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ON EXPONENTIAL CONTRACTION OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH MARKOVIAN SWITCHING

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Abstract. This work addresses the exponential contraction of stochastic differential equations with Markovian switching. By a novel approach, new explicit criteria for the exponential contraction in mean square are derived. An example is given to illustrate the effectiveness of the criteria.

***Keywords**:* exponential contraction, stochastic differential equations, Markovian switching.
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1. INTRODUCTION

Contraction theory is a powerful tool for analyzing non-linear dynamical systems, with applications in biology [17], control theory [12], observer design [3], synchronization of coupled oscillators [19], traffic networks [5], etc. Recently, various problems of contraction of dynamical systems has attracted much attention from researchers, see e.g. [1, 4, 8–10, 13, 15, 16, 18]. For comprehensive references on them, ones refer to very nice recent review papers [8, 18].

Generally speaking, a dynamical system is called exponentially contractive if any two trajectories converge to one other at an exponential rate. For example, by definition, the differential system $\dot{x}(t) = f(x(t),t),\ t \geq t_0$, is exponentially contractive (or incrementally exponentially stable [18, Definition 2.2]), if there are positive constants β and K such that

$$\|x(t,t_0,x_0) - x(t,t_0,y_0)\| \leq Ke^{-\beta(t-t_0)}\|x_0 - y_0\|, \quad t \geq t_0,$$

for any $x_0, y_0 \in \mathbb{R}^d$. A well-known explicit criterion for the contraction of solutions of the differential equation is stated as follows:

THEOREM 1 [1]. *Assume that $f(x,t)$ is continuous in $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ and is continuously differentiable in x. If the Jacobian $J_f(x,t)$ satisfies*

$$\mu(J_f(x,t)) \leq -\beta, \quad x \in \mathbb{R}^d, t \geq t_0,$$

for some $\beta > 0$, then $\dot{x}(t) = f(x(t),t)$, is exponentially contractive. Here, $\mu(A)$ is the matrix measure of $A \in \mathbb{R}^{d\times d}$, defined by $\mu(A) := \lim_{h\to 0^+} \dfrac{\|I_d + hA\| - 1}{h}$.

Recently, extensions of Theorem 1 to stochastic differential equations, to switching jump diffusions with a hidden Markov chains, to functional differential equations, and to hybrid systems, have been reported in [16], [10], [15, Theorem 3.5], [4, Theorem 1], respectively. As far as, we know, except the results mentioned here, there are no further extensions of Theorem 1, see [8] for updated and detailed information.

In this paper, we are concerned with the exponential contraction of stochastic differential equations with Markovian switching. To the best of our knowledge, this problem is open and so the main purpose of our paper is to fill this gap. We propose a novel approach to the problem which does not involve Lyapunov functions. Our approach is based on a comparison principle and a proof by reductio ad absurdum. Consequently, several explicit criteria for the exponential contraction in mean square are derived. As far as we know, the main results of the present paper (Theorem 2 and Corollaries 1, 2) cannot be found in the literature. In particular, Corollary gives extensions of Theorem 1 [1] mentioned above and of [16, Theorem 2.2] to stochastic differential equations with Markovian switching.

2. MAIN RESULTS

Let $\mathbb{R}^{n\times m}$ be the set of all $n\times m$ matrices. For a matrix $A\in\mathbb{R}^{n\times n}$, $A^\top$ denotes its transpose. For $A=(a_{ij})\in\mathbb{R}^{n\times n}$ and $B=(b_{ij})\in\mathbb{R}^{n\times n}$, we write $A\geq B$ if $a_{ij}\geq b_{ij}$ for any (i,j) and $A\gg B$ if $a_{ij}>b_{ij}$ for any (i,j). Similar notations are adopted for vectors. Let I_n be the $n\times n$ identity matrix. For $x=(x_1,\ldots,x_d)^\top\in\mathbb{R}^d$, $|x|:=\left(\sum_{i=1}^d x_i^2\right)^{1/2}$.

Let $s(A):=\max\{\Re\lambda : \lambda \text{ is an eigenvalue of } A\}$ be the spectral abscissa of $A\in\mathbb{R}^{n\times n}$. Then A is called Hurwitz stable if $s(A)<0$. A matrix $A\in\mathbb{R}^{n\times n}$ is called a Metzler matrix if all off-diagonal elements of A are nonnegative, see e.g. [7]. Then, A is Hurwitz stable if and only if $Ap\ll 0$, for some $p\in\mathbb{R}^n_+, p\gg 0$, see e.g. [7].

