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(N,ε)-PSEUDOSPECTRA AND CONDITION (n,ε)-PSEUDOSPECTRA OF AN ELEMENT PENCIL IN A UNITAL COMPLEX BANACH ALGEBRA

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Abstract. As a continuation of the paper J. Math. Anal. Appl. 464(2018), 939-954. We will define and examine (n,ε)-pseudospectra and condition (n,ε)-pseudospectra of an element pencil in a unital complex Banach algebra. We characterize (n,ε)-pseudospectra of an element pencil in a unital complex Banach algebra. In particular, we demonstrate some properties of (n,ε)-pseudospectra and condition (n,ε)-pseudospectra of an element pencil in unital complex Banach algebras.

***Keywords**:* complex Banach algebras, pseudospectra, condition pseudospectra.
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1. INTRODUCTION AND PRELIMINARIES

The pseudospectrum of an element in a unital complex Banach algebra was studied by A. Krishnan and S. H. Kulkarni [11]. In [12], S. H. Kulkarni and D. Sukumar introduced and studied the condition pseudospectra of an element in a unital complex Banach algebra. Recently, K. Dhara and S. H. Kulkarni [7] extended and studied the (n,ε)-pseudospectra of an element in a unital complex Banach algebra.

In Banach spaces, M. Seidel [16] extended and studied the (n,ε)-pseudospectra of operators on complex Banach spaces. The approximation of spectra of linear operators on Hilbert spaces enabled A. C. Hansen [10] to introduce the concept of (n,ε)-pseudospectra of operators in complex Hilbert spaces. The generalized eigenvalue problem is one of important problems in functional analysis that is the dynamical problem related to many engineering structures. Linear operator pencils become apparent in control theory, quantum mechanics, numerical solutions to differential equations and then play a central role in perturbation theory and numerical analysis, e.g., [9, 13, 18].

In this study, we define and examine the (n,ε)-pseudospectra and condition (n,ε)-pseudospectra of an element pencil in unital complex Banach algebras. In particular, we demonstrate some properties of (n,ε)-pseudospectra and condition (n,ε)-pseudospectra of an element pencil in unital complex Banach algebras. We start with:

Definition 1 [14]. A unital Banach algebra is a unital algebra $\mathscr{A}$ with identity 1 together with a complete norm $\|\cdot\|$ satisfying the following conditions:

(i) $\|1\| = 1$;

(ii) $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathscr{A}$.

Definition 2 [14]. An element of a unital complex Banach algebra $\mathscr{A}$ for which there exists an inverse (left inverse, right inverse) will be called invertible (left invertible, right invertible). The unique inverse of an invertible element x will be denoted by x^{-1}. The set of all invertible elements in an algebra $\mathscr{A}$ will be denoted by $Inv(\mathscr{A})$.

PROPOSITION 1. *[14] Let $\mathscr{A}$ be a unital complex Banach algebra and let $x \in \mathscr{A}$ such that $\|x\| < 1$, then $(x-1)^{-1}$ exists in $\mathscr{A}$ and $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$.*

Definition 3 [14]. Let $\mathscr{A}$ be a unital complex Banach algebra and let $x \in \mathscr{A}$, the spectrum $\sigma(x)$ of the element x is defined by

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \notin Inv(\mathscr{A})\}.$$

The resolvent set $\rho(x)$ of x is $\mathbb{C}\backslash\sigma(x)$.

We define the spectrum of an element pencil in a unital complex Banach algebra.

Definition 4. Let $\mathscr{A}$ be a unital complex Banach algebra and let $x, y \in \mathscr{A}$, the spectrum $\sigma(x,y)$ of the element pencil of the form $x - \lambda y$ is defined by

$$\sigma(x,y) = \{\lambda \in \mathbb{C} : x - \lambda y \notin Inv(\mathscr{A})\}.$$

The resolvent set $\rho(x,y)$ of the element pencil of the form $x-\lambda y$ is $\mathbb{C}\backslash\sigma(x,y)$. Set for all $\lambda \in \rho(x,y), R(\lambda,x,y) = (x-\lambda y)^{-1}$, $R(\lambda,x,y)$ is called the resolvent of the element pencil (x,y) of the form $x - \lambda y$.

Note that $\sigma(x,1) = \sigma(x)$.

2. MAIN RESULTS

2.1. (N,ε)-pseudospectra of an element pencil in unital complex Banach algebras

Now, we define the (n,ε)-pseudospectra of an element pencil in unital complex Banach algebras.

Definition 5. Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $\varepsilon > 0$, the (n,ε)-pseudospectrum $\sigma_{n,\varepsilon}(j,a)$ of the element pencil (j,a) of the form $j - \lambda a$ is

$$\sigma_{n,\varepsilon}(j,a) = \sigma(j,a) \cup \{\lambda \in \mathbb{C} : \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} > \frac{1}{\varepsilon}\},$$

with the convention $\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} = \infty$, if $\lambda \in \sigma(j,a)$.

By Definition 5, we get the following remark.

Remark 1. If $n = 0$, then $\sigma_{0,\varepsilon}(j,a) = \sigma_\varepsilon(j,a)$ is the pseudospectrum of the element pencil (j,a).

We get the following results.

THEOREM 1. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $\varepsilon > 0$, hence*

(i) *For each $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 \le \varepsilon_2$, $\sigma_{n,\varepsilon_1}(j,a) \subset \sigma_{n,\varepsilon_2}(j,a)$;*

(ii) $\sigma(j,a) = \bigcap\limits_{\varepsilon>0} \sigma_{n,\varepsilon}(j,a)$.

Proof. (i) If $\lambda \in \sigma_{n,\varepsilon_1}(j,a)$, hence $\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} > \varepsilon_1^{-1} \ge \varepsilon_2^{-1}$, thus $\lambda \in \sigma_{n,\varepsilon_2}(j,a)$.

