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COEFFICIENT PROBLEMS FOR ANALYTIC FUNCTIONS IN $\mathscr{M}(\alpha)$ AND $\mathscr{N}(\alpha)$

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Abstract. In this paper we consider coefficients problems in classes $\mathscr{M}(\alpha)$ and $\mathscr{N}(\alpha)$, $\alpha > 0$, which are defined in a similar way to the class of starlike and convex functions. We generalize the known results concerning the bounds of coefficients in these classes. We also discuss the difference of successive coefficients for functions in $\mathscr{M}(\alpha)$ and $\mathscr{N}(\alpha)$. Obtained results are then compared with those proved by Obradović et al., Schild as well as Sim and Thomas.

Keywords: coefficient problem, difference of successive coefficients, starlike functions of order α, convex functions of order α, functions convex in one direction.
Mathematics Subject Classification (MSC2020): 30C45, 30C50

1. INTRODUCTION

Let $\mathscr{A}$ be the family of all functions analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, having the power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ . \tag{1}$$

Robertson (1936, [1]) introduced the class of starlike functions of order α

$$\mathscr{S}^*(\alpha) = \left\{ f \in \mathscr{A} : \mathfrak{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \right\} \ , \ \alpha < 1 \ .$$

To compare the results we establish in Section 3 with the results for starlike functions of order α, we slightly modify the above definition. Instead of $\mathscr{S}^*(\alpha)$, we consider the class $\tilde{\mathscr{S}}^*(\alpha)$

$$\tilde{\mathscr{S}}^*(\alpha) = \left\{ f \in \mathscr{A} : \mathfrak{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 1 - \frac{\alpha}{2} \right\} \ , \ \alpha > 0 \ .$$

Schild (1965, [2]) proved that if $f \in \mathscr{S}^*(\alpha)$, $\alpha \in [0,1)$, then

$$|a_n| \le \frac{1}{(n-1)!} \prod_{k=2}^{n} (k - 2\alpha) \ . \tag{2}$$

Hence, if $f \in \tilde{\mathscr{S}}^*(\alpha)$, $\alpha \in (0,2]$, then

$$|a_n| \le \frac{1}{(n-1)!} \prod_{k=0}^{n-2} (\alpha + k) \ . \tag{3}$$

From the original proof of Schild's result it follows that the results (2) and (3) are also valid for $\alpha < 0$ and $\alpha > 2$, respectively.

Similarly to the class $\tilde{\mathscr{S}}^*(\alpha)$, $\alpha > 0$, one can consider the class

$$\mathscr{M}(\alpha) = \left\{ f \in \mathscr{A} : \mathfrak{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\alpha}{2} \right\} , \alpha > 0 \tag{4}$$

and the relative class

$$\mathscr{N}(\alpha) = \left\{ f \in \mathscr{A} : \mathfrak{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < 1 + \frac{\alpha}{2} \right\} , \alpha > 0 \tag{5}$$

corresponding to the class of convex functions of order α

$$\tilde{\mathscr{K}}(\alpha) = \left\{ f \in \mathscr{A} : \mathfrak{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 1 - \frac{\alpha}{2} \right\} , \ \alpha > 0 \, .$$

The class $\mathscr{N}(1) = \mathscr{N}$ was introduced by Ozaki (1941, [3]). He proved that functions in $\mathscr{N}$ are univalent in $\mathbb{D}$. Later, Umezawa (1952, [4]) discussed a general version of this class and showed that functions in $\mathscr{N}$ are convex in one direction. Next, Ponnusamy and Rajasekaran (1995, [5]) showed that functions in $\mathscr{N}(\alpha)$ are starlike in $\mathbb{D}$ for $\alpha \in (0,1]$. They also proved that if $f \in \mathscr{N}(\alpha)$, $\alpha > 1$, then f is not necessarily univalent in $\mathbb{D}$.

Coefficient problems for the class $\mathscr{N}(\alpha)$ have been considered in very few papers. Obradović, Ponnusamy and Wirths (2013, [6]) found the estimate of $|a_n|$ in the class $\mathscr{N}(\alpha)$, $\alpha \in (0,1]$

$$|a_n| \le \frac{\alpha}{n(n-1)} \, . \tag{6}$$

From the Alexander relation, $f \in \mathscr{N}(\alpha)$ if and only if $zf'(z) \in \mathscr{M}(\alpha)$. Hence, for $f \in \mathscr{M}(\alpha)$ given by (1) and $g \in \mathscr{N}(\alpha)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{7}$$

we have

$$a_n = n b_n \, . \tag{8}$$

Thus, from (6) and (8) we obtain that if $f \in \mathscr{M}(\alpha)$, $\alpha \in (0,1]$, then

$$|a_n| \le \frac{\alpha}{n-1} \, .$$

In this paper, we consider coefficients problems in the classes $\mathscr{M}(\alpha)$ and $\mathscr{N}(\alpha)$ for all $\alpha > 0$ (not restricted to $\alpha \in (0,1]$). In Section 3 we generalize the known results concerning the bounds of coefficients in these classes. In Section 4 we discuss the difference of successive coefficients for functions in $\mathscr{M}(\alpha)$ and $\mathscr{N}(\alpha)$.

