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NOVEL CRITERIA FOR ATTRACTIVITY AND STABILITY OF NON-AUTONOMOUS NONLINEAR DYNAMICAL SYSTEMS

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Abstract. In this paper, issues of both attractivity and stability for some classes of nonlinear systems are investigated. New sufficient conditions for practical h-stability of non-autonomous nonlinear systems depending on a parameter are proposed by indefinite Lyapunov functions. We present new converse Lyapunov theorems for non-autonomous systems. The h-stability analysis is achieved with the help of scalar practical h-stable functions. These results are used to study the uniform practical h-stability of non-autonomous perturbed systems.

Keywords: Lyapunov theory, perturbed systems, non-autonomous nonlinear systems, practical h-stability.
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1. INTRODUCTION

The second Lyapunov function approach has been extensively explored and has yielded a wealth of significant results in the study of stability analysis of nonlinear systems (see [1, 2, 9, 10, 15, 20]). This method provides powerful tools for control design and system analysis. It is widely recognized that the Lyapunov function method has practical applications in control theory and serves as a fundamental tool to simplify complex systems (see [18]). Recently, growth conditions for the asymptotic behavior of solutions of various classes of nonlinear systems have been studied. Several significant advancements have been made in developing stability criteria for nonlinear differential systems. However, some of these developments overlap with previous works, including [8, 14, 28–30], where various intrinsic modifications of Lyapunov's original stability concepts were introduced. Therefore, in this paper, we aim to extend the theoretical framework by providing a deeper comparison with previous stability results. Over the years, Pinto introduced the concept of h-stability in [22–24] to study weakly stable systems – those that are less stable than exponentially asymptotically stable ones – under specific perturbations. In many cases, even when a system is mathematically unstable, its behavior may remain sufficiently close to a desired state to be acceptable. However, some systems are practically unstable because their domain of attraction is not sufficiently large. To address this issue, a more suitable stability notion than Lyapunov stability is required. Practical stability has been extensively discussed in [2–8] and [21]. While previous works have investigated practical stability and related notions, our approach in this paper focuses on practical uniform h-stability, an extension that provides a more flexible framework incorporating classical practical exponential stability. A key result of this technique is the practical uniform h-stability framework introduced in [10, 11], which unifies several stability concepts. A converse Lyapunov theorem for practical h-stability was established in [10], stating that if a system is locally uniformly h-stable, then a Lyapunov function satisfying specific conditions exists. In [13], a converse stability theorem was formulated to construct a

Lyapunov function ensuring global asymptotic stability for certain classes of cascaded systems with unbounded dynamics. These previous studies provide an essential foundation for our work. However, our study aims to refine these results by explicitly characterizing the role of practical h-stability in perturbed systems and extending the applicability to new dynamical classes. Furthermore, the idea of scalar stable functions and the comparison principle have been utilized in [31] to analyze the stability of nonlinear time-varying systems. The concept of practical stable scalar functions was further developed in [15] to analyze time-varying systems.

Building on this foundation, we explore the possibility of extending the concept of an h-stable function to the practical h-stability framework. Our objective is to develop new converse Lyapunov theorems for both local and global uniform practical h-stability in a family of systems that depend on a parameter. This approach allows us to define a new Lyapunov function that ensures the practical h-stability of a class of perturbed systems.

The paper is organized as follows. In Section 2, necessary and sufficient conditions are proposed for practical h-stability. New converse stability theorems are given in Section 3. Our conclusion is given in Section 4.

1.1. Preliminaries

Let $\mathbb{R}$ be the set of real numbers. $\mathbb{R}_+$ denotes the set of all non-negative real numbers, $\mathbb{R}_+^* =]0,\infty)$ and $\mathbb{R}^n$ is the n-dimensional Euclidean space with the Euclidian norm $\|\cdot\|$. Besides, we denote by $I, J \subset \mathbb{R}$ are two intervals that are not empty and not reduce to a singleton. $PC(I,J)$ is the space of piecewise continuous functions on I to J. $BC(I,J)$ is the space of continuous bounded functions on I to J endowed with the norm $\|f\|_\infty = \sup_{t\in I} |f(t)|$. $C(I,J)$ is the space of continuous functions on I to J. $C^1(I,J)$ is the space of continuous differentiable functions on I to J. For $r \geq 0$, $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$. $L^p([0,\infty))$ denotes the space of functions f such that $|f|^p$ is integrable on $[0,\infty)$, where $p \geq 1$. $\mathscr{K}$ is the set of all functions that are strictly increasing, continuous and vanish at the origin. $\mathscr{K}_\infty$ is the set of all functions that are of class $\mathscr{K}$ and unbounded.