Let $(\Omega,\mathscr{F},\mathbb{P},\{\mathscr{F}_t\})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}$ satisfying the usual conditions. Let $B(t)$, $t\geq 0$, be an m-dimensional standard Brownian motion defined on the probability space. Assume that $r(t)$, $t\geq t_0$, is a continuous time Markov chain on the probability space taking values in a finite state space $\mathscr{M}=\{1,\ldots,N\}$ with generator $\Gamma=(\gamma_{ij})\in\mathbb{R}^{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta)=j|r(t)=i\}=\begin{cases}\gamma_{ij}\Delta+o(\Delta) & \text{if } i\neq j,\\ 1+\gamma_{ii}\Delta+o(\Delta) & \text{if } i=j,\end{cases}$$

where $\gamma_{ij}\geq 0$ if $i\neq j$ and $\gamma_{ii}=-\sum_{i\neq j}\gamma_{ij}$ for any $i\in\mathscr{M}$, see e.g. [2]. The Markov chain $r(\cdot)$ is $\mathscr{F}_t$-adapted but independent of the Brownian motion $B(\cdot)$.

Let $t_0\geq 0$ be a given constant. Let $f:[t_0,\infty)\times\mathbb{R}^d\times\mathscr{M}\to\mathbb{R}^d$, and $g:[t_0,\infty)\times\mathbb{R}^d\times\mathscr{M}\to\mathbb{R}^{d\times m}$ be Borel functions. Consider the nonlinear stochastic differential equation with Markovian switching

$$dx(t)=f(t,x(t),r(t))dt+g(t,x(t),r(t))dB(t),\ \ t\in[t_0,\infty), \tag{1}$$

with the initial condition

$$x(t_0)=x_0, \text{ and } r(t_0)=i_0. \tag{2}$$

Throughout, it is assumed that $f(t,x,i)$ and $g(t,x,i)$ satisfy the local Lipschitz condition. That is, for each $n\in\mathbb{N}$ and $T>t_0$, there exists a constant $\widetilde{K}_{n,T}>0$ such that

$$|f(t,x,i)-f(t,\overline{x},i)|+|g(t,x,i)-g(t,\overline{x},i)|\leq\widetilde{K}_{n,T}|x-\overline{x}|$$

for any $t\in[t_0,T]$, $|x|\vee|\overline{x}|\leq n$, $i\in\mathscr{M}$. Moreover, for each $T>t_0$, there exists a constant $\overline{K}_T>0$ such that

$$\left[x^\top f(t,x,i)\right]\vee|g(t,x,i)|^2\leq\overline{K}_n(1+|x|^2),\ \ (t,x,i)\in[0,T]\times\mathbb{R}^d\times\mathscr{M}.$$

Under the above condition, Eq. (1) has a unique global and continuous solution $x(\cdot;t_0,x_0,i_0)$ for any initial condition $(x_0, i_0) \in \mathbb{R}^d \times \mathscr{M}$. Moreover,

$$\mathbb{E}\Big[\sup_{t_0 \le t \le T} |x(t;t_0,x_0,i_0)|^q\Big] < \infty \text{ for any } T > 0, q \ge 2; \tag{3}$$

see [14, Theorem 3.24, p. 99].

The following is a natural generalization of the exponential contraction to (1).

Definition 1. Equation (1) is said to be exponentially contractive in mean square if there exist constants $K > 0$, $\lambda > 0$ such that

$$\mathbb{E}|x(t;t_0,x_0,i_0) - x(t;t_0,y_0,i_0)|^2 \le Ke^{-\lambda(t-t_0)}|x_0 - y_0|^2$$

for any $(t,x_0,i_0),(t,y_0,i_0) \in [t_0,\infty) \times \mathbb{R}^d \times \mathscr{M}$.

The following theorem gives an explicit criterion for the exponential contraction of (1).

