



NECESSARY AND SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF COMMUTATORS OF MULTI-SUBLINEAR MAXIMAL FUNCTIONS

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Abstract. In this paper, we investigated two different kinds of commutators associated with the multi-sublinear maximal function \mathcal{M}_α when symbol functions belong to Lipschitz spaces. We give some necessary and sufficient conditions for the boundedness of these two different kinds of commutators on the products of Slice space.

Keywords: multi-sublinear maximal function, Lipschitz space, commutator, slice space.

Mathematics Subject Classification (MSC2020): 42B20, 42B25, 47B47.

1. INTRODUCTION AND MAIN RESULTS

Let T a classical singular integral operator, the commutator $[T, b]$ generated by T and a suitable function b as follows

$$[T, b]f(x) = T((b(x) - b)f)(x) = b(x)T(f)(x) - T(bf)(x). \quad (1)$$

A well-known result due to Coifman, Rochberg and Weiss [1] states that $[T, b]$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$, where $1 < p < \infty$. Moreover, Janson [2] also gave some characterizations of Lipschitz spaces via the commutator. It was proved that $[T, b]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, where $0 < \beta < 1$, $1 < p < n/\beta$, $\beta = n(1/p - 1/q)$.

Let $0 \leq \alpha < n$, for a locally integrable function f , the maximal function is defined by

$$M_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x .

The maximal commutator of M_α with a locally integrable function b is defined by

$$M_{\alpha,b}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |b(x)f(y) - b(y)f(y)| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x . $[M_\alpha, b]$ can refer to (1).

When $\alpha = 0$, we simply denote by $[M_\alpha, b] = [M, b]$ and $M_{\alpha,b} = M_b$.

In 2000, Bastero, Milman and Ruiz [3] gave the necessary and sufficient conditions of the boundedness of $[M, b]$ and $[M^\#, b]$ on Lebesgue spaces. In 2009, the authors [4] considered the same problem for the fractional maximal function. Then, the commutators theory of maximal functions have been studied intensively by many authors (see [5–18]). In 2017, Zhang [8] gave the necessary and sufficient conditions of the boundedness of M_b and $[M, b]$ on Lebesgue spaces and Morrey spaces when the symbol b belong to Lipschitz spaces, by which some new characterizations of Lipschitz functions are given. In 2018, Zhang, Wu and Sun [11] considered

the boundedness for the commutator of the fractional maximal function and sharp maximal function on Orlicz spaces. Recently, Yang and Zhou [14, 17] studied necessary and sufficient conditions for the boundedness of M_b , $[M, b]$, $M_{\alpha, b}$ and $[M_{\alpha}, b]$ on Slice spaces when the function b belongs to Lipschitz spaces.

On the other hand, multilinear Calderón-Zygmund operators were introduced and first studied by Coifman and Meyer [19]. The theory was then further investigated in the last few decades. In 2014, Lu and Zhang [20] studied the multilinear commutators of multilinear Calderón-Zygmund operators. In 2022, Lu, Wang and Zhou [21] considered some classical multilinear operators on weighted amalgam spaces, such as the multilinear Hardy-Littlewood maximal operator, the multilinear Calderón-Zygmund operator and so on. Moreover, it turns out that the multi-sublinear maximal functions play an important role in establishing the multiple-weight theory. Now, we recall the definition of the multi-sublinear maximal functions.

Definition 1. Let $0 \leq \alpha < mn$, for a collection of integrable functions $\vec{f} = (f_1, \dots, f_m)$, the multi-sublinear maximal function \mathcal{M}_{α} is defined by $\mathcal{M}_{\alpha}(\vec{f})(x) = \sup_{Q \ni x} \frac{1}{|Q|^{m-\alpha/n}} \prod_{j=1}^m \int_Q |f_j(y_j)| dy_j$, where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x .