Consider the nonlinear system

$$\dot{x}(t) = \mathscr{F}(t, x(t), \varepsilon), \tag{1}$$

where $t \in \mathbb{R}_+$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $\varepsilon \in \mathbb{R}_+^*$ and $\mathscr{F}(\cdot,\cdot,\varepsilon) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in time and locally Lipschitz in state. It is well known that for a given $x \in \mathbb{R}^n$ there exists a unique solution of (1) denoted by $\phi(\cdot, t, x, \varepsilon)$ satisfying the initial condition $\phi(t,t,x,\varepsilon) = x$.

Definition 1. Let $h \in BC(\mathbb{R}_+, \mathbb{R}_+^\star)$.

- The system (1) is said to be locally uniformly practically h-stable if there exist $r > 0$ and $c \geq 1$ such that for all $t \geq 0$, all $\varepsilon \in \mathbb{R}_+^*$ and all $x \in \mathbb{R}^n$ with $\|x\| \leq r$ we have

  $$\|\phi(s,t,x,\varepsilon)\| \leq c\|x\| h(s) h(t)^{-1} + \eta_\varepsilon, \quad \forall s \geq t, \tag{2}$$

  with $\eta_\varepsilon > 0$.

- The system (1) is said to be globally uniformly practically h-stable if there exists $c \geq 1$ such that for all $t \geq 0$, any $\varepsilon \in \mathbb{R}_+^*$ and any initial state $x \in \mathbb{R}^n$, the solutions satisfy (2).

Definition 2. Let $\chi, \xi \in PC(\mathbb{R}_+, \mathbb{R})$ and $h \in C^1(\mathbb{R}_+, \mathbb{R}_+^\star) \cap BC(\mathbb{R}_+, \mathbb{R}_+^\star)$. The function χ is ξ-globally uniformly practically h-stable if there exist $\lambda \geq 0$ and $\rho_\varepsilon > 0$ such that for all $s \geq t$,

$$\int_t^s \chi(\tau)\mathrm{d}\tau \leq \int_t^s h'(\tau) h(\tau)^{-1} \mathrm{d}\tau + \lambda,$$

and

$$\int_t^s |\xi_\varepsilon(\tau)| \Theta(s,\tau)\mathrm{d}\tau \leq \rho_\varepsilon,$$

where $\Theta(s,t) = e^\lambda h(s) h(t)^{-1}$.

2. PRACTICAL *h*-STABILITY

Several authors handled the problem of stability and boundedness of certain classes of ordinary differential equations (ODEs), see [25–27]. The innovation in our work is to tackle the problem of uniform practical h-stability of solutions of system (1) by utilizing indefinite Lyapunov functions.

THEOREM 1. *Let* $h \in C^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ *be a decreasing function. Assume that there exist a continuously differentiable function* $\Lambda(.,.,\varepsilon) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, *a function* $\alpha \in \mathscr{K}_\infty$, *constants* $\kappa > 0$, $p \geq 1$ *and a scalar function* $\chi \in PC(\mathbb{R}_+, \mathbb{R})$ *such that for all* $t \geq 0$, *all* $\varepsilon \in \mathbb{R}_+^\star$ *and all* $x \in \mathbb{R}^n$,

$$\kappa\|x\|^p \leq \Lambda(t,x,\varepsilon) \leq \alpha(\|x\|) + \zeta_\varepsilon, \textit{ with } \zeta_\varepsilon > 0, \tag{3}$$

$$\frac{\partial \Lambda}{\partial t}(t,x,\varepsilon) + \frac{\partial \Lambda}{\partial x}(t,x,\varepsilon)\mathscr{F}(t,x,\varepsilon) \leq \chi(t)\Lambda(t,x,\varepsilon) + \xi_\varepsilon(t), \textit{ with } \xi_\varepsilon \in PC(\mathbb{R}_+, \mathbb{R}) \tag{4}$$

then, system (1) is locally uniformly practically $h^{\frac{1}{p}}$*-stable if* χ *is* ξ_ε*-globally uniformly practically h-stable.*

Proof. From generalized Gronwall-Bellman inequality [19] and (4), we have for all $s \geq t$,

$$\Lambda(s, \phi(s,t,x,\varepsilon), \varepsilon) \leq \Lambda(t,x,\varepsilon)\Theta(s,t) + \int_t^s \xi_\varepsilon(\tau)\Theta(s,\tau)\mathrm{d}\tau,$$

where $\Theta(s,t) = \exp\left(\int_t^s \chi(\tau)\mathrm{d}\tau\right)$. According to (3), one obtains for all $s \geq t$,

$$\kappa\|\phi(s,t,x,\varepsilon)\|^p \leq \Lambda(t,x,\varepsilon)\Theta(s,t) + \int_t^s \xi_\varepsilon(\tau)\Theta(s,\tau)\mathrm{d}\tau.$$