The problem of estimating

$$|a_{n+1}| - |a_n| \tag{9}$$

for a given family of all functions analytic in $\mathbb{D}$ having the Taylor series expansion (1) is strictly related to the famous Bieberbach conjecture for the class $\mathscr{S}$ of all univalent functions. If

$$||a_{n+1}| - |a_n|| \le 1 \tag{10}$$

were true, it would imply that $|a_n| \le n$ for all univalent functions.

Now, it is known that (10) is not true in $\mathscr{S}$ (see, [8]), contrary to the Bieberbach conjecture. For $\mathscr{S}$ the best result up to date was obtained by Grinspan in 1976. He proved [7] that

$$||a_{n+1}| - |a_n|| \le 3.61 \quad \text{for} \quad f \in \mathscr{S} \, .$$

On the other hand, the inequality in (10) holds true for some sublasses of $\mathscr{S}$: the class of starlike functions $\mathscr{S}^*$ (Leung, 1978, [9]), the class of spirallike functions $\mathscr{S}_\gamma$ (Hamilton, 1980, [10]), the class of spirallike functions of order α $\mathscr{S}_\gamma(\alpha)$ (Arora, Ponnusamy, Sahoo, 2019, [11]), and the class of close-to-convex functions $\mathscr{K}_0(k)$ related to the Koebe function $k(z) = \frac{z}{(1-z)^2}$ (Zaprawa, 2020, [13]). For the class $\mathscr{K}$ of convex functions the problem is more complicated. Li and Sugawa in 2017 [12] proved that for $f \in \mathscr{K}$ the following sharp inequality holds

$$|a_{n+1}| - |a_n| \leq \tfrac{1}{n+1} \ .$$

In numerous cases, analogously to the class $\mathscr{K}$, obtaining the sharp bounds of (9) for all $n \in \mathbb{N}$ is impossible. In these cases it is interesting to derive the sharp bound of (9) for initial values of n. For example, Li and Sugawa in [12] proved additionaly that for $f \in \mathscr{K}$

$$|a_3| - |a_2| \geq -\tfrac{1}{2} \ , \qquad |a_4| - |a_3| \geq -\tfrac{1}{3} \ .$$

Moreover,

$$\inf\{|a_{n+1}| - |a_n| : f \in \mathscr{K}\} = d_n \qquad \text{where} \qquad -\tfrac{2}{n+1} < d_n < -\tfrac{1}{n} \ .$$

It is worth recalling the result by Sim and Thomas, who proved that if $f \in \mathscr{S}^*(\alpha)$, $\alpha \in [0,1)$, then

$$\frac{-\sqrt{2}(1-\alpha)}{\sqrt{2-\alpha}} \leq |a_3| - |a_2| \leq 1-\alpha \ .$$

Modifying the definition of $\mathscr{S}^*(\alpha)$ as above, we can write that if $f \in \tilde{\mathscr{S}}^*(\alpha)$, $\alpha \in (0,2]$, then

$$\frac{-\alpha}{\sqrt{2+\alpha}} \leq |a_3| - |a_2| \leq \frac{\alpha}{2} \ . \tag{11}$$

2. PRELIMINARY RESULTS

Let $\mathscr{P}$ be the class of analytic functions h with a positive real part in $\mathbb{D}$, having the Taylor series expansion

$$h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ . \tag{12}$$

By $\mathscr{B}$ we denote the class of analytic self-maps of $\mathbb{D}$. The symbol $\mathscr{B}_0$ stands for the class of Schwarz functions, i.e., $\omega \in \mathscr{B}$ and $\omega(0) = 0$. Hence, the function $\omega \in \mathscr{B}_0$ has the Taylor series expansion

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n \ . \tag{13}$$

It is known that for h and ω given by (12) and (13), respectively, we have

$$p_1 = 2c_1 \ , \qquad p_2 = 2({c_1}^2 + c_2) \ , \qquad p_3 = 2({c_1}^3 + 2c_1c_2 + c_3) \ . \tag{14}$$

To prove our results, we need two lemmas concerning functions in the classes $\mathscr{P}$ and $\mathscr{B}_0$. The first one is the result obtained by Prokhorov and Szynal [14]. Since the result is lengthy, we are quoting here some specific part of their result which we need in our further investigation.