THEOREM 2. *Assume that there exist continuous functions $\theta_i : [t_0,\infty) \to \mathbb{R}, i \in \mathscr{M}$ such that*

$$2(x-y)^\top (f(t,x,i) - f(t,y,i)) + |g(t,x,i) - g(t,y,i)|^2 \le \theta_i(t)|x-y|^2, \tag{4}$$

for any $(t,x,i),(t,y,i) \in [t_0,\infty) \times \mathbb{R}^d \times \mathscr{M}$. Let

$$A(t) := \operatorname{diag}(\theta_1(t),\ldots,\theta_N(t)) + \Gamma, \;\; t \ge t_0. \tag{5}$$

If there exist a constant $\beta > 0$ and a vector $p := (p_1,\ldots,p_N)^\top \in \mathbb{R}^N$, $p \gg 0$, such that

$$A(t)p \le -\beta p, \;\; t \ge t_0, \tag{6}$$

then Eq. (1) *is exponentially contractive in mean square.*

Proof. Let $(x_0,i_0),(y_0,i_0) \in \mathbb{R}^d \times \mathscr{M}$. For notational simplicity, let $x(t) := x(t;t_0,x_0,i_0), y(t) := x(t;t_0,y_0,i_0)$. Without loss of generality, suppose $|x_0 - y_0| > 0$. Let $K > \max_{i\in\mathscr{M}} p_i$. Define

$$X(t) = \mathbb{E}\big[|x(t) - y(t)|^2 p_{r(t)}\big],\; t \ge t_0; \quad Y(t) = Ke^{-\beta(t-t_0)}|x_0 - y_0|^2,\;\; t \ge t_0.$$

Note that the trajectories of $x(t), y(t)$ are continuous, while the trajectories of $r(t)$ are right-continuous. Combining this observation with (3), we can conclude that $X(t)$ is right-continuous on the interval $[t_0,\infty)$. It is also clear that $Y(\cdot)$ is continuous on $[t_0,\infty)$.

Since $K > \max_{i\in\mathscr{M}} p_i$, then

$$X(t_0) < Y(t_0)$$

.

We will prove that

$$X(t) < Y(t),\;\; t \ge t_0. \tag{7}$$

If this statement were false, by continuity, there would exist a number $t_* > t_0$ such that

$$X(t) < Y(t) \;\text{ for }\; t \in [t_0,t_*),\;\; X(t_*) \ge Y(t_*). \tag{8}$$

Fix $\alpha > \beta$. Consider the function $V(t,x,i) = e^{\alpha t}|x|^2 p_i$ for $(t,x,i) \in [t_0,\infty) \times \mathbb{R}^d \times \mathscr{M}$. By the generalized Itô formula [14],

$$e^{\alpha t_*}\mathbb{E}\big[|x(t_*) - y(t_*)|^2 p_{r(t_*)}\big] = e^{\alpha t_0}|x_0 - y_0|^2 p_{i_0} + \mathbb{E}\int_{t_0}^{t_*} e^{\alpha s}\Big\{\alpha|x(s) - y(s)|^2 p_{r(s)} +$$

$$p_{r(s)}\Big[2(x(s)-y(s))^\top\left(f(s,x(s),r(s))-f(s,y(s),r(s))\right)+\left|g(s,x(s),r(s))-g\left(s,y(s),r(s)\right)\right|^2\Big]+$$
$$\sum_{j\in\mathscr{M}}\gamma_{r(s)j}|x(s)-y(s)|^2p_j\Big\}\mathrm{d}s, \tag{9}$$

which together with (4) implies that

$$e^{\alpha t_*}\mathbb{E}\left[|x(t_*)-y(t_*)|^2p_{r(t_*)}\right]\le e^{\alpha t_0}|x_0-y_0|^2p_{i_0}+\mathbb{E}\int_{t_0}^{t_*}e^{\alpha s}\Big\{\Big(\alpha p_{r(s)}+\theta_{r(s)}(s)p_{r(s)}+\sum_{j\in\mathscr{M}}\gamma_{r(s)j}p_j\Big)|x(s)-y(s)|^2\Big\}\mathrm{d}s. \tag{10}$$

In view of (8), we have

$$\mathbb{E}\left[|x(s)-y(s)|^2p_{r(s)}\right]<Ke^{-\beta(s-t_0)}|x_0-y_0|^2,\ \ s\in[t_0,t_*). \tag{11}$$

