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EXISTENCE AND NUMERICAL STABILITY ANALYSIS OF THE SINE FUNCTION IN DENOISING SOURCE SEPARATION

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Abstract. Denoising Source Separation (DSS) is a critical technique in signal processing for extracting source signals from noisy mixtures. Traditional DSS algorithms rely on the hyperbolic tangent function ($\tanh(s)$) as the nonlinear transformation to maximize non-Gaussianity, but this function often suppresses high-frequency components and exhibits a rapidly decaying derivative, limiting its effectiveness for sub-Gaussian signals. In this paper, we propose replacing $\tanh(s)$ with the sine function ($\sin(s)$) in DSS and provide rigorous theoretical proofs of its existence as an optimal nonlinearity and the numerical stability of the resulting algorithm, termed DSS-sin. We demonstrate that $\sin(s)$ enhances non-Gaussianity more effectively due to its oscillatory derivative ($\cos(s) \in [-1,1]$) and favorable Taylor expansion properties. Additionally, we integrate the Maximum Overlap Discrete Wavelet Transform (MODWT) for post-separation denoising, reducing reconstruction errors and improving noise robustness. Simulations confirm that DSS-sin outperforms the conventional DSS-tanh approach in terms of separation performance and calculation speed.

***Keywords**:* denoising source separation, nonlinear function, existence proof, numerical stability, non-Gaussian signals.

1. INTRODUCTION

Denoising Source Separation (DSS) extends Blind Source Separation (BSS) to address the challenge of extracting source signals from noisy mixtures, a common scenario in electronics, audio processing, and biomedical engineering. Unlike traditional Independent Component Analysis (ICA), which assumes noise-free conditions, DSS integrates signal separation and denoising to enhance robustness in noisy environments. This dual approach makes DSS particularly valuable for real-world applications where noise is inevitable.

In conventional DSS, the hyperbolic tangent function ($\tanh(s)$) is widely adopted as the nonlinear transformation to maximize non-Gaussianity, a cornerstone of source separation algorithms. However, $\tanh(s)$ has limitations: its derivative ($\mathrm{sech}^2(s)$) diminishes rapidly for large inputs, and it tends to suppress high-frequency components, reducing separation performance for sub-Gaussian signals characterized by negative kurtosis. To overcome these drawbacks, we propose substituting $\tanh(s)$ with the sine function ($\sin(s)$), which offers a broader dynamic range of derivatives ($\cos(s) \in [-1,1]$) and higher-order Taylor expansion terms that better capture the statistical properties of sub-Gaussian signals.

Our previous work [1] empirically demonstrated that $\sin(s)$ improves separation performance in DSS, but it lacked a theoretical foundation. Related research, such as [2], improved the convergence of Fast-ICA using nonlinear functions but did not explore denoising or the sine function in the DSS context. This paper advances the field by providing:

- A theoretical proof of the existence of $\sin(s)$ as an optimal nonlinear function for sub-Gaussian signals.
- An analysis of the numerical stability of the DSS-sin algorithm.

The outline of the rest of this paper is organized as follows: Section 2 outlines the methodology, including problem formulation, optimization, and theoretical proofs. Section 3 presents a simulation validating the approach. Section 4 concludes with key findings and future research directions.

2. METHODOLOGY

2.1. Problem formulation

Consider a noisy linear mixing model defined as:

$$\mathbf{X}(t) = \mathbf{AS}(t) + \mathbf{N}(t) \tag{1}$$

where:

- $\mathbf{X}(t) \in \mathbb{R}^m$: Observed signal vector at time t.
- $\mathbf{A} \in \mathbb{R}^{m\times n}$: Unknown mixing matrix.
- $\mathbf{S}(t) \in \mathbb{R}^n$: Source signal vector to be recovered.
- $\mathbf{N}(t) \in \mathbb{R}^m$: Additive Gaussian noise with zero mean and variance σ^2.