When $\alpha = 0$, \mathcal{M}_0 is the multilinear maximal function denoted by \mathcal{M} , and \mathcal{M}_{α} is the multilinear fractional maximal function when $0 < \alpha < mn$. When $\alpha = 0, m = 1$, \mathcal{M}_0 is the classical Hardy-Littlewood maximal function, and \mathcal{M}_{α} is the classical fractional maximal function when $0 < \alpha < n$.

As usual, Denote by $|Q|$ the Lebesgue measure of Q , χ_Q the characteristic function of Q and write $Q^m = (Q)^m = \underbrace{Q \times \dots \times Q}_m$, $\vec{f} = (f_1, \dots, f_m)$, $\vec{y} = (y_1, \dots, y_m)$ and $d\vec{y} = dy_1 \dots dy_m$. We also use $\vec{b} = (b_1, \dots, b_m) \in (\dot{\Lambda}_{\beta})^m$ to stand for $b_j \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$ for $j = 1, \dots, m$.

Definition 2. Let $0 \leq \alpha < mn$, $\vec{f} = (f_1, \dots, f_m)$ and $\vec{b} = (b_1, \dots, b_m)$ be two collections of locally integrable functions. The maximal commutator of \mathcal{M}_{α} with \vec{b} is defined by $\mathcal{M}_{\alpha, \vec{b}}(\vec{f})(x) = \sum_{i=1}^m \mathcal{M}_{\alpha, \vec{b}}^i(\vec{f})(x)$, where

$$\mathcal{M}_{\alpha, \vec{b}}^i(\vec{f})(x) = \sup_{Q \ni x} \frac{1}{|Q|^{m-\alpha/n}} \int_{Q^m} |b_i(x) - b_i(y_i)| \prod_{j=1}^m |f_j(y_j)| d\vec{y},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x .

The nonlinear commutators of \mathcal{M}_{α} with \vec{b} is given by $[\mathcal{M}_{\alpha}, \vec{b}](\vec{f})(x) = \sum_{i=1}^m [\mathcal{M}_{\alpha}, \vec{b}]_i(\vec{f})(x)$, where

$$[\mathcal{M}_{\alpha}, \vec{b}]_i(\vec{f})(x) = [\mathcal{M}_{\alpha}, b_i]_i(\vec{f})(x) = b_i \mathcal{M}_{\alpha}(\vec{f})(x) - \mathcal{M}_{\alpha}(f_1, \dots, b_i f_i, \dots, f_m)(x).$$

When $\alpha = 0$, we simply denote by $[\mathcal{M}_{\alpha}, \vec{b}] = [\mathcal{M}, \vec{b}]$ and $\mathcal{M}_{\alpha, \vec{b}} = \mathcal{M}_{\vec{b}}$. We would like to remark that operators $\mathcal{M}_{\alpha, \vec{b}}$ and $[\mathcal{M}_{\alpha}, \vec{b}]$ essentially differ from each other. For example, $\mathcal{M}_{\alpha, \vec{b}}$ is positive and sublinear, but $[\mathcal{M}_{\alpha}, \vec{b}]$ is neither positive nor sublinear.

In 2015, Zhang [22] considered the mapping properties of the commutators $\mathcal{M}_{\vec{b}}$ and $[\mathcal{M}, \vec{b}]$ when the symbols belongs to $BMO(\mathbb{R}^n)$. In 2019, Zhang [23] studied the boundedness of $\mathcal{M}_{\alpha, \vec{b}}$ and $[\mathcal{M}_{\alpha}, \vec{b}]$ when the symbols belongs to Lipschitz spaces. Recently, Zhang and Ağcayazi gave necessary and sufficient conditions for the boundedness of $\mathcal{M}_{\alpha, \vec{b}}$ and $[\mathcal{M}_{\alpha}, \vec{b}]$ on the products of Morrey spaces when the symbol belongs to Lipschitz spaces.

Our main aim of this paper is to find the necessary and sufficient conditions for the boundedness of $\mathcal{M}_{\alpha, \vec{b}}$ and $[\mathcal{M}_{\alpha}, \vec{b}]$ on the products of Slice spaces, where symbols belongs to Lipschitz spaces. Our results extend the results in [17] to the multilinear setting and also extend the results in [23] to the Slice spaces.