(ii) Since for any $\varepsilon > 0$, $\sigma(j,a) \subseteq \sigma_{n,\varepsilon}(j,a)$, hence $\sigma(j,a) \subseteq \bigcap\limits_{\varepsilon>0} \sigma_{n,\varepsilon}(j,a)$. For the converse inclusion, if $\lambda \in \bigcap\limits_{\varepsilon>0} \sigma_{n,\varepsilon}(j,a)$, hence for any $\varepsilon > 0$, $\lambda \in \sigma_{n,\varepsilon}(j,a)$. If $\lambda \notin \sigma(j,a)$, thus $\lambda \in \{\lambda \in \mathbb{C} : \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} > \varepsilon^{-1}\}$, for $\varepsilon \to 0^+$, we obtain $\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} = \infty$ which is a contradiction with $\lambda \notin \sigma(j,a)$. Consequently $\lambda \in \sigma(j,a)$.

□

COROLLARY 1. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A})$ and $\varepsilon > 0$, then for any $n \in \mathbb{N}$, $\sigma_{n+1,\varepsilon}(j,a) \subset \sigma_{n,\varepsilon}(j,a)$.*

Proof. If $\lambda \in \sigma_{n+1,\varepsilon}(j,a)$, hence

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(j-\lambda a)^{-2^{n+1}}\|^{\frac{1}{2^{n+1}}} && (1)\\ &\leq \|(j-\lambda a)^{-2^{n}}\|^{\frac{1}{2^{n}}}, && (2) \end{aligned}$$

thus $\lambda \in \sigma_{n,\varepsilon}(j,a)$. □

PROPOSITION 2. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $\varepsilon > 0$. Then for any $\lambda \in \mathbb{C}^*$ and for each $\alpha \in \mathbb{C}$, $\sigma_{n,\varepsilon}(\lambda j + \alpha a, a) = \alpha + \lambda \sigma_{n,\frac{\varepsilon}{|\lambda|}}(j,a)$.*

Proof. If $\mu \notin \sigma(\lambda j + \alpha a, a)$, then

$$\begin{aligned} (\lambda j + \alpha a - \mu a)^{-1} \in \mathscr{A} &\Leftrightarrow \lambda^{-1}(j - \frac{\mu-\alpha}{\lambda}a)^{-1} \in \mathscr{A}\\ &\Leftrightarrow (j - \frac{\mu-\alpha}{\lambda}a)^{-1} \in \mathscr{A}, \end{aligned}$$

hence $\sigma(\lambda j + \alpha a, a) = \alpha + \lambda \sigma(j,a)$. Let $\mu \in \sigma_{n,\varepsilon}(\lambda j + \alpha a, a)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(\lambda j + \alpha a - \mu a)^{-2^n}\|^{\frac{1}{2^n}} && (3)\\ &\leq |\lambda|^{-1}\|(j - \frac{\mu-\alpha}{\lambda}a)^{-2^n}\|^{\frac{1}{2^n}}, && (4) \end{aligned}$$

hence $\frac{\mu-\alpha}{\lambda} \in \sigma_{n,\frac{\varepsilon}{|\lambda|}}(j,a)$, thus $\mu \in \alpha + \lambda \sigma_{n,\frac{\varepsilon}{|\lambda|}}(j,a)$. Similarly, we obtain the converse inclusion. □

THEOREM 2. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $\varepsilon > 0$. Then,*

$$\bigcup_{d \in \mathscr{A}: \|d\| < \varepsilon^{2^n}} \sigma_n(j+d, a) \subseteq \sigma_{n,\varepsilon}(j,a),$$

where $\sigma_n(j+d,a) = \{\lambda \in \mathbb{C} : (j-\lambda a)^{2^n} + d \quad \text{is not invertible}\}$.

Proof. Let $\bigcup_{d \in \mathscr{A}: \|d\| < \varepsilon^{2^n}} \sigma_n(j+d,a)$. We argue by contradiction. Suppose that $\lambda \in \rho(j,a)$ and $\|(j-\lambda a)^{-2^n}\| \leq \varepsilon^{-2^n}$.

Consider f defined on $\mathscr{A}$ by

$$f = \sum_{k=0}^{\infty} (j-\lambda a)^{-2^n} \Big(-d(j-\lambda a)^{-2^n} \Big)^k.$$

From

$$\|d(j-\lambda a)^{-2^n}\| \leq \|d\| \|(j-\lambda a)^{-2^n}\| < 1$$

and Proposition 1, f is well-defined and in $\mathscr{A}$. It is easy to see that f can be written as follows $f = ((j-\lambda a)^{2^n} + d)^{-1}$ which is a contradiction. Thus $\lambda \in \sigma_{n,\varepsilon}(j,a)$. □

PROPOSITION 3. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), ja = aj, n \in \mathbb{N}$ and $\varepsilon > 0$. Then*

$$\sigma_{n,\frac{\varepsilon}{\|a\|}}(a^{-1}j) \subseteq \sigma_{n,\varepsilon}(j,a) \subseteq \sigma_{n,\varepsilon\|a^{-1}\|}(a^{-1}j).$$

Proof. We get $\sigma(a^{-1}j)=\sigma(j,a)$. If $\lambda\in\sigma_{n,\frac{\varepsilon}{\|a\|}}(a^{-1}j)$, then

$$\begin{aligned}\frac{\|a\|}{\varepsilon} &< \|(a^{-1}j-\lambda 1)^{-2^n}\|^{\frac{1}{2^n}}\\ &= \|(a^{-1}(j-\lambda a))^{-2^n}\|^{\frac{1}{2^n}}\\ &\leq \|a\|\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}.\end{aligned}$$

From a is invertible, then $\|a\|>0$, so $\frac{1}{\varepsilon}<\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}$, hence $\lambda\in\sigma_{n,\varepsilon}(j,a)$. If $\lambda\in\sigma_{n,\varepsilon}(j,a)$, hence

$$\begin{aligned}\frac{1}{\varepsilon} &< \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}\\ &= \|(a(a^{-1}j-\lambda 1))^{-2^n}\|^{\frac{1}{2^n}}\\ &\leq \|a^{-1}\|\|(a^{-1}j-\lambda 1)^{-2^n}\|^{\frac{1}{2^n}},\end{aligned}$$

thus $\frac{1}{\varepsilon\|a^{-1}\|}<\|(a^{-1}j-\lambda 1)^{-2^n}\|^{\frac{1}{2^n}}$, hence $\lambda\in\sigma_{n,\varepsilon\|a^{-1}\|}(a^{-1}j)$. □