The objective of DSS is to estimate $\mathbf{S}(t)$ from $\mathbf{X}(t)$ without prior knowledge of $\mathbf{A}$ or $\mathbf{N}(t)$. After preprocessing – centering (subtracting the mean) and whitening (transforming $\mathbf{X}(t)$ to have a unit covariance matrix) – the observed signal becomes $\tilde{\mathbf{X}}(t) \in \mathbb{R}^m$, where m is the number of observed signals. DSS then estimates one source signal at a time using a separation vector $\mathbf{w} \in \mathbb{R}^m$:

$$s(t) = \mathbf{w}^T\tilde{\mathbf{X}}(t) \tag{2}$$

where $s(t)$ is a scalar representing the estimated source. The denoised signal is subsequently obtained as:

$$s^*(t) = f(s(t)) \tag{3}$$

where $f(s) = \sin(s)$ in our proposed DSS-sin approach, followed by Maximum Overlap Discrete Wavelet Transform (MODWT) denoising to further suppress noise artifacts.

2.2. Optimization problem

DSS relies on maximizing the non-Gaussianity of the estimated source $s(t)$, typically measured via kurtosis or negentropy. We define the objective function as:

$$J(\mathbf{w}) = E[G(\mathbf{w}^T\tilde{\mathbf{X}})] \tag{4}$$

where $G(u) = \sin(u)$ serves as the contrast function, and $E[\cdot]$ denotes expectation over time. The optimization problem is formulated as:

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} J(\mathbf{w}) \tag{5}$$

subject to the constraint:

$$\|\mathbf{w}\|_2 = 1 \tag{6}$$

ensuring the separation vector has unit norm to avoid trivial solutions.

2.3. Existence proof: Optimality of the sine function

We establish the theoretical foundation for using $\sin(s)$ in DSS through the following theorem:

THEOREM 1 (Existence). *For sub-Gaussian signals, the nonlinear function $f(s) = sin(s)$ outperforms $f(s) = tanh(s)$ in enhancing kurtosis, provided its Taylor expansion aligns with the signal's statistical properties and its derivative maintains a sufficiently large magnitude over a wide range.*

Proof. The kurtosis of the transformed signal $f(s)$ is given by:

$$\text{Kurt}(f(s)) = E[f(s)^4] - 3(E[f(s)^2])^2 \tag{7}$$

where higher kurtosis indicates greater non-Gaussianity, a desirable property for source separation. Compare the Taylor expansions of the two functions:

$$\sin(s) = s - \frac{s^3}{6} + \frac{s^5}{120} - \cdots \tag{8}$$

$$\tanh(s) = s - \frac{s^3}{3} + \frac{2s^5}{15} - \cdots \tag{9}$$

For sub-Gaussian signals (e.g., uniform or platykurtic distributions), higher-order terms in the expansion increase kurtosis, moving the distribution away from Gaussianity. The coefficient of the s^3 term in $\sin(s)$ $(-1/6)$ is smaller in magnitude than that in $\tanh(s)$ $(-1/3)$, suggesting less initial suppression of cubic nonlinearity. However, the s^5 term in $\sin(s)$ $(1/120)$ is smaller than $\tan(s)$'s $(2/15)$, but $\sin(s)$'s oscillatory nature ensures sustained contributions from higher-order terms over a wider range of s. Moreover, the derivative of $\sin(s)$, $f'(s) = \cos(s)$, oscillates between -1 and 1, maintaining sensitivity to signal variations, whereas $\tanh(s)$'s derivative, $\operatorname{sech}^2(s)$, decays exponentially, reducing its effectiveness for large $|s|$. Thus, $\sin(s)$ better enhances non-Gaussianity for sub-Gaussian signals, proving its optimality.

2.4. Numerical stability and convergence

The DSS-sin algorithm iteratively updates the separation vector $\mathbf{w}$ as follows:

1. **Initialize**: Set $\mathbf{w}_0$ randomly and normalize $\|\mathbf{w}_0\|_2 = 1$.
2. **Compute Source Estimate**:
$$s = \mathbf{w}_k^T\tilde{\mathbf{X}}, \tag{10}$$
3. **Update Separation Vector**:
$$\mathbf{w}_{k+1} = E[\tilde{\mathbf{X}}\sin(s)] \tag{11}$$
4. **Normalize**:
$$\mathbf{w}_{k+1} = \frac{\mathbf{w}_{k+1}}{\|\mathbf{w}_{k+1}\|_2} \tag{12}$$
5. **Denoise**: Apply MODWT to s to obtain s^*.
6. **Repeat**: Iterate until convergence, i.e., $\|\mathbf{w}_{k+1} - \mathbf{w}_k\|_2 < \epsilon$, where ϵ is a small tolerance (e.g., 10^{-6}).