To state the results, we first give some definitions and notations.

Definition 3. The Lipschitz space of order β , $0 < \beta < 1$, is the space of function f , such that

$$\dot{\Lambda}_{\beta} = \{f : |f(x) - f(y)| \leq C|x - y|^{\beta}\},$$

and the smallest constant $C > 0$ is the Lipschitz norm $\|\cdot\|_{\dot{\Lambda}_{\beta}}$.

In 2019, Auscher and Mourougolou [25] introduced the slice space $(E_2^p)_t$ with $0 < t < \infty$ and $1 < p < \infty$, they studied the weak solutions of boundary value problems with a t -independent elliptic systems in the upper half plane. Recently, Auscher and Prisuelos-Arribas [26] studied the boundedness of some classical operators on the Slice space $(E_r^p)_t$ with $0 < t < \infty$ and $1 < p, r < \infty$. Lu, Zhou and Wang [27] established the necessary and sufficient conditions for boundedness of commutators associated with the Calderón-Zygmund operators on Slice spaces. We shall recall the definition of the Slice space.

Definition 4. Let $0 < t < \infty$ and $1 < r, p < \infty$. The slice space $(E_r^p)_t(\mathbb{R}^n)$ is defined as the set of all locally r -integrable functions f on \mathbb{R}^n such that

$$\|f\|_{(E_r^p)_t(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(\frac{1}{|Q(x,t)|} \int_{Q(x,t)} |f(y)|^r dy \right)^{p/r} dx \right)^{1/p} < \infty.$$

If we take $r = p$, then the Slice space $(E_r^p)_t(\mathbb{R}^n)$ is the Lebesgue space $L^p(\mathbb{R}^n)$. For a cube Q , we denote by $\|f\|_{(E_r^p)_t(Q)} = \|f\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)}$.

Definition 5. Let $0 \leq \alpha < mn$ and $\vec{f} = (f_1, \dots, f_m)$ be a collection of locally integrable functions, the multi-sublinear maximal function with respect to Q is given by

$$\mathcal{M}_{\alpha,Q}(\vec{f})(x) = \sup_{\substack{Q_0 \ni x \\ Q_0 \subset Q}} \frac{1}{|Q_0|^{m-\alpha/n}} \prod_{j=1}^m \int_{Q_0} |f_j(y)| dy, \quad x \in Q.$$

where the supremum is taken over all the cubes $Q_0 \subset Q$ and $Q_0 \ni x$.

When $\alpha = 0$, we denote by $\mathcal{M}_Q = \mathcal{M}_{0,Q}$; when $m = 1$, we denote by $M_{\alpha,Q} = \mathcal{M}_{\alpha,Q}$; when $\alpha = 0$ and $m = 1$, we denote by $M_Q = \mathcal{M}_{0,Q}$.

Our main results in this paper can be stated as follows.

THEOREM 1. Let $0 < \beta < 1$, $0 \leq \alpha < mn$, $0 < \alpha + \beta < mn$, $0 < t < \infty$ and $\vec{b} = (b_1, \dots, b_m)$ be a collection of locally integrable function. If $1 < p, p_1, \dots, p_m < \infty$, $1 < q, q_1, \dots, q_m < \infty$ and $(\alpha + \beta)/n = 1/p - 1/r = 1/q - 1/s$, where $1/p = \sum_{i=1}^m 1/p_i$ and $1/q = \sum_{i=1}^m 1/q_i$. Then the following assertions are equivalent:

- (i) $\vec{b} \in (\dot{\Lambda}_\beta)^m$;
- (ii) $\mathcal{M}_{\alpha,\vec{b}}$ is bounded from $(E_{p_1}^{q_1})_t(\mathbb{R}^n) \times \dots \times (E_{p_m}^{q_m})_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that for all $f_j \in (E_{p_j}^{q_j})_t(\mathbb{R}^n)$ ($j = 1, \dots, m$),

$$\|\mathcal{M}_{\alpha,\vec{b}}(\vec{f})\|_{(E_r^s)_t(\mathbb{R}^n)} \leq C \|\vec{b}\|_{(\dot{\Lambda}_\beta)^m} \prod_{j=1}^m \|f_j\|_{(E_{p_j}^{q_j})_t(\mathbb{R}^n)}.$$

Remark 1. When $m = 1$, the result was proved in [17] (see Theorem 1.4). When $p_j = q_j$ and $s = r$ the result was proved in [23] (see Theorem 1.1).