PROPOSITION 4. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j\in\mathscr{A}, a\in Inv(\mathscr{A})$ and $\lambda\notin\sigma(j,a)$, then $\mu\in\sigma((j-\lambda a)^{-1},a)$ if and only if $\frac{1}{\mu}\in\sigma((j-\lambda a)a)$.*

Proof. For each $\lambda\notin\sigma(j,a)$,

$$(j-\lambda a)^{-1}-\mu a=(j-\lambda a)^{-1}[1-\mu(j-\lambda a)a].$$

Hence $\mu\in\sigma((j-\lambda a)^{-1},a)$ if and only if $\frac{1}{\mu}\in\sigma((j-\lambda a)a)$. □

Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j\in\mathscr{A}, a\in Inv(\mathscr{A})$ and $n\in\mathbb{N}$. Define $\gamma^n_{j,a}:\mathbb{C}\longrightarrow\mathbb{R}_+$ as

$$\gamma^n_{j,a}(\lambda)=\begin{cases}\|(j-\lambda a)^{-2^n}\|^{\frac{-1}{2^n}}, & \text{if } \lambda\notin\sigma(j,a)\\ 0, & \text{if } \lambda\in\sigma(j,a).\end{cases}$$

THEOREM 3. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j\in\mathscr{A}, a\in Inv(\mathscr{A})$ and $n\in\mathbb{N}$, we have*

(i) *$\gamma^n_{j,a}$ is continuous.*

(ii) *$\sigma_{n,\varepsilon}(j,a)=\{\lambda\in\mathbb{C}:\gamma^n_{j,a}(\lambda)<\varepsilon\}$.*

(iii) *$\sigma_{n,\varepsilon}(\alpha j,\alpha a)=\sigma_{n,\frac{\varepsilon}{|\alpha|}}(j,a)$ for all $\alpha\in\mathbb{C}\backslash\{0\}$.*

(iv) *If $\mathscr{A}$ is a C^*-aglebra, then $\lambda\in\sigma_{n,\varepsilon}(j,a)$ if and only if $\overline{\lambda}\in\sigma_{n,\varepsilon}(j^*,a^*)$.*

Proof. (i) We have

$$\begin{aligned}\sigma(j,a) &= \{\lambda\in\mathbb{C}: j-\lambda a \text{ is not invertible}\}\\ &= \{\lambda\in\mathbb{C}: ja^{-1}-\lambda \text{ is not invertible}\}\\ &= \sigma(ja^{-1}).\end{aligned}$$

Assume that $\lambda_k\in\mathbb{C}\backslash\sigma(j,a)$ such that $\lambda_k\longrightarrow\lambda$ for certain $\lambda\notin\sigma(j,a)$. Hence

$$\gamma^n_{j,a}(\lambda_k)=\|(j-\lambda_k a)^{-2^n}\|^{\frac{-1}{2^n}}\longrightarrow\|(j-\lambda a)^{-2^n}\|^{\frac{-1}{2^n}}=\gamma^n_{j,a}(\lambda). \tag{5}$$

If $\lambda_k\in\mathbb{C}\backslash\sigma(j,a)$ such that $\lambda_k\longrightarrow\lambda$ for certain $\lambda\in\sigma(j,a)$. From Lemma 10.17 of [15], $\|(j-\lambda_k a)^{-2^n}\|\longrightarrow\infty$ and $\gamma^n_{j,a}(\lambda_k)\longrightarrow 0=\gamma^n_{j,a}(\lambda)$. Then $\gamma^n_{j,a}(\lambda)$ is continuous.

(ii) It follows from Definition 5 and the definition of $\gamma^n_{j,a}$.
(iii) Let $\alpha \in \mathbb{C}\backslash\{0\}$, we get $\sigma(\alpha j, \alpha a) = \sigma(j,a)$. Then

$$\begin{aligned} \lambda \in \sigma_{n,\varepsilon}(\alpha j, \alpha a)\backslash\sigma(\alpha j, \alpha a) &\iff \frac{1}{\varepsilon} < \|(\alpha j - \lambda \alpha a)^{-2^n}\|^{\frac{1}{2^n}} \\ &\iff \frac{|\alpha|}{\varepsilon} < \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} \\ &\iff \lambda \in \sigma_{n,\frac{\varepsilon}{|\alpha|}}(j,a)\backslash\sigma(j,a). \end{aligned}$$

(iv) From Theorem 11.15 of [15], $\lambda \in \sigma(j,a)$ if and only if $\overline{\lambda} \in \sigma(j^*,a^*)$. If $\lambda \in \sigma_{n,\varepsilon}(j,a)$, then by Proposition 1.4 and Proposition 1.7 of [5],

$$\frac{1}{\varepsilon} < \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} = \|(j^* - \overline{\lambda} a^*)^{-2^n}\|^{\frac{1}{2^n}},$$

thus $\overline{\lambda} \in \sigma_{n,\varepsilon}(j^*,a^*)$. Similarly, if $\overline{\lambda} \in \sigma_{n,\varepsilon}(j^*,a^*)$, then $\lambda \in \sigma_{n,\varepsilon}(j,a)$. □

THEOREM 4. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A})$ and $n \in \mathbb{N}$. We have*

(i) *If $ja = aj$ and $\|a^{2^n}\|^{\frac{1}{2^n}} \neq 0$, then*

$$\sigma_{n,\frac{\varepsilon}{\|a^{2^n}\|^{\frac{1}{2^n}}}}(a^{-1}j) \subseteq \sigma_{n,\varepsilon}(j,a) \subseteq \sigma_{n,\varepsilon\|a^{-2^n}\|^{\frac{1}{2^n}}}(a^{-1}j).$$

(ii) *If j is invertible, $\|a^{2^n}\|^{\frac{1}{2^n}}, \|j^{2^n}\|^{\frac{1}{2^n}} \neq 0$, $ja = aj$ and $k_n = |\lambda^{-1}|\|j^{2^n}\|^{\frac{1}{2^n}}\|a^{-2^n}\|^{\frac{1}{2^n}}$ for certain $\lambda \neq 0$, then*

$$\lambda \in \sigma_{n,\varepsilon}(j^{-1},a) \Rightarrow \lambda^{-1} \in \sigma_{n,\varepsilon k_n}(j,a^{-1}).$$