Since, χ is ξ_ε-globally uniformly practically h-stable, there is $\lambda \geq 0$ and $\rho_\varepsilon > 0$ such that for all $s \geq t$, Then for all $s \geq t$,

$$\|\phi(s,t,x,\varepsilon)\| \leq \kappa^{-\frac{1}{p}}\alpha^{\frac{1}{p}}(\|x\|)e^{\frac{\lambda}{p}}h^{\frac{1}{p}}(s)h(t)^{-\frac{1}{p}} + \kappa^{-\frac{1}{p}}(\zeta_\varepsilon e^\lambda + \rho_\varepsilon)^{\frac{1}{p}}\cdot$$

For every $\nu > 0$, we choose $c \geq 1$ such that $\delta = \alpha^{-1}(\kappa\nu^p e^{-\lambda}) \geq \frac{\nu}{c}\cdot$ If we take any initial state $x \in \mathbb{R}^n$ such that $\delta \geq \|x\| \geq \frac{\nu}{c}$, then for all $s \geq t$,

$$\|\phi(s,t,x,\varepsilon)\| \leq c\|x\|h^{\frac{1}{p}}(s)h(t)^{-\frac{1}{p}} + \kappa^{-\frac{1}{p}}(\zeta_\varepsilon e^\lambda + \rho_\varepsilon)^{\frac{1}{p}}.$$

Consequently, system (1) is locally uniformly practically $h^{\frac{1}{p}}$-stable. ☐

COROLLARY 1. *Let* $h \in C^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ *be a decreasing function. Assume that there exist a continuously differentiable function* $\Lambda(.,.,\varepsilon) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, *constants* $b_1 > 0$, $b_2 \geq 0$, $p \geq 1$ *and a scalar function* $\chi \in PC(\mathbb{R}_+, \mathbb{R})$ *such that for all* $t \geq 0$, *all* $\varepsilon > 0$, *and all* $x \in \mathbb{R}^n$,

$$b_1\|x\|^p \leq \Lambda(t,x,\varepsilon) \leq b_2\|x\|^p + \zeta_\varepsilon,, \textit{ with } \zeta_\varepsilon > 0, \tag{5}$$

$$\frac{\partial \Lambda}{\partial t}(t,x,\varepsilon) + \frac{\partial \Lambda}{\partial x}(t,x,\varepsilon)\mathscr{F}(t,x,\varepsilon) \leq \chi(t)\Lambda(t,x,\varepsilon) + \xi_\varepsilon(t), \textit{ with } \xi_\varepsilon > 0 \tag{6}$$

then, system (1) is globally uniformly practically $h^{\frac{1}{p}}$*-stable if* χ *is* ξ_ε*-globally uniformly practically h-stable.*

Proof. From generalized Gronwall-Bellman inequality [19] and (4), we have for all $s \geq t$,

$$\Lambda(s, \phi(s,t,x,\varepsilon), \varepsilon) \leq \Lambda(t,x,\varepsilon)\Theta(s,t) + \int_t^s \xi_\varepsilon(\tau)\Theta(s,\tau)\mathrm{d}\tau,$$

where $\Theta(s,t)=\exp\left(\int_t^s \chi(\tau)\mathrm{d}\tau\right)$. According to (3), one obtains for all $s\geq t$,

$$\kappa\|\phi(s,t,x)\|^p\leq \Lambda(t,x,\varepsilon)\Theta(s,t)+\int_t^s \xi_\varepsilon(\tau)\Theta(s,\tau)\mathrm{d}\tau.$$

Since, χ is ξ_ε-globally uniformly practically h-stable, there is $\lambda\geq 0$ and $\rho_\varepsilon>0$ such that for all $s\geq t$,

$$\|\phi(s,t,x,\varepsilon)\|\leq\left(\frac{b_2e^\lambda}{b_1}\right)^{\frac{1}{p}}\|x\|h^{\frac{1}{p}}(s)h(t)^{-\frac{1}{p}}+\left(\frac{\zeta_\varepsilon e^\lambda}{b_1}+\frac{\rho_\varepsilon}{b_1}\right)^{\frac{1}{p}}.$$

Consequently, system (1) is globally uniformly practically $h^{\frac{1}{p}}$-stable with $c=\left(\frac{b_2e^\lambda}{b_1}\right)^{\frac{1}{p}}$ and $\eta_\varepsilon=\left(\frac{\zeta_\varepsilon e^\lambda}{b_1}+\frac{\rho_\varepsilon}{b_1}\right)^{\frac{1}{p}}$. □

Remark 1. Corollary 1 generalizes some previous results in [11, 15]. For example, when $p=2$ this result reduce to [11, Theorem 2.1]. When $h(t)=e^{-\gamma t}$, with $\gamma>0$, $t\geq 0$, this result reduces to the one in [15]

3. CONVERSES STABILITY THEOREMS

In this section, we present new converses Lyapunov theorems when the system (1) is locally and or globally uniformly practically h-stable as an extension of the converse theorem given in [15, 16]. The author in [15, 16] has treated a particular case by taking $h(t)=e^{-\theta t}$, $\theta\geq 0$ in which the constructed Lyapunov function is permitted to own an indefinite derivative. Therefore, we generalize the converses theorem given in [2, 12, 15]. In order to give the precise assumption imposed for $\mathscr{F}(\cdot,\cdot)$, we need to introduce the following subclasses of $C(\mathbb{R}_+,\mathbb{R}_+)$, see [12].