By (6),

$$\alpha p_{r(s)}+\theta_{r(s)}(s)p_{r(s)}+\sum_{j\in\mathscr{M}}\gamma_{r(s)j}p_j\le(\alpha-\beta)p_{r(s)}.$$

This together with (11) leads to

$$\mathbb{E}\Big[\Big(\alpha p_{r(s)}+\theta_{r(s)}(s)p_{r(s)}+\sum_{j\in\mathscr{M}}\gamma_{r(s)j}p_j\Big)|x(s)-y(s)|^2\Big]\le(\alpha-\beta)Ke^{-\beta(s-t_0)}|x_0-y_0|^2,\ \ s\in[t_0,t_*). \tag{12}$$

By the Fubini theorem and (12), we have from (10) that

$$e^{\alpha t_*}\mathbb{E}\left[|x(t_*)-y(t_*)|^2p_{r(t_*)}\right]\le e^{\alpha t_0}|x_0-y_0|^2p_{i_0}+\int_{t_0}^{t_*}e^{\alpha s}(\alpha-\beta)Ke^{-\beta(s-t_0)}|x_0-y_0|^2\mathrm{d}s=e^{\alpha t_0}|x_0-y_0|^2p_{i_0}$$
$$+Ke^{\beta t_0}|x_0-y_0|^2\int_{t_0}^{t_*}e^{(\alpha-\beta)s}(\alpha-\beta)\mathrm{d}s=e^{\alpha t_0}|x_0-y_0|^2p_{i_0}+Ke^{\beta t_0+(\alpha-\beta)t_*}|x_0-y_0|^2-Ke^{\alpha t_0}|x_0-y_0|^2. \tag{13}$$

Since $p_{i_0}<K$, and $|x_0-y_0|>0$, we have $e^{\alpha t_0}|x_0-y_0|^2p_{i_0}<Ke^{\alpha t_0}|x_0-y_0|^2$. It follows from (13) that

$$e^{\alpha t_*}\mathbb{E}\left[|x(t_*)-y(t_*)|^2p_{r(t_*)}\right]<Ke^{\beta t_0+(\alpha-\beta)t_*}|x_0-y_0|^2.$$

Consequently,

$$\mathbb{E}\left[|x(t_*)-y(t_*)|^2p_{r(t_*)}\right]<Ke^{-\beta(t_*-t_0)}|x_0-y_0|^2,$$

which contradicts the second statement in (8). Thus, (7) holds; that is,

$$\mathbb{E}\left[|x(t)-y(t)|^2p_{r(t)}\right]<Ke^{-\beta(t-t_0)}|x_0-y_0|^2,\ \ t\ge t_0.$$

Hence,

$$\mathbb{E}\left[|x(t)-y(t)|^2\right]<\frac{K}{\widehat{p}}e^{-\beta(t-t_0)}|x_0-y_0|^2,\ \ t\ge t_0,$$

where $\widehat{p}=\min_{i\in\mathscr{M}}p_i$. This completes the proof. □

COROLLARY 1. *Suppose* (4) *holds for some constants* $\theta_i, i\in\mathscr{M}$. *Let* A *be the matrix given by* (5). *If* A *is Hurwitz stable, then* (1) *is exponentially contractive in mean square.*

Proof. Since A is a Hurwitz stable Metzler matrix, $Ap\ll 0$, for some $p\in\mathbb{R}^N, p\gg 0$, see e.g. [7]. This gives $Ap\ll-\beta p$ for a sufficiently small $\beta>0$. Then, (6) holds. By Theorem 2, Eq. (1) is exponentially contractive in mean square. □

Recall that $\mu(A) := \lim_{h\to 0^+} \frac{\|I_d+hA\|-1}{h}$, is the matrix measure of $A\in\mathbb{R}^{d\times d}$. Note that

$$x^\top Ax \le \mu(A)|x|^2, \quad x\in\mathbb{R}^d, \tag{14}$$

see e.g. [6].