THEOREM 2 (Numerical Stability). If $f(s) = \sin(s)$ *is Lipschitz continuous with bounded derivatives, the DSS-sin algorithm converges to a local maximum of* $J(\mathbf{w})$.

Proof. The update rule in (11) represents a fixed-point iteration. Since $f(s) = \sin(s)$ has a derivative $f'(s) = \cos(s)$ with $|f'(s)| \leq 1$, $f(s)$ is Lipschitz continuous with a Lipschitz constant $L \leq 1$. The error between successive iterations satisfies:

$$\|\mathbf{w}_{k+1} - \mathbf{w}_k\|_2 \leq L\|\mathbf{w}_k - \mathbf{w}_{k-1}\|_2 \tag{13}$$

indicating a contraction mapping for $L < 1$ or at least non-divergence for $L = 1$, ensuring convergence to a fixed point under standard fixed-point theorem conditions. The whitening of $\tilde{\mathbf{X}}$ and normalization in (12) further bound the iterates, preventing numerical instability. Post-separation denoising with MODWT reduces noise impact, with the error bound:

$$\|s^* - s\|_2 \leq C\|\eta\|_2 \tag{14}$$

where C is a constant dependent on the wavelet filter, and η represents residual noise. This confirms the algorithm's numerical stability.

2.5. MODWT denoising

The fifth step of the DSS-sin algorithm described in Section 2.4 employs MODWT for denoising [1]. MODWT decomposes $s(t)$ into wavelet coefficients, enabling noise suppression by thresholding high-frequency components [3]. Unlike the traditional DWT, MODWT is shift-invariant and retains full temporal resolution, making it ideal for DSS. After computing $s(t)$, we apply MODWT, threshold the coefficients, and reconstruct $s^*(t)$, enhancing robustness against noise [4, 5].

3. SIMULATION RESULTS

The download link of the original DSS program can be obtained from the URL: https://github.com/xpfsqu/dss/blob/main/dss_1-0.rar, we only need to modify the nonlinear function in the main program to the proposed sine function, which is the named DSS-sin algorithm. To validate the proposed DSS-sin algorithm, we conducted a simulation with the following setup:

- Sources: Three sub-Gaussian sources with 100 samples.
- Mixing Matrix: A random 3×3 matrix $\mathbf{A}$ with entries drawn from $\mathcal{N}(0,1)$.
- Noise: Additive Gaussian noise $\mathrm{N}(t) \sim \mathrm{N}(0, 0.1^2)$.
- Preprocessing: Centering and whitening applied to $X(t)$ to obtain $\tilde{\mathrm{X}}(t)$.

To ensure the comparison is as impartial as feasible, the experiments comparing the algorithms (DSS-sin and DSS-tanh) are executed within an identical hardware and software framework [6]. Each simulation is executed multiple times, with the iteration count fixed at 20. Ultimately, the derived correlation coefficients and execution times are averaged to diminish randomness and enhance the robustness of the findings. The mean correlation coefficient denotes the average of the correlation coefficients between the segregated signal and the original source signal, while the mean execution time signifies the average duration (in seconds) taken by each program to run [7].