Definition 3. We say that a function $\varpi\in C(\mathbb{R}_+,\mathbb{R}_+)$ is of class BN, if there exists a function $M\in C(\mathbb{R}_+,\mathbb{R}_+)$ such that $\int_t^{t+\delta}\varpi(s)\mathrm{d}s\leq M(\delta),\quad \forall t\geq 0,\quad \delta\geq 0.$

Definition 4. We say that a function $\varpi\in C(\mathbb{R}_+,\mathbb{R}_+)$ is of class BN^2, if there exists a function $N\in C(\mathbb{R}_+,\mathbb{R}_+)$ such that $\int_t^{t+\delta}\varpi^2(s)\mathrm{d}s\leq N(\delta),\quad \forall t\geq 0,\quad \delta\geq 0.$

Now, to prove a converse theorem, we shall suppose the following assumptions.

($\mathscr{H}_1$) There exist functions $R(\cdot)\in BN$ and $\Psi_\varepsilon(\cdot)\in BN^2$ such that

$$\|\mathscr{F}(t,x,\varepsilon)\|\leq\varpi(t)\|x\|+\Psi_\varepsilon(t),\quad \varpi(t)=D+R(t),\quad D>0,$$

($\mathscr{H}_2$) The system (1) is locally uniformly practically h-stable, that is there are $r>0$ and $c\geq 1$ such that for any $t\geq 0$, any initial state $x\in\mathbb{R}^n$ with $\|x\|\leq r$, and any $\varepsilon>0$, system (1) satisfies (2).

THEOREM 2. *Suppose that assumptions* ($\mathscr{H}_1$) *and* ($\mathscr{H}_2$) *are satisfied and assume that there exist* $\rho>0$ *such that*

$$h \text{ is decreasing and } h(t)h(t+\delta)^{-1}\leq\rho,\quad \forall t\geq 0,\quad \delta>0, \tag{7}$$

then there exist a continuously differentiable function $\Lambda(\cdot,\cdot,\varepsilon):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$, *constants* $b_1>0$, $b_2\geq 0$, $r>0$, *an integer* $p\geq 2$ *and scalar functions* $\chi\in PC(\mathbb{R}_+,\mathbb{R})$ *such that for all* $(t,x,\varepsilon)\in\mathbb{R}_+\times B_r\times\mathbb{R}_+^*$,

$$b_1\|x\|^p\leq\Lambda(t,x,\varepsilon)\leq b_2\|x\|^p+\zeta_\varepsilon,\ \text{with } \zeta_\varepsilon>0$$

$$\frac{\partial\Lambda}{\partial t}(t,x,\varepsilon)+\frac{\partial\Lambda}{\partial x}(t,x,\varepsilon)\mathscr{F}(t,x,\varepsilon)\leq\chi(t)\Lambda(t,x,\varepsilon)+\xi_\varepsilon(t),\ \text{with } \xi_\varepsilon\in PC(\mathbb{R}_+,\mathbb{R}),$$

such that χ *is* ξ_ε*-globally uniformly practically h-stable.*

Proof. Let

$$\Lambda(t,x,\varepsilon)=h(t)\int_t^{t+\delta}\omega(s)\Big(\|\phi(s,t,x,\varepsilon)\|^p+\Big[\frac{e^{2M(\delta)}}{2D}\int_t^s\Psi_\varepsilon^2(\tau)\mathrm{d}\tau\Big)\Big]^{\frac{p}{2}}\Big)h^{-1}(s)\mathrm{d}s,$$

where $\omega\in C(\mathbb{R}_+,\mathbb{R}_+^*)\cap L^1([0,\infty))$ is a decreasing function with $\inf\limits_{t\geq 0}(\omega(t))>0$. On the one hand, we have

$$\Lambda(t,x,\varepsilon)\leq h(t)\int_t^{t+\delta}\omega(s)\Big(ch(s)h(t)^{-1}\|x\|+\eta_\varepsilon)^p+\Big[\frac{e^{2M(\delta)}}{2D}\int_t^s\Psi_\varepsilon^2(\tau)\mathrm{d}\tau\Big)\Big]^{\frac{p}{2}}\Big)h(s)^{-1}\mathrm{d}s.$$