Also, if $k'_n = |\lambda^{-1}|\|j^{-2^n}\|^{\frac{1}{2^n}}\|a^{2^n}\|^{\frac{1}{2^n}}$ for certain $\lambda \neq 0$, then

$$\lambda \in \sigma_{n,\varepsilon}(j,a^{-1}) \Rightarrow \lambda^{-1} \in \sigma_{n,\varepsilon k'_n}(j^{-1},a).$$

Proof. (i) We get $\sigma(a^{-1}j) = \sigma(j,a)$. If

$$\begin{aligned} \frac{\|a^{2^n}\|^{\frac{1}{2^n}}}{\varepsilon} &< \|(a^{-1}j - \lambda 1)^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|(a^{-1}(j-\lambda a))^{-2^n}\|^{\frac{1}{2^n}} \\ &\leq \|a^{2^n}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}. \end{aligned}$$

So $\lambda \in \sigma_{n,\varepsilon}(j,a)$. Further presume that $ja = aj$ and $\lambda \in \sigma_{n,\varepsilon}(j,a)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|(a(a^{-1}j - \lambda 1))^{-2^n}\|^{\frac{1}{2^n}} \\ &\leq \|a^{-2^n}\|^{\frac{1}{2^n}}\|(a^{-1}j-\lambda 1)^{-2^n}\|^{\frac{1}{2^n}}, \end{aligned}$$

thus $\lambda \in \sigma_{n,\varepsilon\|a^{-2^n}\|^{\frac{1}{2^n}}}(a^{-1}j)$.

(ii) Let us assume that j,a are invertible and $ja=aj$. Let $\lambda \neq 0$, then

$$\begin{aligned} \lambda \in \sigma(j^{-1},a) &\iff \lambda a - j^{-1} \text{ is not invertible} \\ &\iff -\lambda a(\frac{a^{-1}}{\lambda} - j)j^{-1} \text{ is not invertible} \\ &\iff \frac{1}{\lambda} \in \sigma(j,a^{-1}). \end{aligned}$$

If $\lambda \in \sigma_{n,\varepsilon}(j^{-1},a)\backslash\sigma(j^{-1},a)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(j^{-1}-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|(\lambda a(\frac{a^{-1}}{\lambda} - j)j^{-1})^{-2^n}\|^{\frac{1}{2^n}} \\ &\leq |\lambda|^{-1}\|a^{-2^n}\|^{\frac{1}{2^n}}\|j^{2^n}\|^{\frac{1}{2^n}}\|(\frac{a^{-1}}{\lambda} - j)^{-2^n}\|^{\frac{1}{2^n}}. \end{aligned}$$

So $\lambda^{-1} \in \sigma_{n,\varepsilon k_n}(j,a^{-1})$.

On the other hand, if $\lambda \in \sigma_{n,\varepsilon}(j,a^{-1})$, then

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(j-\lambda a^{-1})^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|(\lambda a^{-1}(\frac{a}{\lambda} - j^{-1})j)^{-2^n}\|^{\frac{1}{2^n}} \\ &\leq |\lambda|^{-1}\|a^{2^n}\|^{\frac{1}{2^n}}\|j^{-2^n}\|^{\frac{1}{2^n}}\|(\frac{a}{\lambda} - j^{-1})^{-2^n}\|^{\frac{1}{2^n}}. \end{aligned}$$

So $\lambda^{-1} \in \sigma_{n,\varepsilon k'_n}(j^{-1},a)$. □

THEOREM 5. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a,w \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $k_n = \|w\|^{\frac{1}{2^n}}\|w^{-1}\|^{\frac{1}{2^n}}$. Set $i = wjw^{-1}$ and $l = waw^{-1}$, then*

$$\sigma_{n,\frac{\varepsilon}{k_n}}(i,l) \subset \sigma_{n,\varepsilon}(j,a) \subset \sigma_{n,\varepsilon k_n}(i,l)$$

and

$$\sigma_{n,\frac{\varepsilon}{k_n}}(j,a) \subset \sigma_{n,\varepsilon}(i,l) \subset \sigma_{n,\varepsilon k_n}(j,a).$$

Proof. We have $\sigma(i,l) = \sigma(j,a)$. If $\lambda \in \sigma_{n,\frac{\varepsilon}{k_n}}(i,l)$, then

$$\begin{aligned} \frac{\|w\|^{\frac{1}{2^n}}\|w^{-1}\|^{\frac{1}{2^n}}}{\varepsilon} &< \|(i-\lambda l)^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|((w(j-\lambda a)w^{-1})^{-1})^{2^n}\|^{\frac{1}{2^n}} \\ &\leq \|w\|^{\frac{1}{2^n}}\|w^{-1}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}, \end{aligned}$$

hence $\lambda \in \sigma_{n,\varepsilon}(j,a)$. Then $\sigma_{n,\frac{\varepsilon}{k_n}}(i,l) \subset \sigma_{n,\varepsilon}(j,a)$. Analogously, if $\lambda \in \sigma_{n,\varepsilon}(j,a)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|((w^{-1}(i-\lambda l)w)^{-1})^{2^n}\|^{\frac{1}{2^n}} \\ &\leq \|w\|^{\frac{1}{2^n}}\|w^{-1}\|^{\frac{1}{2^n}}\|(i-\lambda l)^{-2^n}\|^{\frac{1}{2^n}}, \end{aligned}$$

hence $\lambda \in \sigma_{n,\varepsilon k_n}(i,l)$. Thus $\sigma_{n,\varepsilon}(j,a) \subset \sigma_{n,\varepsilon k_n}(i,l)$. Utilizing analogous reasoning, we demonstrate that

$$\sigma_{n,\frac{\varepsilon}{k_n}}(j,a) \subset \sigma_{n,\varepsilon}(i,l) \subset \sigma_{n,\varepsilon k_n}(j,a).$$

$\square$

2.2. Condition (n,ε)-pseudospectra of an element pencil in unital complex Banach algebras

Now, we define the condition (n,ε)-pseudospectra of an element pencil in unital complex Banach algebras.