THEOREM 2. Let $0 < \beta < 1$, $0 \leq \alpha < mn$, $0 < \alpha + \beta < mn$, $0 < t < \infty$ and $\vec{b} = (b_1, \dots, b_m)$ be a collection of locally integrable function. If $1 < p, p_1, \dots, p_m < \infty$, $1 < q, q_1, \dots, q_m < \infty$ and $(\alpha + \beta)/n = 1/p - 1/r = 1/q - 1/s$, where $1/p = \sum_{i=1}^m 1/p_i$ and $1/q = \sum_{i=1}^m 1/q_i$.

- (i) If $\vec{b} \in (\dot{\Lambda}_\beta)^m$ and $b_j \geq 0$ for $j = 1, \dots, m$, then $[\mathcal{M}_{\alpha,\vec{b}}]$ is bounded from $(E_{p_1}^{q_1})_t(\mathbb{R}^n) \times \dots \times (E_{p_m}^{q_m})_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$;
- (ii) If $[\mathcal{M}_{\alpha,\vec{b}}]_i$ is bounded from $(E_{p_1}^{q_1})_t(\mathbb{R}^n) \times \dots \times (E_{p_m}^{q_m})_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, then $b_i \in \dot{\Lambda}_\beta$ and $b_i \geq 0$ for $i = 1, \dots, m$.

Remark 2. When $m = 1$, the result was proved in [17] (see Theorem 1.3). When $p_j = q_j$ and $s = r$ the result was proved in [23] (see Theorem 1.4).

2. PRELIMINARIES AND LEMMAS

In this section, we recall some known preliminaries and lemmas. It is known that the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ coincides with some Morrey-Companato space (see [28] for example) and can be characterized by mean oscillation as follows, which is due to Janson, Taibleson and Weiss [28] and Paluszyński [29].

LEMMA 1. *Let $0 < \beta < 1$ and $1 \leq q < \infty$. Define*

$$\dot{\Lambda}_{\beta,q}(\mathbb{R}^n) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(y) - f_Q|^q dy \right)^{\frac{1}{q}} < \infty \right\}.$$

Then, for all $0 < \beta < 1$ and $1 \leq p < \infty$, $\dot{\Lambda}_\beta(\mathbb{R}^n) = \dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ with equivalent norms.

From [13], we have the following characterization of non-negative Lipschitz functions.

LEMMA 2 [13]. *Let $0 < \beta < 1$ and b be a locally integrable function. Then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$ if and only if there exists a constant $C > 0$ such that $\sup_Q |Q|^{-1-\beta/n} \int_Q |b(x) - M_Q(b)(x)| dx \leq C$.*

The following the lemma is given in [21](see Theorem 2.1 and Theorem 6.1), they obtained that the boundedness of the multi-sublinear maximal operator $\mathcal{M}_\alpha(0 \leq \alpha < mn)$ on Slice spaces.

LEMMA 3. *Let $0 < t < \infty$, $0 \leq \alpha < mn$, $1 < p, p_1, \dots, p_m < \infty$ and $1 < q, q_1, \dots, q_m < \infty$ with $\alpha/n = 1/p - 1/r = 1/q - 1/s$, where $1/p = \sum_{j=1}^m 1/p_j$ and $1/q = \sum_{j=1}^m 1/q_j$. If $f_j \in (E_{p_j}^{q_j})_t(\mathbb{R}^n)$ ($j = 1, \dots, m$), then*

$$\left\| \mathcal{M}_\alpha(\vec{f}) \right\|_{(E_r^s)_t(\mathbb{R}^n)} \leq C \prod_{j=1}^m \|f_j\|_{(E_{p_j}^{q_j})_t(\mathbb{R}^n)},$$

where the positive constant C is independent of \vec{f} and t .