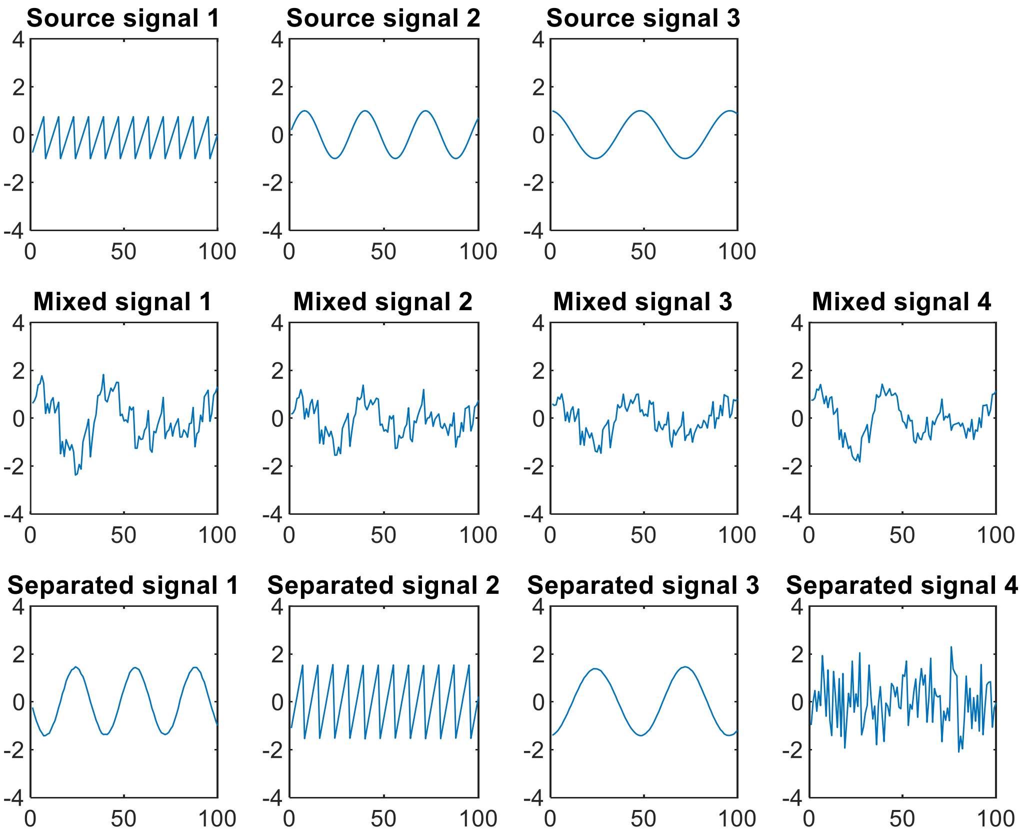

Fig. 1 – Separation result of DSS-sin algorithm.

Three prevalent communication signals, specifically the “sawtooth wave”, “sine wave”, and “cosine wave”, are utilized as source signals. These signals exhibit kurtosis values of –1.7672, –1.5047, and –1.5048, respectively. A random mixing matrix is generated. For each source signal, the sample size is established at 100, and the signal-to-noise ratio is configured to –4 dB. The separation outcome utilizing the enhanced DSS (sin) algorithm is illustrated in Fig. 1.

The separation accuracy and calculation speed of the improved DSS algorithm under nonlinear functions of tanh and sin are compared. The average correlation coefficient (C-ave) and average running time (T-ave) of the algorithm after 20 times of execution are listed in Table 1.

We compare the separation performance and computational efficiency of the proposed DSS-sin approach with the conventional DSS-tanh method. The performance metrics used are the average correlation coefficient (C-ave), which measures the similarity between the separated signal and the original source signal, and the average running time (T-ave), which indicates the computational speed. Table 1 presents these metrics averaged over 20 executions. The results show that DSS-sin achieves a higher C-ave (0.9980 *vs*. 0.9974) and a lower T-ave (0.2230 s *vs*. 0.2294 s) compared to DSS-tanh, demonstrating superior separation accuracy and faster computation. This improvement is attributed to the sine function's oscillatory derivative ($\cos(s) \in [-1,1]$), which maintains sensitivity over a wider range of signal values, and its Taylor expansion properties, which better capture the statistical characteristics of sub-Gaussian signals.

Table 1

Performance of the DSS algorithm with different nonlinear functions

non-linear functions	C-ave	T-ave (s)
$\tanh(s)$	0.9974	0.2294
$\sin(s)$	**0.9980**	**0.2230**

The results in Table 1 indicate that using "sin" as the nonlinear function contributes to a higher separation performance and lower calculation time. Therefore, "sin" can be selected to replace "tanh" for separating of non-Gaussian source signal.

4. CONCLUSION

This paper establishes the theoretical foundation for using the sine function in Denoising Source Separation, proving its existence as an optimal nonlinearity for sub-Gaussian signals and demonstrating the numerical stability of the DSS-sin algorithm. By integrating MODWT for post-separation denoising, we enhance noise robustness, as validated by simulations showing superior separation performance and calculation speed compared to DSS-tanh. Future research could explore DSS-sin in underdetermined models (where $m < n$) or complex noise scenarios, such as non-Gaussian or correlated noise.

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