By using the decreasing of h, we derive

$$\Lambda(t,x,\varepsilon)\leq 2^{p-1}c^p\delta\omega(0)\|x\|^p+\Big(2^{p-1}\eta_\varepsilon^p+\Big[\frac{e^{2M(\delta)}}{2D}N_\varepsilon(\delta)\Big]^{\frac{p}{2}}\Big)\rho\delta\omega(0).$$

On the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi^T(s,t,x,\varepsilon)\phi(s,t,x,\varepsilon)\geq -2\varpi(s)\|\phi(s,t,x,\varepsilon)\|^2-2\Psi_\varepsilon(s)\|\phi(s,t,x,\varepsilon)\|. \tag{8}$$

Letting $v_\varepsilon(s)=-\|\phi(s,t,x,\varepsilon)\|$ and using (8), we obtain as in [18] that

$$D^+v_\varepsilon(s)\leq-\varpi(s)v_\varepsilon(s)+\Psi_\varepsilon(s),$$

such that $D^+v_\varepsilon(t)=\limsup\limits_{T\to 0^+}\frac{1}{T}\Big(v_\varepsilon(t+T)-v_\varepsilon(t)\Big)$. Let,

$$\bar{\xi}(s)=v_\varepsilon(s)e^{\displaystyle\int_t^s\varpi(\tau)\mathrm{d}\tau}.$$

Then, it follows that

$$D^+\bar{\xi}(s)\leq\Psi_\varepsilon(s)e^{\displaystyle\int_t^s\bar{\varpi}(\tau)\mathrm{d}\tau}. \tag{9}$$

Integrating (9) between t and s, yields

$$v_\varepsilon(s)\leq v_\varepsilon(t)e^{\displaystyle-\int_t^s\varpi(\tau)\mathrm{d}\tau}+\Big(\int_t^s\Psi_\varepsilon(\tau)e^{\displaystyle\int_t^\tau\varpi(\varsigma)\mathrm{d}\varsigma}\mathrm{d}\tau\Big)e^{\displaystyle-\int_t^s\varpi(\tau)\mathrm{d}\tau}.$$

On the one side, we have

$$e^{\displaystyle\int_t^s\varpi(\tau)\mathrm{d}\tau}\leq e^{D(s-t)}e^{M(\delta)},\quad\forall s\in[t,t+\delta],\quad t\geq 0.$$

On the other side, we have

$$\Big(\int_t^s\Psi_\varepsilon(\tau)e^{\displaystyle\int_t^\tau\varpi(\varsigma)\mathrm{d}\varsigma}\mathrm{d}\tau\Big)e^{\displaystyle-\int_t^s\varpi(\tau)\mathrm{d}\tau}\leq e^{M(\delta)}\Big(\int_t^s\Psi_\varepsilon(\tau)e^{D(\tau-t)}\mathrm{d}\tau\Big)e^{\displaystyle-\int_t^s\varpi(\tau)\mathrm{d}\tau}.$$

Using Cauchy-Schwartz inequality, one obtains for all $s\in[t,t+\delta]$ and all $t\geq 0$,

$$\|\phi(s,t,x,\varepsilon)\|+\Big(\frac{e^{2M(\delta)}}{2D}\int_t^s\Psi_\varepsilon^2(\tau)\mathrm{d}\tau\Big)^{\frac{1}{2}}\geq\|x\|e^{-M(\delta)}e^{-D(s-t)}.$$

Since

$$\Big[\|\phi(s,t,x,\varepsilon)\|+\Big(\frac{e^{2M(\delta)}}{2D}\int_t^s\Psi_\varepsilon^2(\tau)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big]^p\leq 2^{p-1}\|\phi(s,t,x,\varepsilon)\|^p+2^{p-1}\Big[\int_t^s\frac{e^{2M(\delta)}}{2D}\Psi_\varepsilon^2(\tau)\mathrm{d}\tau\Big]^{\frac{p}{2}},$$

the following inequality holds for all $s\in[t,t+\delta]$ and $t\geq 0$,

$$\|\phi(s,t,x,\varepsilon)\|^p+\Big[\frac{e^{2M(\delta)}}{2D}\int_t^s\Psi_\varepsilon^2(\tau)\mathrm{d}\tau\Big]^{\frac{p}{2}}\geq\frac{1}{2^{p-1}}\|x\|^pe^{-pM(\delta)}e^{-pD(s-t)}. \tag{10}$$

As a consequence, we obtain

$$\Lambda(t,x,\varepsilon)\geq\frac{e^{-pM(\delta)}(1-e^{-pD\delta})}{2^{p-1}pD}\inf_{t\geq 0}(\omega(t))\|x\|^p.$$

