-p_1}} \Lambda(n) = H(\pi(N-N_1) - \pi(N-2N_1)) - W + \mathcal{O} \left(\frac{H}{\ln(N)} \right) \tag{12}$$

where

$$W = \sum_{N-2N_1 < p_1 \leq N-N_1} \sum_{0 < |\gamma| \leq T} \frac{(N-p_1)^\rho - (N-p_1-H)^\rho}{\rho}.$$

Next, Lemma 2 yields

$$\pi(N-N_1) - \pi(N-2N_1) \gg \frac{N_1}{\ln(N)}. \tag{13}$$

By combining (11), (12) and (13) we obtain

$$J(N, H) \gg \frac{HN_1}{\ln(N)} - W + \mathcal{O} \left(\frac{H\sqrt{N_1}}{\ln(N)} + N_1 \right) + \mathcal{O} \left(\frac{H}{\ln(N)} \right). \tag{14}$$

To prove Theorem 1 it suffices to show that

$$W \ll HN_1 \exp \left(-\sqrt[10]{\ln(N)} \right).$$

First we write

$$\begin{aligned} \sum_{0 < |\gamma| \leq T} \frac{(N - p_1)^\rho - (N - p_1 - H)^\rho}{\rho} &= \sum_{0 < |\gamma| \leq T} \int_{N - p_1 - H}^{N - p_1} x^{\rho-1} dx \\ &= \int_0^H \sum_{0 < |\gamma| \leq T} (x_1 + N - p_1 - H)^{\rho-1} dx \\ &\ll \frac{H}{N_1} \left| \sum_{0 < |\gamma| \leq T} (x_1 + N - p_1 - H)^\rho \right| \end{aligned}$$

here $x_1 \in [0, H]$ which maximize the modulus of the sum over zeros ρ of the Riemann's zeta function. Thus, we have

$$W \ll \frac{H}{N_1} \sum_{N - 2N_1 < p_1 \leq N - N_1} \left| \sum_{0 < |\gamma| \leq T} (x + N - p_1 - H)^\rho \right|.$$

The Cauchy's inequality give us

$$W^2 \ll \frac{H^2}{N_1} \sum_{N - 2N_1 < p_1 \leq N - N_1} \left| \sum_{0 < |\gamma| \leq T} (x_1 + N - p_1 - H)^\rho \right|^2.$$

Put

$$f(t) = \sum_{0 < |\gamma| \leq T} (x_1 + N - t - H)^\rho.$$

Lemma 4 give us

$$W^2 \ll \frac{H^2}{N_1} \left(0.5 \int_{N - 2N_1}^{N - N_1} |f(t)|^2 dt + 2 \left(\int_{N - 2N_1}^{N - N_1} |f(t)|^2 dt \right)^{0.5} \left(\int_{N - 2N_1}^{N - N_1} |f'(t)|^2 dt \right)^{0.5} \right) \quad (15)$$

It follows from Lemma 7 that

$$\begin{aligned} \int_{N - 2N_1}^{N - N_1} |f(t)|^2 dt &= \int_{N_1 + x_1 - H}^{2N_1 + x_1 - H} \left| \sum_{0 < |\gamma| \leq T} t^\rho \right|^2 dt \\ &\leq \int_{N_1/2}^{2N_1} \left| \sum_{0 < |\gamma| \leq T} t^\rho \right|^2 dt \\ &\ll N_1^2 (\ln(N))^3 \max_{0.5 \leq \alpha < 1} N_1^{2\alpha-1} N(\alpha, T). \end{aligned} \quad (16)$$

Similarly, we have

$$\begin{aligned} \int_{N - 2N_1}^{N - N_1} |f'(t)|^2 dt &= \int_{N_1 + x_1 - H}^{2N_1 + x_1 - H} \left| \sum_{0 < |\gamma| \leq T} \rho t^{\rho-1} \right|^2 dt \\ &\ll N_1^{-2} \int_{N_1/2}^{2N_1} \left| \sum_{0 < |\gamma| \leq T} \rho t^\rho \right|^2 dt \\ &\ll T^2 (\ln(N))^3 \max_{0.5 \leq \alpha < 1} N_1^{2(\alpha-1)} N(\alpha, T). \end{aligned} \quad (17)$$

By combining (15), (16) and (17) we obtain

$$W^2 \ll H^2 N_1^2 (\ln(N))^3 \max_{0.5 \leq \alpha < 1} N_1^{2(\alpha-1)} N(\alpha, T). \quad (18)$$

We recall that $T = N_1 (\ln(N))^3 / H$ and $H \geq N^{0.07+\varepsilon} = N_1^{2/15+0.5\varepsilon}$. It follows from Lemma 8 that

$$\begin{aligned} N_1^{2(\alpha-1)} N(\alpha, T) &\ll N_1^{2(\alpha-1)} T^{(30/13+\varepsilon)(1-\alpha)} (\ln(T))^{14} \\ &\ll N_1^{2(\alpha-1)} \left(\frac{N_1 (\ln(N))^3}{H} \right)^{(30/13+\varepsilon)(1-\alpha)} (\ln(N))^{14} \\ &\ll \left(\frac{N_1^{2/15+13\varepsilon/30}}{H} \right)^{(30/13+\varepsilon)(1-\alpha)} (\ln(N))^{19} \\ &\ll N_1^{-\varepsilon(30/13+\varepsilon)(1-\alpha)/15} (\ln(N))^{19} \\ &\ll N_1^{-2/13\varepsilon(1-\alpha)} (\ln(N))^{19}. \end{aligned} \quad (19)$$

From (18) and (19),

$$W^2 \ll H^2 N_1^2 (\ln(N))^{22} \max_{0.5 \leq \alpha < 1} N_1^{-2/13\varepsilon(1-\alpha)}.$$

Therefore, by Lemma 3,

$$\begin{aligned} W^2 &\ll H^2 N_1^2 \exp \left(-\frac{c_1 \varepsilon \ln(N)}{(\ln(\ln(N)))^{1/3} (\ln(N))^{2/3}} + 22 \ln(\ln(N)) \right) \\ &\ll H^2 N_1^2 \exp \left(-\sqrt[10]{\ln(N)} \right). \end{aligned} \quad (20)$$

We finish the proof by combining (14) and (20).

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