COROLLARY 2. *Suppose $f(x,t,i)$ is continuously differentiable in x for any $t\ge t_0, i\in\mathscr{M}$ and it obeys*

$$\mu(J_f(x,t,i)) \le \gamma_i(t), \qquad x\in\mathbb{R}^d, t\in[t_0,\infty), i\in\mathscr{M}, \tag{15}$$

where $J_f(x,t,i)$ is the Jacobian matrix of $f(x,t,i)$ at x. Assume that

$$|g(t,x,i)-g(t,y,i)|^2 \le \psi_i(t)|x-y|^2, \quad t\in[t_0,\infty), x,y\in\mathbb{R}^d, i\in\mathscr{M}. \tag{16}$$

Let

$$\theta_i(t) := 2\gamma_i(t)+\psi_i(t), \quad t\ge t_0, i\in\mathscr{M}$$

and let $A(t)$ be defined by (5). *If* (6) *holds then Eq.* (1) *is exponentially contractive in mean square.*

Proof. Since $f(x,t,i)$ is continuously differentiable in x, by the mean value theorem, we get

$$f(x,t,i)-f(y,t,i) = \left(\int_0^1 J_f\left(y+s(y-x),t,i\right)\mathrm{d}s\right)(x-y), \quad x,y\in\mathbb{R}^d, t\in[t_0,\infty), i\in\mathscr{M}.$$

It follows from (14) and (15) that

$$(x-y)^\top(f(x,t,i)-f(y,t,i)) = (x-y)^\top\left(\int_0^1 J_f\left(y+s(y-x),t,i\right)\mathrm{d}s\right)(x-y) \le \gamma_i(t)|x-y|^2,$$

for any $x,y\in\mathbb{R}^d, t\in[t_0,\infty), i\in\mathscr{M}$. Corollary 2 follows from Theorem 2. □

Remark 1. Corollary 1 gives an extension of [16, Theorem 2.2] to stochastic differential equations with Markovian switching. Particularly, if $g(x,t,i)=0$, for each $i\in\mathscr{M}$ then Corollary 2 reduces to a criterion for the contraction of the differential equation with Markovian switching

$$\dot{x}(t) = f(x,t,r(t)), \quad t\ge 0. \tag{17}$$

This gives an extension of Theorem 1 to (17).

Remark 2. In [11], the authors addressed the exponential stability in mean square of stochastic delay differential equations with Markovian switching. Actually, the exponential contraction and the exponential stability are different issues.

Roughly speaking, the exponential contraction means that any two trajectories of a system converge to one other at an exponential rate. In addition, if the system has an equilibrium (saying x^*), then any trajectory converges to x^*. So the system is exponentially stable. Consequently, the exponential contraction is the more general one.

Example 1. Consider a scalar stochastic differential equation with Markovian switching given by

$$dx(t) = (1.5-r(t))x(t)dt + 0.2e^{-t}r(t)\cos x(t)dB(t), \;\; t\ge 0, \tag{18}$$

where

$$f(t,x,i) = (1.5-i)x, \quad g(t,x,i) = 0.2ie^{-t}\cos x, \; t\ge 0, \; x\in\mathbb{R}, \; i\in\mathscr{M} := \{1,2\}$$

and

$$\Gamma = \begin{pmatrix} -2.5 & 2.5 \\ 0.5 & -0.5 \end{pmatrix}.$$

Clearly,

$$\begin{aligned} 2(x-y)(f(t,x,i)-f(t,y,i))+|g(t,x,i)-g(t,y,i)|^2 &= (3-2i)(x-y)^2+0.04i^2e^{-2t}(\cos x-\cos y)^2 \\ &\le \left(3-2i+0.04i^2\right)(x-y)^2, \end{aligned} \tag{19}$$

for any $t \geq 0, x, y \in \mathbb{R}, i \in \mathscr{M}$. Then, we have $\theta_i = 3-2i+0.04i^2, i \in \mathscr{M}$, and

$$A := \operatorname{diag}(\theta_1, \theta_2) + \Gamma = \begin{pmatrix} -2.5+1.04 & 2.5 \\ 0.5 & -0.5-0.84 \end{pmatrix}.$$

Since A is a Metzler matrix and $Ap \ll 0$ for $p = (1, 0.5)^\top \in \mathbb{R}^2_+$, A is Hurwitz stable. By Corollary 1, Eq. (18) is exponentially contractive in mean square.

3. CONCLUDING REMARKS

This work has focused on exponential contraction in mean square of stochastic differential equations with Markovian switching. A novel approach has been developed to establish explicit criteria for the contraction. The techniques for analyzing contraction developed in this paper seems to be effective in treating more general stochastic hybrid systems, which will be studied in a near future.

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