Definition 6. Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $0 < \varepsilon < 1$, the condition (n,ε)-pseudospectrum $\Lambda_{n,\varepsilon}(j,a)$ of an element pencil (j,a) of the form $j-\lambda a$ is

$$\Lambda_{n,\varepsilon}(j,a) = \sigma(j,a) \cup \{\lambda \in \mathbb{C} : \|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} > \frac{1}{\varepsilon}\},$$

with the convention $\|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} = \infty$, if $\lambda \in \sigma(j,a)$.

By Definition 6, we get the following remark.

Remark 2. If $n=0$, then $\Lambda_{0,\varepsilon}(j,a) = \Lambda_\varepsilon(j,a)$ is the condition pseudospectrum of an element pencil (j,a).

We get the following results.

THEOREM 6. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $0 < \varepsilon < 1$, then*

(i) *For each $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 \leq \varepsilon_2$, $\sigma(j,a) \subset \Lambda_{n,\varepsilon_1}(j,a) \subset \Lambda_{n,\varepsilon_2}(j,a)$;*

(ii) $\sigma(j,a) = \bigcap_{\varepsilon>0} \Lambda_{n,\varepsilon}(j,a)$.

Proof. (i) If $\lambda \in \Lambda_{n,\varepsilon_1}(j,a)$, hence $\|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} > \varepsilon_1^{-1} \geq \varepsilon_2^{-1}$, thus $\lambda \in \Lambda_{n,\varepsilon_2}(j,a)$.

(ii) Since for any $\varepsilon > 0$, $\sigma(j,a) \subseteq \Lambda_{n,\varepsilon}(j,a)$, hence $\sigma(j,a) \subseteq \bigcap_{\varepsilon>0} \Lambda_{n,\varepsilon}(j,a)$. For the converse inclusion, if $\lambda \in \bigcap_{\varepsilon>0} \Lambda_{n,\varepsilon}(j,a)$, hence for any $\varepsilon > 0$, $\lambda \in \Lambda_{n,\varepsilon}(j,a)$. If $\lambda \notin \sigma(j,a)$, thus $\lambda \in \{\lambda \in \mathbb{C} : \|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} > \varepsilon^{-1}\}$, for $\varepsilon \to 0^+$, we obtain $\|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} = \infty$ which is a contradiction with $\lambda \notin \sigma(j,a)$. Consequently $\lambda \in \sigma(j,a)$. $\square$

COROLLARY 2. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $0 < \varepsilon < 1$, then for any $n \in \mathbb{N}$, $\Lambda_{n+1,\varepsilon}(j,a) \subset \Lambda_{n,\varepsilon}(j,a)$.*

Proof. If $\lambda \in \Lambda_{n+1,\varepsilon}(j,a)$, hence

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(j-\lambda a)^{2^{n+1}}\|^{\frac{1}{2^{n+1}}} \|(j-\lambda a)^{-2^{n+1}}\|^{\frac{1}{2^{n+1}}} && (6) \\ &\leq \|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}, && (7) \end{aligned}$$

thus $\lambda \in \Lambda_{n,\varepsilon}(j,a)$. $\square$

PROPOSITION 5. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A}), n \in \mathbb{N}$ and $0 < \varepsilon < 1$. Then for any $\lambda \in \mathbb{C}^*$ and for each $\alpha \in \mathbb{C}, \Lambda_{n,\varepsilon}(\lambda j + \alpha a, a) = \alpha + \lambda \Lambda_{n,\varepsilon}(j,a)$.*

Proof. If $\mu \notin \sigma(\lambda j+\alpha a,a)$, then

$$\begin{aligned}(\lambda j+\alpha a-\mu a)^{-1}\in\mathscr{A} &\Leftrightarrow \lambda^{-1}(j-\frac{\mu-\alpha}{\lambda}a)^{-1}\in\mathscr{A}\\ &\Leftrightarrow (j-\frac{\mu-\alpha}{\lambda}a)^{-1}\in\mathscr{A},\end{aligned}$$

hence $\sigma(\lambda j+\alpha a,a)=\alpha+\lambda\sigma(j,a)$. Let $\mu\in\Lambda_{n,\varepsilon}(\lambda j+\alpha a,a)$, then

$$\begin{aligned}\frac{1}{\varepsilon} &< \|(\lambda j+\alpha a-\mu a)^{2^n}\|^{\frac{1}{2^n}}\|(\lambda j+\alpha a-\mu a)^{-2^n}\|^{\frac{1}{2^n}} && (8)\\ &\le |\lambda|^{-1}\|(j-\frac{\mu-\alpha}{\lambda}a)^{-2^n}\|^{\frac{1}{2^n}}|\lambda|\|(j-\frac{\mu-\alpha}{\lambda}a)^{2^n}\|^{\frac{1}{2^n}}, && (9)\end{aligned}$$

hence $\frac{\mu-\alpha}{\lambda}\in\Lambda_{n,\varepsilon}(j,a)$, thus $\mu\in\alpha+\lambda\Lambda_{n,\varepsilon}(j,a)$. Similarly, we obtain the converse inclusion. □

THEOREM 7. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j\in\mathscr{A}, a\in Inv(\mathscr{A})$, $n\in\mathbb{N}$ and $0<\varepsilon<1$. Then,*

$$\bigcup_{d\in\mathscr{A}:\|d\|<\varepsilon^{2^n}\|(j-\lambda a)^{2^n}\|}\sigma_n(j+d,a)\subset\Lambda_{n,\varepsilon}(j,a),$$

where $\sigma_n(j+d,a)=\{\lambda\in\mathbb{C}:(j-\lambda a)^{2^n}+d \quad \text{is not invertible}\}$.

Proof. Let $\bigcup_{d\in\mathscr{A}:\|d\|<\varepsilon^{2^n}\|(j-\lambda a)^{2^n}\|}\sigma_n(j+d,a)$. We argue by contradiction. Suppose that $\lambda\in\rho(j,a)$ and $\|(j-\lambda a)^{2^n}\|\|(j-\lambda a)^{-2^n}\|\le\varepsilon^{-2^n}$.