LEMMA 4 [27]. *Let $0 < t < \infty$, $1 < p, r < \infty$ and Q be a cube in \mathbb{R}^n , then $\|\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)} \approx |Q|^{1/p}$.*

LEMMA 5 [23]. *Let $0 \leq \alpha < mn$ and $\vec{b} = (b_1, \dots, b_m)$ be a collection of nonnegative locally integrable function. Then for any $\vec{f} = (f_1, \dots, f_m)$ with $f_j \in L_{\text{loc}}(\mathbb{R}^n)$ ($j = 1, \dots, m$),*

$$|[\mathcal{M}_\alpha, \vec{b}](\vec{f})(x)| \leq \mathcal{M}_{\alpha, \vec{b}}(\vec{f})(x).$$

LEMMA 6 [23]. *Let $0 \leq \gamma < mn$, f be a locally integrable function and Q_0 be a cube in \mathbb{R}^n . Then for any $1 \leq j \leq m$ and all $x \in Q_0$,*

$$\mathcal{M}_{\gamma, Q_0}(\underbrace{\chi_{Q_0}, \dots, \chi_{Q_0}}_{j-1}, \underbrace{f \chi_{Q_0}, \chi_{Q_0}, \dots, \chi_{Q_0}}_{m-j})(x) = \mathcal{M}_\gamma(\underbrace{\chi_{Q_0}, \dots, \chi_{Q_0}}_{j-1}, \underbrace{f \chi_{Q_0}, \chi_{Q_0}, \dots, \chi_{Q_0}}_{m-j})(x)$$

and

$$\mathcal{M}_{\gamma, Q_0}(\underbrace{\chi_{Q_0}, \dots, \chi_{Q_0}}_m)(x) = \mathcal{M}_\gamma(\underbrace{\chi_{Q_0}, \dots, \chi_{Q_0}}_m)(x) = |Q_0|^{\gamma/n}.$$

3. PROOF OF MAIN RESULTS

In this section, we prove Theorems 1 and 2. For notational convenience, we will write $\vec{\chi}_Q = (\underbrace{\chi_Q, \dots, \chi_Q}_m)$

and $f \circ \vec{\chi}_Q^i = (\underbrace{\chi_Q, \dots, \chi_Q}_{i-1}, \underbrace{f \chi_Q, \chi_Q, \dots, \chi_Q}_{m-i})$ in the following.

Proof of Theorem 1. (i) \Rightarrow (ii) Assume $\vec{b} \in (\dot{\Lambda}_\beta)^m$, for any $x \subset \mathbb{R}^n$, we have

$$\begin{aligned} \mathcal{M}_{\alpha, \vec{b}}^i(\vec{f})(x) &= \sup_{Q \ni x} \frac{1}{|Q|^{m-\alpha/n}} \int_{Q^m} |b_i(x) - b_i(y_j)| \prod_{j=1}^m |f_j(y_j)| dy_j \\ &\leq C \|b_i\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \mathcal{M}_{\alpha+\beta}(\vec{f})(x). \end{aligned}$$

Because of $\mathcal{M}_{\alpha, \vec{b}}(\vec{f})(x) = \sum_{i=1}^m \mathcal{M}_{\alpha, \vec{b}}^i(\vec{f})(x) \leq C (\sum_{i=1}^m \|b_i\|_{\dot{\Lambda}_\beta}) \mathcal{M}_{\alpha+\beta}(\vec{f})(x)$, together with Lemma 3, shows that $\mathcal{M}_{\alpha, \vec{b}}$ is bounded from $(E_{p_1}^{q_1})_t(\mathbb{R}^n) \times \cdots \times (E_{p_m}^{q_m})_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(ii) \Rightarrow (i) It suffices to prove $b_i \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ for $i = 1, \dots, m$. For any fixed cube Q , we have ([23], page 9)