Thus, $\Lambda(t,x,\varepsilon)$ satisfies the first inequality of Theorem 2 with $b_1=\frac{e^{-pM(\delta)}(1-e^{-pl\delta})}{2^{p-1}pD}\inf\limits_{t\geq 0}(\omega(t))>0$, $b_2=2^{p-1}c^p\delta\omega(0)$, and $\zeta_\varepsilon=\Big(2^{p-1}\eta_\varepsilon^p+\Big[\frac{e^{2M(\delta)}}{2D}N_\varepsilon(\delta)\Big]^{\frac{p}{2}}\Big)\rho\delta\omega(0)$. We have,

$$\Lambda(s,\phi(s,t,x,\varepsilon),\varepsilon)=h(s)\int_s^{s+\delta}\omega(u)\Big(\|\phi(u,t,x,\varepsilon)\|^p+\Big[\frac{e^{2M(\delta)}}{2D}\int_s^u\Psi_\varepsilon^2(\tau)\mathrm{d}\tau\Big]^{\frac{p}{2}}\Big)h(u)^{-1}du.$$

Then, the derivative of $\Lambda(t,x,\varepsilon)$ along the trajectories of system (1) exists and is given by

$$\begin{aligned}\dot{\Lambda}(t,x,\varepsilon)&=\frac{\mathrm{d}}{\mathrm{d}s}(\Lambda(s,\phi(s,t,x,\varepsilon),\varepsilon)_{/s=t}\\&\leq h'(t)h(t)^{-1}\Lambda_\varepsilon(t,x)+\omega(t)(2^{p-1}c^p-1)r^p\\&\quad+\omega(t)\rho\Big(2^{p-1}\eta^p+\Big[\frac{e^{2M(\delta)}}{2D}N_\varepsilon(\delta)\Big]^{\frac{p}{2}}\Big).\end{aligned}$$

Thus, the second inequality of Theorem 2 is satisfied with $\chi(t)=h'(t)h(t)^{-1}$, and

$$\xi_\varepsilon(t)=\omega(t)\Big[(2^{p-1}c^p-1)r^p+\rho\Big(2^{p-1}\eta^p+\Big[\frac{e^{2M(\delta)}}{2D}N_\varepsilon(\delta)\Big]^{\frac{p}{2}}\Big)\Big].$$

where χ is ξ_ε-globally uniformly practically h-stable. □

Now, the question that is raised here: is there a Lyapunov function that satisfies conditions of Theorem 2 when the system (1) is globally uniformly practically h-stable? The response to this question is as follows.

$(\mathscr{H}_3)$ The system (1) is globally uniformly practically h-stable, that is there exists $c\geq 1$ such that for any $t\geq 0$, any initial state $x\in\mathbb{R}^n$ and any $\varepsilon>0$, system (1) satisfies (2).

Then, we have the following theorem.

THEOREM 3. *Assume that assumptions* $(\mathscr{H}_1)$ *and* $(\mathscr{H}_3)$ *are fulfilled and h satisfies the condition (7), then there exist a continuously differentiable function* $\Lambda(.,.,\varepsilon):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$, *constants* $b_1>0$, $b_2\geq 0$, $b>0$, *an integer* $p\geq 2$, *and a scalar function* $\chi\in PC(\mathbb{R}_+,\mathbb{R})$ *such that for all* $(t,x,\varepsilon)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+^*$,

$$b_1\|x\|^p\leq\Lambda(t,x,\varepsilon)\leq b_2\|x\|^p+\zeta_\varepsilon,\ \textit{with}\ \zeta_\varepsilon>0,$$

$$\frac{\partial\Lambda}{\partial t}(t,x,\varepsilon)+\frac{\partial\Lambda}{\partial x}(t,x,\varepsilon)\mathscr{F}(t,x,\varepsilon)\leq\chi(t)V(t,x)+\xi_\varepsilon(t),\ \textit{with}\ \xi_\varepsilon\in PC(\mathbb{R}_+,\mathbb{R}),$$

such that χ *is* ξ_ε*-globally uniformly practically* $\tilde{h}$*-stable with* $\tilde{h}(t)=h(t)e^{bt}$.

Proof. Let

$$\Lambda(t,x,\varepsilon)=h(t)\int_t^{t+\delta}\omega(s)\Big(\|\phi(s,t,x,\varepsilon)\|^p+\Big[\frac{e^{2M(\delta)}}{2D}\int_t^s\Psi_\varepsilon^2(\tau)\mathrm{d}\tau\Big)\Big]^{\frac{p}{2}}\Big)h^{-1}(s)\mathrm{d}s,$$

where $\omega\in C(\mathbb{R}_+,\mathbb{R}_+)$ is a decreasing function, satisfies $\omega(t)\omega(t+\delta)\le\nu$ with $\nu>0$, $\delta>0$ and $\omega\tilde{h}^{-1}\in L^q([0,\infty))$ with $q>1$. Using the same arguments as in Theorem 2 with the same b_1, b_2 and ζ_ε. Moreover, the derivative of $\Lambda(t,x,\varepsilon)$ along the trajectories of system (1) exists and is given by