Consider f defined on $\mathscr{A}$ by

$$f=\sum_{k=0}^{\infty}(j-\lambda a)^{-2^n}\Big(-d(j-\lambda a)^{-2^n}\Big)^k.$$

From

$$\|d(j-\lambda a)^{-2^n}\|\le\|d\|\|(j-\lambda a)^{-2^n}\|<1$$

and Proposition 1, f is well-defined and in $\mathscr{A}$. Hence f can be written as follows $f=((j-\lambda a)^{2^n}+d)^{-1}$ which is a contradiction. Then $\lambda\in\Lambda_{n,\varepsilon}(j,a)$. □

PROPOSITION 6. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j\in\mathscr{A}, a\in Inv(\mathscr{A})$ such that $ja=aj$, $n\in\mathbb{N}$ and $0<\varepsilon<1$. Then*

$$\Lambda_{n,\frac{\varepsilon}{k}}(a^{-1}j)\subseteq\Lambda_{n,\varepsilon}(j,a)\subseteq\Lambda_{n,\varepsilon k}(a^{-1}j)$$

in which $k=\|a\|\|a^{-1}\|$.

Proof. We get $\sigma(a^{-1}j)=\sigma(j,a)$. If $\lambda\in\Lambda_{n,\frac{\varepsilon}{k}}(a^{-1}j)$, then

$$\begin{aligned}\frac{\|a\|\|a^{-1}\|}{\varepsilon} &< \|(a^{-1}j-\lambda 1)^{2^n}\|^{\frac{1}{2^n}}\|(a^{-1}j-\lambda 1)^{-2^n}\|^{\frac{1}{2^n}}\\ &= \|(a^{-1}(j-\lambda a))^{2^n}\|^{\frac{1}{2^n}}\|(a^{-1}(j-\lambda a))^{-2^n}\|^{\frac{1}{2^n}}\\ &\le \|a\|\|a^{-1}\|\|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}.\end{aligned}$$

From $\|a\|>0$, so $\frac{1}{\varepsilon}<\|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}$, hence $\lambda\in\Lambda_{n,\varepsilon}(j,a)$. If $\lambda\in\Lambda_{n,\varepsilon}(j,a)$, hence

$$\begin{aligned}\frac{1}{\varepsilon} &< \|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}\\ &= \|(a(a^{-1}j-\lambda 1))^{2^n}\|^{\frac{1}{2^n}}\|(a(a^{-1}j-\lambda 1))^{-2^n}\|^{\frac{1}{2^n}}\\ &\leq \|a^{-1}\|\|a\|\|(a^{-1}j-\lambda 1)^{2^n}\|^{\frac{1}{2^n}}\|(a^{-1}j-\lambda 1)^{-2^n}\|^{\frac{1}{2^n}},\end{aligned}$$

thus $\frac{1}{\varepsilon\|a^{-1}\|\|a\|}<\|(a^{-1}j-\lambda 1)^{2^n}\|^{\frac{1}{2^n}}\|(a^{-1}j-\lambda 1)^{-2^n}\|^{\frac{1}{2^n}}$, hence $\lambda\in\Lambda_{n,\varepsilon k}(a^{-1}j)$. □

Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j,a\in\mathscr{A}$, $n\in\mathbb{N}$ and $0<\varepsilon<1$. Define $\Gamma^n_{j,a}:\mathbb{C}\longrightarrow\mathbb{R}_+$ as

$$\Gamma^n_{j,a}(\lambda)=\begin{cases}\|(j-\lambda a)^{2^n}\|^{\frac{-1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{-1}{2^n}}, & \text{if } \lambda\notin\sigma(j,a)\\ 0, & \text{if } \lambda\in\sigma(j,a).\end{cases}$$

THEOREM 8. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j\in\mathscr{A}, a\in Inv(\mathscr{A})$, $n\in\mathbb{N}$ and $0<\varepsilon<1$. We have*

(i) $\Gamma^n_{j,a}$ *is continuous.*

(ii) $\Lambda_{n,\varepsilon}(j,a)=\{\lambda\in\mathbb{C}:\Gamma^n_{j,a}(\lambda)<\varepsilon\}$.

(iii) $\Lambda_{n,\varepsilon}(\alpha j,\alpha a)=\Lambda_{n,\varepsilon}(j,a)$ *for all* $\alpha\in\mathbb{C}\backslash\{0\}$.

(iv) *If $\mathscr{A}$ is a C^*-algebra, then $\lambda\in\Lambda_{n,\varepsilon}(j,a)$ if and only if $\overline{\lambda}\in\Lambda_{n,\varepsilon}(j^*,a^*)$.*

Proof. (i) We have

$$\begin{aligned}\sigma(j,a) &= \{\lambda\in\mathbb{C}: j-\lambda a \text{ is not invertible}\}\\ &= \{\lambda\in\mathbb{C}: ja^{-1}-\lambda \text{ is not invertible}\}\\ &= \sigma(ja^{-1}).\end{aligned}$$

Assume that $\lambda_k\in\mathbb{C}\backslash\sigma(j,a)$ such that $\lambda_k\longrightarrow\lambda$ for certain $\lambda\notin\sigma(j,a)$. Hence

$$\begin{aligned}\Gamma^n_{j,a}(\lambda_k) &= \|(j-\lambda_k a)^{2^n}\|^{\frac{-1}{2^n}}\|(j-\lambda_k a)^{-2^n}\|^{\frac{-1}{2^n}}\\ &\longrightarrow \|(j-\lambda a)^{2^n}\|^{\frac{-1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{-1}{2^n}}\\ &= \Gamma^n_{j,a}(\lambda).\end{aligned}$$

If $\lambda_k\in\mathbb{C}\backslash\sigma(j,a)$ such that $\lambda_k\longrightarrow\lambda$ for certain $\lambda\in\sigma(j,a)$. From Lemma 10.17 of [15], $\|(j-\lambda_k a)^{2^n}\|\|(j-\lambda_k a)^{-2^n}\|\longrightarrow\infty$ and $\Gamma^n_{j,a}(\lambda_k)\longrightarrow 0=\Gamma^n_{j,a}(\lambda)$. Then $\Gamma^n_{j,a}(\lambda)$ is continuous.