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - (b_i)_Q| dx \leq \frac{1}{|Q|^{1+(\alpha+\beta)/n}} \int_Q \mathcal{M}_{\alpha, \vec{b}}(\vec{\chi}_Q)(x) dx.$$

Since $\mathcal{M}_{\alpha, \vec{b}}$ is bounded from $(E_{p_1}^{q_1})_t(\mathbb{R}^n) \times \cdots \times (E_{p_m}^{q_m})_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, by Hölder's inequality, Lemma 4 and noting that $(\alpha + \beta)/n = 1/q - 1/s$, it is easy to see that

$$\begin{aligned} \frac{1}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - (b_i)_Q| dx &\leq \frac{C}{|Q|^{1+(\alpha+\beta)/n}} \left\| \mathcal{M}_{\alpha, \vec{b}}(\vec{\chi}_Q) \right\|_{(E_r^s)_t(Q)} \|\chi_Q\|_{(E_{r'}^s)_t(Q)} \\ &\leq \frac{C}{|Q|^{1/s+(\alpha+\beta)/n}} \prod_{j=1}^m \|\chi_Q\|_{(E_{p_j}^{q_j})_t(Q)} \leq C, \end{aligned}$$

where $1/s + 1/s' = 1, 1/r + 1/r' = 1$. This together with Lemma 1 gives $b_i \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

Proof of Theorem 2. (i) \Rightarrow (ii) Obviously, it follows from Lemma 5 and (i) \Rightarrow (ii) of Theorem 1.

(ii) \Rightarrow (i) By Lemma 2, it suffices to prove

$$\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - M_Q(b_i)(x)| dx \leq C. \quad (2)$$

We divide the proof into two cases according to the value of α .

Case 1. Let $\alpha = 0$. For any fixed cube Q and any $x \in Q$, Lemma 6 gives

$$\mathcal{M}(b_i \circ \vec{\chi}_Q^i)(x) = \mathcal{M}_Q(b_i \circ \vec{\chi}_Q^i)(x) \text{ and } \mathcal{M}(\vec{\chi}_Q)(x) = 1.$$

For any $x \in Q$, we note that $\mathcal{M}_Q(b_i \circ \vec{\chi}_Q^i)(x) = M_Q(b_i)(x)$, then

$$b_i(x) - M_Q(b_i)(x) = b_i(x) \mathcal{M}(\vec{\chi}_Q)(x) - \mathcal{M}(b_i \circ \vec{\chi}_Q^i)(x) = [\mathcal{M}, \vec{b}]_i(\vec{\chi}_Q)(x).$$

Since $[\mathcal{M}, \vec{b}]_i$ is bounded from $(E_{p_1}^{q_1})_t(\mathbb{R}^n) \times \cdots \times (E_{p_m}^{q_m})_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, by Hölder's inequality, Lemma 4 and $\beta/n = 1/q - 1/s = 1/q_1 + \cdots + 1/q_m - 1/s$, we have

$$\begin{aligned} \frac{1}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - M_Q(b_i)(x)| dx &= \frac{1}{|Q|^{1+\beta/n}} \int_Q |[\mathcal{M}, \vec{b}]_i(\vec{\chi}_Q)(x)| dx \\ &\leq \frac{C}{|Q|^{1+\beta/n}} \left\| [\mathcal{M}, \vec{b}]_i(\vec{\chi}_Q) \right\|_{(E_r^s)_t(Q)} \|\chi_Q\|_{(E_{r'}^s)_t(Q)} \\ &\leq \frac{C}{|Q|^{1/s+\beta/n}} \prod_{j=1}^m \|\chi_Q\|_{(E_{p_j}^{q_j})_t(Q)} \leq C, \end{aligned}$$

which implies (ii) since Q is arbitrary and the constant C is independent of Q .