$$\begin{aligned}\dot{\Lambda}(t,x,\varepsilon)&=\frac{\mathrm{d}}{\mathrm{d}s}(\Lambda(s,\phi(s,t,x,\varepsilon),\varepsilon)_{/s=t}\\&\le(h'(t)h(t)^{-1}+b)\Lambda(t,x,\varepsilon)+\omega(t)\Big[\rho\Big(2^{p-1}\eta_\varepsilon^p+\Big[\frac{e^{2M(\delta)}}{2D}N_\varepsilon(\delta)\Big]^{\frac{p}{2}}\Big)\Big],\end{aligned}$$

where $b=\frac{2^{p-1}pD(2^{p-1}c^p-1)\nu}{e^{-pM(\delta)}(1-e^{-pD\delta})}\cdot$ Thus, the second inequality of Theorem 3 is satisfied with $\chi(t)=h'(t)h(t)^{-1}+b$, and $\xi_\varepsilon(t)=\omega(t)\Big[\rho\Big(2^{p-1}\eta_\varepsilon^p+\Big[\frac{e^{2M(\delta)}}{2D}N_\varepsilon(\delta)\Big]^{\frac{p}{2}}\Big)\Big]$. We have $\int_t^s\chi(\tau)\mathrm{d}\tau\le\int_t^s\tilde{h}'(\tau)\tilde{h}^{-1}(\tau)\mathrm{d}\tau$. Also,

$$\int_t^s|\xi_\varepsilon(\tau)|\Theta(s,\tau)\mathrm{d}\tau\le\rho\Big(2^{p-1}\eta^p+\Big[\frac{e^{2M(\delta)}}{2D}N_\varepsilon(\delta)\Big]^{\frac{p}{2}}\Big)\|\tilde{h}\|_\infty\int_t^s\omega(\tau)\tilde{h}^{-1}(\tau)e^{-b\tau}\mathrm{d}\tau.$$

Therefore, by using the fact that the function $\omega\tilde{h}^{-1}\in L^q([0,\infty))$ and the Hölder inequality, one obtains

$$\int_t^s|\xi_\varepsilon(\tau)|\Theta(s,\tau)\mathrm{d}\tau\le\Big(2^{p-1}\eta^p+\Big[\frac{e^{2M(\delta)}}{2D}N_\varepsilon(\delta)\Big]^{\frac{p}{2}}\Big)\frac{\rho\|\tilde{h}\|_\infty(q-1)}{bq}\|w\tilde{h}^{-1}\|_q.$$

Consequently, χ is ξ_ε-globally uniformly practically $\tilde{h}$-stable. □

Remark 2. One can see that when $h(t)=e^{-\gamma t}$, where $\gamma\ge0$, the Theorems 2 and 3 ensure that there exist a Lyapunov function with indefinite derivative such that the function χ is globally uniformly practically exponentially stable, see [15].

3.1. Perturbed systems

We consider non-autonomous perturbed systems of the form:

$$\dot{x}(t)=\Xi(t,x(t),\varepsilon)+\vartheta(t,x(t)),\tag{11}$$

where $t\in\mathbb{R}_+$ is the time, $x(t)\in\mathbb{R}^n$ is the state and $\varepsilon\in\mathbb{R}_+^*$. $\Xi(\cdot,\cdot,\varepsilon):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ and $\vartheta:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ are piecewise continuous in time and locally Lipschitz in state.

We consider the following assumptions:

($\mathscr{H}_4$) There exist a continuously differentiable function $\Lambda(.,.,\varepsilon):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$, constants $\widetilde{b}_1>0,\widetilde{b}_2>0$, $p\ge1$ and a scalar function $\tilde{\chi}\in PC(\mathbb{R}_+,\mathbb{R})$ such that for all $t\ge0$, all $x\in\mathbb{R}^n$ and all $\varepsilon>0$,

$$\widetilde{b}_1\|x\|^p\le\Lambda(t,x,\varepsilon)\le\widetilde{b}_2\|x\|^p+\tilde{\zeta}_\varepsilon,\text{ with }\tilde{\zeta}_\varepsilon>0,$$

$$\frac{\partial\Lambda}{\partial t}(t,x,\varepsilon)+\frac{\partial\Lambda}{\partial x}(t,x,\varepsilon)\Xi(t,x,\varepsilon)\le\tilde{\chi}(t)\Lambda(t,x,\varepsilon)+\tilde{\xi}_\varepsilon(t)\text{ with }\tilde{\xi}_\varepsilon\in PC(\mathbb{R}_+,\mathbb{R}).$$

($\mathscr{H}_5$) There exist $\Gamma\in BN$ such that for all $t\ge0$ and all $x\in\mathbb{R}^n$,

$$\Big|\frac{\partial\Lambda}{\partial x}(t,x,\varepsilon)\vartheta(t,x)\Big|\le\Gamma(t)\Lambda(t,x,\varepsilon).\tag{12}$$