(ii) It follows from Definition 6 and the definition of $\Gamma^n_{j,a}$.

(iii) Let $\alpha\in\mathbb{C}\backslash\{0\}$, we get $\sigma(\alpha j,\alpha a)=\sigma(j,a)$. Then

$$\begin{aligned}\lambda\in\Lambda_{n,\varepsilon}(\alpha j,\alpha a)\backslash\sigma(\alpha j,\alpha a) &\iff \frac{1}{\varepsilon}<\|(\alpha j-\lambda\alpha a)^{2^n}\|^{\frac{1}{2^n}}\|(\alpha j-\lambda\alpha a)^{-2^n}\|^{\frac{1}{2^n}}\\ &\iff \frac{1}{\varepsilon}<\|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}\\ &\iff \lambda\in\Lambda_{n,\varepsilon}(j,a)\backslash\sigma(j,a).\end{aligned}$$

(iv) From Theorem 11.15 of [15], $\lambda\in\sigma(j,a)$ if and only if $\overline{\lambda}\in\sigma(j^*,a^*)$. If $\lambda\in\Lambda_{n,\varepsilon}(j,a)$, then by Proposition 1.4 and Proposition 1.7 of [5],

$$\|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}}=\|(j^*-\overline{\lambda}a^*)^{2^n}\|^{\frac{1}{2^n}}\|(j^*-\overline{\lambda}a^*)^{-2^n}\|^{\frac{1}{2^n}},$$

hence $\overline{\lambda}\in\Lambda_{n,\varepsilon}(j^*,a^*)$. Similarly, if $\overline{\lambda}\in\Lambda_{n,\varepsilon}(j^*,a^*)$, then $\lambda\in\Lambda_{n,\varepsilon}(j,a)$. □

THEOREM 9. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}, a \in Inv(\mathscr{A})$, $n \in \mathbb{N}$ and $0 < \varepsilon < 1$. We have*

(i) *If $ja = aj$ and $\|a^{2^n}\|^{\frac{1}{2^n}} \neq 0$, then*

$$\Lambda_{n, \frac{\varepsilon}{\|a^{-2^n}\|^{\frac{1}{2^n}} \|a^{2^n}\|^{\frac{1}{2^n}}}}(a^{-1}j) \subseteq \Lambda_{n,\varepsilon}(j,a) \subseteq \Lambda_{n,\varepsilon \|a^{2^n}\|^{\frac{1}{2^n}} \|a^{-2^n}\|^{\frac{1}{2^n}}}(a^{-1}j).$$

(ii) *If j is invertible, $\|a^{2^n}\|^{\frac{1}{2^n}}, \|j^{2^n}\|^{\frac{1}{2^n}} \neq 0$, $ja = aj$ and $k_n = \|j^{2^n}\|^{\frac{1}{2^n}} \|a^{-2^n}\|^{\frac{1}{2^n}} \|j^{-2^n}\|^{\frac{1}{2^n}} \|a^{2^n}\|^{\frac{1}{2^n}}$ for certain $\lambda \neq 0$, then*

$$\lambda \in \Lambda_{n,\varepsilon}(j^{-1},a) \Rightarrow \lambda^{-1} \in \Lambda_{n,\varepsilon k_n}(j,a^{-1})$$

and

$$\lambda \in \Lambda_{n,\varepsilon}(j,a^{-1}) \Rightarrow \lambda^{-1} \in \Lambda_{n,\varepsilon k_n}(j^{-1},a).$$

Proof. (i) We get $\sigma(a^{-1}j) = \sigma(j,a)$. If $\lambda \in \Lambda_{n, \frac{\varepsilon}{\|a^{2^n}\|^{\frac{1}{2^n}} \|a^{-2^n}\|^{\frac{1}{2^n}}}}(a^{-1}j)$, then

$$\begin{aligned} \frac{\|a^{2^n}\|^{\frac{1}{2^n}} \|a^{-2^n}\|^{\frac{1}{2^n}}}{\varepsilon} &< \|(a^{-1}j - \lambda 1)^{2^n}\|^{\frac{1}{2^n}} \|(a^{-1}j - \lambda 1)^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|(a^{-1}(j - \lambda a))^{2^n}\|^{\frac{1}{2^n}} \|(a^{-1}(j - \lambda a))^{-2^n}\|^{\frac{1}{2^n}} \\ &\leq \|a^{2^n}\|^{\frac{1}{2^n}} \|a^{-2^n}\|^{\frac{1}{2^n}} \|(j - \lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j - \lambda a)^{-2^n}\|^{\frac{1}{2^n}}. \end{aligned}$$

So $\lambda \in \Lambda_{n,\varepsilon}(j,a)$.

Further presume that $ja = aj$ and $\lambda \in \Lambda_{n,\varepsilon}(j,a)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(j - \lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j - \lambda a)^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|(a(a^{-1}j - \lambda 1))^{2^n}\|^{\frac{1}{2^n}} \|(a(a^{-1}j - \lambda 1))^{-2^n}\|^{\frac{1}{2^n}} \\ &\leq \|a^{-2^n}\|^{\frac{1}{2^n}} \|a^{2^n}\|^{\frac{1}{2^n}} \|(a^{-1}j - \lambda 1)^{2^n}\|^{\frac{1}{2^n}} \|(a^{-1}j - \lambda 1)^{-2^n}\|^{\frac{1}{2^n}}, \end{aligned}$$

thus $\lambda \in \Lambda_{n,\varepsilon \|a^{-2^n}\|^{\frac{1}{2^n}} \|a^{2^n}\|^{\frac{1}{2^n}}}(a^{-1}j)$.