Case 2. Let $0 < \alpha < mn$. For any fixed Q and any $x \in Q$, by Hölder's inequality and Lemma 4, then

$$\begin{aligned} \frac{1}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - M_Q(b_i)(x)| dx &\leq \frac{1}{|Q|^{\beta/n+1/s}} \|b_i - M_Q(b_i)\|_{(E_r^s)_t(Q)} \\ &\leq \frac{1}{|Q|^{\beta/n+1/s}} \left\| b_i - |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i) \right\|_{(E_r^s)_t(Q)} \\ &\quad + \frac{1}{|Q|^{\beta/n+1/s}} \left\| |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i) - M_Q(b_i) \right\|_{(E_r^s)_t(Q)} \\ &:= I_1 + I_2, \end{aligned}$$

For I_1 , for any fixed $Q \subset \mathbb{R}^n$ and $x \in Q$, by Lemma 6, it is not difficult to check that

$$\mathcal{M}_\alpha(\vec{\chi}_Q)(x) = \mathcal{M}_{\alpha,Q}(\vec{\chi}_Q)(x) = |Q|^{\alpha/n} \text{ and } \mathcal{M}_\alpha(b_i \circ \vec{\chi}_Q^i)(x) = \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i)(x).$$

Since $[\mathcal{M}_\alpha, \vec{b}]_i$ is bounded from $(E_{p_1}^{q_1})_t(\mathbb{R}^n) \times \cdots \times (E_{p_m}^{q_m})_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$ and noting that $(\alpha + \beta)/n = 1/q - 1/s = 1/q_1 + \cdots + 1/q_m - 1/s$, then

$$\begin{aligned} I_1 &= \frac{1}{|Q|^{\beta/n+1/s}} \left\| b_i - |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i) \right\|_{(E_r^s)_t(Q)} \\ &= \frac{1}{|Q|^{(\alpha+\beta)/n+1/s}} \left\| [\mathcal{M}_\alpha, \vec{b}]_i(\vec{\chi}_Q) \right\|_{(E_r^s)_t(Q)} \\ &\leq \frac{C}{|Q|^{1/q}} \prod_{j=1}^m \|\chi_Q\|_{(E_{p_j}^{q_j})_t(Q)} \leq C, \end{aligned} \tag{3}$$

where in the last step we have applied Lemma 4.

Next, we estimate I_2 . For any $x \in Q$, by Lemma 6, we obtain

$$\begin{aligned} &\left| |Q|^{-\frac{\alpha}{n}} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i)(x) - M_Q(b_i)(x) \right| \\ &\leq |Q|^{-\frac{\alpha}{n}} \left| \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i)(x) - |b_i(x)| |Q|^{\frac{\alpha}{n}} \right| + \left| |b_i(x)| - M_Q(b_i)(x) \right| \\ &= |Q|^{-\frac{\alpha}{n}} \left| \mathcal{M}_\alpha(b_i \circ \vec{\chi}_Q^i)(x) - |b_i(x)| \mathcal{M}_\alpha(\vec{\chi}_Q)(x) \right| + \left| |b_i(x)| \mathcal{M}(\vec{\chi}_Q)(x) - \mathcal{M}(b_i \circ \vec{\chi}_Q^i)(x) \right| \\ &= |Q|^{-\frac{\alpha}{n}} \left| [|b_i|, \mathcal{M}_\alpha]_i(\vec{\chi}_Q)(x) \right| + \left| [|b_i|, \mathcal{M}]_i(\vec{\chi}_Q)(x) \right|. \end{aligned} \tag{4}$$

For any fixed cube Q , by Hölder's inequality, Lemma 4 and (3), it is easy to see that

$$\begin{aligned} &\frac{1}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i)(x)| dx \\ &\leq \frac{C}{|Q|^{1+\beta/n}} \left\| b_i - |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i) \right\|_{(E_r^s)_t(Q)} \|\chi_Q\|_{(E_r^s)_t(Q)} \\ &\leq \frac{C}{|Q|^{1/s+\beta/n}} \left\| b_i - |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i) \right\|_{(E_r^s)_t(Q)} \leq C. \end{aligned} \tag{5}$$