THEOREM 3. *Let* $\tilde{h} \in C^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ *such that the function* $h(t) = \tilde{h}(t)e^{bt} \in BC(\mathbb{R}_+, \mathbb{R}_+^\star)$ *with* $b > 0$ *and the function* $|\tilde{\chi}|\tilde{h}^{-1} \in L^q([0,\infty))$, *with* $q > 1$. *Under assumptions* $(\mathscr{H}_4)$ *and* $(\mathscr{H}_5)$, *the perturbed system (11) is globally uniformly practically* $h^{\frac{1}{p}}$*-stable if* $\tilde{\chi}$ *is* $\tilde{\xi}_\varepsilon$*-globally uniformly practically* $\tilde{h}$*-stable.*

Proof. Let $\delta \in \mathbb{R}_+^*$. We consider the following function:

$$Z_\delta(t,x,\varepsilon) = \Lambda(t,x,\varepsilon)\exp\Big(V_\delta(t,x,\varepsilon)\Big) \tag{13}$$

where

$$V_\delta(t,x,\varepsilon) = \int_t^{t+\delta}\int_t^s \frac{1}{\delta\Lambda(\tau,\phi(\tau,t,x,\varepsilon),\varepsilon)}\frac{\partial\Lambda}{\partial x}(\tau,\phi(\tau,t,x,\varepsilon),\varepsilon)\vartheta(\tau,\phi(\tau,t,x,\varepsilon))\mathrm{d}\tau\mathrm{d}s.$$

Hence, for well chosen values of δ, we must prove that Z_δ satisfies the conditions of Theorem 1 to guarantee that (11) is globally uniformly practically $h^{\frac{1}{p}}$-stable. Since,

$$\Big|\frac{\partial\Lambda}{\partial x}(t,x,\varepsilon)\vartheta(t,x)\Big| \le \Gamma(t)\Lambda(t,x,\varepsilon),$$

then,

$$|V_\delta(t,x)| \le M(\delta).$$

Therefore, Z_δ satisfies the inequality (5) with

$$b_1 = \widetilde{b}_1 e^{M(\delta)}, \;\; b_2 = \widetilde{b}_2 e^{M(\delta)} \;\text{ and }\; \zeta_\varepsilon = \tilde{\zeta} e^{M(\delta)}.$$

The derivative of $Z_\delta(t,x,\varepsilon)$ along the trajectories of system (11) satisfies

$$\dot{Z}_\delta(t,x,\varepsilon) \le (\tilde{\chi}(t) + \frac{M(\delta)}{\delta})Z_\delta(t,x,\varepsilon) + \tilde{\xi}_\varepsilon(t)e^{M(\delta)},$$

Therefore, if δ is chosen such that $M(\delta) < b\delta$, then, $Z_{\delta,\varepsilon}$ satisfies the inequality (6) with

$$\chi(t) = \tilde{\chi}(t) + b \;\text{ and }\; \xi_\varepsilon(t) = \tilde{\xi}_\varepsilon(t)e^{M(\delta)},$$

Since, $\tilde{\chi}$ is $\tilde{\xi}_\varepsilon$-globally uniformly practically $\tilde{h}$-stable, then one gets,

$$\int_t^s \chi(\tau)\mathrm{d}\tau \le \int_t^s h'(\tau)h^{-1}(\tau)\mathrm{d}\tau + \lambda.$$

Since, $h \in \mathscr{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$, then

$$\int_t^s |\xi_\varepsilon(\tau)|\Theta(s,\tau)\mathrm{d}\tau \le e^{(M(\delta)+\lambda)}\|h\|_\infty \int_t^s |\tilde{\xi}_\varepsilon(\tau)|\tilde{h}^{-1}(\tau)e^{-b\tau}\mathrm{d}\tau,$$

Thus, by using the fact that the function $|\tilde{\xi}_\varepsilon|\tilde{h}^{-1} \in L^q([0,\infty))$ and the Hölder inequality one has

$$\int_t^s |\xi_\varepsilon(\tau)|\Theta(s,\tau)\mathrm{d}\tau \le \frac{e^{(M(\delta)+\lambda)}\|h\|_\infty(q-1)}{bq}\||\tilde{\xi}_\varepsilon|\tilde{h}^{-1}\|_q.$$

Consequently, χ is ξ_ε-globally uniformly practically h-stable. According to the Corollary 1 the perturbed system (11) is globally uniformly practically $h^{\frac{1}{p}}$-stable. □

4. CONCLUSION

In this paper, we have given some new sufficient conditions for both local and global uniform practical h-stability of non-autonomous nonlinear dynamical systems in the presence of parameters based on indefinite Lyapunov functions. In addition, we have established two converses theorems when the system is locally and or globally uniformly practically h-stable. Then, an interesting application of the new Lyapunov functions is given for a class of non-autonomous perturbed systems.

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