(ii) Let us assume that j, a are invertible and $ja = aj$. Let $\lambda \neq 0$, then

$$\begin{aligned} \lambda \in \sigma(j^{-1},a) &\iff \lambda a - j^{-1} \text{ is not invertible} \\ &\iff -\lambda a(\frac{a^{-1}}{\lambda} - j)j^{-1} \text{ is not invertible} \\ &\iff \frac{1}{\lambda} \in \sigma(j,a^{-1}). \end{aligned}$$

If $\lambda \in \Lambda_{n,\varepsilon}(j^{-1},a) \backslash \sigma(j^{-1},a)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(j^{-1} - \lambda a)^{2^n}\|^{\frac{1}{2^n}} \|(j^{-1} - \lambda a)^{-2^n}\|^{\frac{1}{2^n}} \\ &= \|(\lambda a(\frac{a^{-1}}{\lambda} - j)j^{-1})^{2^n}\|^{\frac{1}{2^n}} \|(\lambda a(\frac{a^{-1}}{\lambda} - j)j^{-1})^{-2^n}\|^{\frac{1}{2^n}} \\ &\leq |\lambda|^{-1} \|a^{-2^n}\|^{\frac{1}{2^n}} \|j^{2^n}\|^{\frac{1}{2^n}} \|(\frac{a^{-1}}{\lambda} - j)^{-2^n}\|^{\frac{1}{2^n}} |\lambda| \|a^{2^n}\|^{\frac{1}{2^n}} \|j^{-2^n}\|^{\frac{1}{2^n}} \|(\frac{a^{-1}}{\lambda} - j)^{2^n}\|^{\frac{1}{2^n}}. \end{aligned}$$

So $\lambda^{-1} \in \Lambda_{n,\varepsilon k_n}(j,a^{-1})$.

On the other hand, if $\lambda \in \Lambda_{n,\varepsilon}(j,a^{-1})$, then

$$\begin{aligned}
\frac{1}{\varepsilon} &< \|(j-\lambda a^{-1})^{2^n}\|^{\frac{1}{2^n}}\|(j-\lambda a^{-1})^{-2^n}\|^{\frac{1}{2^n}} \\
&= \|(\lambda a^{-1}(\frac{a}{\lambda}-j^{-1})j)^{2^n}\|^{\frac{1}{2^n}}\|(\lambda a^{-1}(\frac{a}{\lambda}-j^{-1})j)^{-2^n}\|^{\frac{1}{2^n}} \\
&\leq \|a^{2^n}\|^{\frac{1}{2^n}}\|j^{-2^n}\|^{\frac{1}{2^n}}\|a^{-2^n}\|^{\frac{1}{2^n}}\|j^{2^n}\|^{\frac{1}{2^n}}\|(\frac{a}{\lambda}-j^{-1})^{2^n}\|^{\frac{1}{2^n}}\|(\frac{a}{\lambda}-j^{-1})^{-2^n}\|^{\frac{1}{2^n}}.
\end{aligned}$$

So $\lambda^{-1} \in \Lambda_{n,\varepsilon k_n}(j^{-1},a)$. □

THEOREM 10. *Let $\mathscr{A}$ be a unital complex Banach algebra. Let $j \in \mathscr{A}$, $a,w \in Inv(\mathscr{A})$, $n \in \mathbb{N}$ and $0 < \varepsilon < 1$. Set $k_n = \|w\|^{\frac{1}{2^n}}\|w^{-1}\|^{\frac{1}{2^n}}$, $i = wjw^{-1}$ and $l = waw^{-1}$, then*

$$\Lambda_{n,\frac{\varepsilon}{k_n^2}}(i,l) \subset \Lambda_{n,\varepsilon}(j,a) \subset \Lambda_{n,\varepsilon k_n^2}(i,l)$$

and

$$\Lambda_{n,\frac{\varepsilon}{k_n^2}}(j,a) \subset \Lambda_{n,\varepsilon}(i,l) \subset \Lambda_{n,\varepsilon k_n^2}(j,a).$$

Proof. One can see that $\sigma(j,a) = \sigma(i,l)$. If $\lambda \in \Lambda_{n,\frac{\varepsilon}{k_n^2}}(i,l)$, then

$$\begin{aligned}
\frac{k_n^2}{\varepsilon} &< \|(i-\lambda l)^{2^n}\|^{\frac{1}{2^n}}\|(i-\lambda l)^{-2^n}\|^{\frac{1}{2^n}} \\
&= \|((w(j-\lambda a)w^{-1}))^{2^n}\|^{\frac{1}{2^n}}\|((w(j-\lambda a)w^{-1})^{-1})^{2^n}\|^{\frac{1}{2^n}} \\
&\leq \|w\|^{\frac{1}{2^n}}\|w^{-1}\|^{\frac{1}{2^n}}\|w\|^{\frac{1}{2^n}}\|w^{-1}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}} \times \\
&\quad \|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}},
\end{aligned}$$

hence $\lambda \in \Lambda_{n,\varepsilon}(j,a)$. Then $\Lambda_{n,\frac{\varepsilon}{k_n^2}}(i,l) \subset \Lambda_{n,\varepsilon}(j,a)$.

Analogously, if $\lambda \in \Lambda_{n,\varepsilon}(j,a)$, then

$$\begin{aligned}
\frac{1}{\varepsilon} &< \|(j-\lambda a)^{2^n}\|^{\frac{1}{2^n}}\|(j-\lambda a)^{-2^n}\|^{\frac{1}{2^n}} \\
&= \|(w^{-1}(i-\lambda l)w)^{2^n}\|^{\frac{1}{2^n}}\|((w^{-1}(i-\lambda l)w)^{-1})^{2^n}\|^{\frac{1}{2^n}} \\
&\leq (\|w\|^{\frac{1}{2^n}}\|w^{-1}\|^{\frac{1}{2^n}})^2\|(i-\lambda l)^{2^n}\|^{\frac{1}{2^n}}\|(i-\lambda l)^{-2^n}\|^{\frac{1}{2^n}},
\end{aligned}$$

hence $\lambda \in \Lambda_{n,\varepsilon k_n^2}(i,l)$. Thus $\Lambda_{n,\varepsilon}(j,a) \subset \Lambda_{n,\varepsilon k_n^2}(i,l)$. Utilizing analogous reasoning, we demonstrate that

$$\Lambda_{n,\frac{\varepsilon}{k_n^2}}(j,a) \subset \Lambda_{n,\varepsilon}(i,l) \subset \Lambda_{n,\varepsilon k_n^2}(j,a).$$

□

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