For any cube $Q \subset \mathbb{R}^n$, noticing the obvious estimate

$$|(b_i)_Q| \leq |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i)(x) \text{ for } x \in Q.$$

Let $E = \{x \in Q, b_i(x) \leq (b_i)_Q\}$ and $F = \{x \in Q, b_i(x) > (b_i)_Q\}$, then

$$\int_F |b_i(x) - (b_i)_Q| dx = \int_E |b_i(x) - (b_i)_Q| dx.$$

Since for any $x \in E$, $b_i(x) \leq (b_i)_Q \leq |(b_i)_Q| \leq |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i)(x)$, by using (5), we have

$$\begin{aligned} \frac{1}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - (b_i)_Q| dx &= \frac{2}{|Q|^{1+\beta/n}} \int_E |b_i(x) - (b_i)_Q| dx \\ &\leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i)(x)| dx \leq C. \end{aligned}$$

Thus, $b_i \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, which implies $|b_i| \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

By the definitions of $[\mathcal{M}_\alpha, \vec{b}]_i$ and \mathcal{M}_α and Lemma 6, we have, for any $x \in Q$,

$$\begin{aligned} |[\mathcal{M}_\alpha, |b_i|]_i(\vec{\chi}_Q)(x)| &= ||b_i(x)| \mathcal{M}_\alpha(\vec{\chi}_Q)(x) - \mathcal{M}_\alpha(|b_i| \circ \vec{\chi}_Q^i)(x)| \\ &\leq \sup_{Q' \ni x} \frac{1}{|Q'|^{m-\alpha/n}} \int_{Q'} |b_i(x) - b_i(y_i)| \prod_{j=1}^m \int_{Q'} |\chi_Q(y_j)| dy_j \\ &\leq C \|b_i\|_{\dot{\Lambda}_\beta} \sup_{Q' \ni x} \frac{1}{|Q'|^{m-(\alpha+\beta)/n}} \prod_{j=1}^m \int_{Q'} |\chi_Q(y_j)| dy_j \\ &= C \|b_i\|_{\dot{\Lambda}_\beta} |Q|^{(\alpha+\beta)/n}, \end{aligned}$$

where

$$\mathcal{M}_\alpha(|b_i| \circ \vec{\chi}_Q^i)(x) = \sup_{Q' \ni x} \frac{1}{|Q'|^{m-\alpha/n}} \left(\int_{Q'} |b_i(y_i)| \chi_Q(y_i) dy_i \right) \prod_{j=1(j \neq i)}^m \int_{Q'} |\chi_Q(y_j)| dy_j.$$

Similarly, we have $|[\mathcal{M}, |b_i|]_i(\vec{\chi}_Q)(x)| \leq C \|b_i\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n}$. So, by (4) we obtain, for any $x \in Q$,

$$||Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i)(x) - M_Q(b_i)(x)| \leq C \|b\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n}.$$

Then, by Lemma 4, for any $x \in Q$, we have

$$\begin{aligned} I_2 &= \frac{1}{|Q|^{\beta/n+1/s}} \left\| |Q|^{-\alpha/n} \mathcal{M}_{\alpha,Q}(b_i \circ \vec{\chi}_Q^i) - M_Q(b_i) \right\|_{(E_i)_t(Q)} \\ &\leq \frac{C}{|Q|^{\beta/n+1/s}} |Q|^{\beta/n} \|\chi_Q\|_{(E_i)_t(Q)} \leq C. \end{aligned}$$

Combining I_1 and I_2 , we may get $\frac{1}{|Q|^{1+\beta/n}} \int_Q |b_i(x) - M_Q(b_i)(x)| dx \leq C$, which implies (ii) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

So far, the proofs of Theorem 1 and 2 are completed.

ACKNOWLEDGEMENTS

This work was supported by the Heilongjiang Natural Science Foundation joint key project (NO. ZL2024A001).

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Received November 2, 2024