LEMMA 1 ([14]). *Let $\omega \in \mathscr{B}_0$ be an analytic function of form* (13). *Then, for any real number μ and ν, the following sharp estimate holds*

$$|c_3 + \mu c_1 c_2 + \nu {c_1}^3| \leq \Phi(\mu, \nu), \tag{15}$$

where

$$\Phi(\mu,\nu) \le \begin{cases} 1\,, & (\mu,\nu)\in E_1 \\ |\nu|\,, & (\mu,\nu)\in E_2 \\ \frac{2}{3}(|\mu|+1)\left(\dfrac{|\mu|+1}{3(|\mu|+1+\nu)}\right)^{1/2}, & (\mu,\nu)\in E_3\cup E_4 \\ \frac{1}{3}\nu\dfrac{\mu^2-4}{\mu^2-4\nu}\left(\dfrac{\mu^2-4}{3(\nu-1)}\right)^{1/2}, & (\mu,\nu)\in E_5 \end{cases}$$

and the sets E_i are defined by:

$$\begin{aligned} E_1 &= \left\{(\mu,\nu)\in\mathbb{R}^2:\ \tfrac{1}{2}\le|\mu|\le 2,\quad \tfrac{4}{27}(|\mu|+1)^3-(|\mu|+1)\le\nu\le 1\right\}\,, \\ E_2 &= \left\{(\mu,\nu)\in\mathbb{R}^2:\ 2\le|\mu|\le 4,\quad \nu\ge\tfrac{1}{12}(\mu^2+8)\right\}\,, \\ E_3 &= \left\{(\mu,\nu)\in\mathbb{R}^2:\ \tfrac{1}{2}\le|\mu|\le 2,\quad -\tfrac{2}{3}(|\mu|+1)\le\nu\le\tfrac{4}{27}(|\mu|+1)^3-(|\mu|+1)\right\}\,, \\ E_4 &= \left\{(\mu,\nu)\in\mathbb{R}^2:\ |\mu|\ge 2,\quad -\tfrac{2}{3}(|\mu|+1)\le\nu\le\frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}\right\}\,, \\ E_5 &= \left\{(\mu,\nu)\in\mathbb{R}^2:\ 2\le|\mu|\le 4,\quad \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}\le\nu\le\tfrac{1}{12}(\mu^2+8)\right\}\,. \end{aligned}$$

In the first case of Lemma 1, the extremal function has the form $\omega(z)=z^3$. In the second case, the extremal function has the form $\omega(z)=z$. In other cases, the extremal functions are complicated, see [14] for details.

The second lemma is due to Libera and Złotkiewicz ([15, 16]).

LEMMA 2 ([15, 16]). *If $h\in\mathcal{P}$ be an analytic function of form* (12)*, then*

$$2p_2={p_1}^2+x(4-{p_1}^2)$$

for some x such that $|x|\le 1$.

From (14) and Lemma 2 we know that for $\omega\in\mathscr{B}_0$ given by (13) we have

$$c_2=(1-|c_1|^2)x \tag{16}$$

for some x such that $|x|\le 1$.

Finally, we need theorem concerning the notion of quasi-subordination. Recall that F is quasi-subordinate to G if there exists φ such that $|F(z)|\le|G(\varphi(z))|$, where $|\varphi(z)|\le|z|$, $z\in\mathbb{D}$.

THEOREM 1 (see for example, Theorem 2.2 in [17]). *Let $F(z)=a_0+a_1z+\ldots$ be quasi-subordinate to $G(z)=b_0+b_1z+\ldots$, then*

$$\sum_{k=0}^{n}|a_k|^2\le\sum_{k=0}^{n}|b_k|^2\,,\quad n=0,1,\ldots\,.$$

3. COEFFICIENTS ESTIMATES

To prove the main theorem of this section, we need the following lemma, which can easily be proved by mathematical induction.

LEMMA 3. *For $\alpha>0$ we have*

$$\alpha^2+\sum_{j=1}^{n-1}\frac{\alpha(\alpha-2j)}{(j!)^2}\prod_{k=0}^{j-1}(\alpha-k)^2=\frac{1}{((n-1)!)^2}\prod_{k=0}^{n-1}(\alpha-k)^2\,,\ n\ge 2\,.$$

THEOREM 2. *If $f \in \mathcal{M}(\alpha)$ is given by* (1)*, then*

$$|a_n| \le \frac{1}{(n-1)(m-1)!}\prod_{k=0}^{m-1}(\alpha-k)\,, \quad \alpha \in (2m-2, 2m]\,, m=1,2,\dots,n-2\,, \tag{17}$$

$$|a_n| \le \frac{1}{(n-1)!}\prod_{k=0}^{n-2}(\alpha-k)\,, \quad \alpha > 2n-4\,. \tag{18}$$

The result is sharp for $\alpha \in (0,2]$ and for $\alpha \in (2n-4,\infty)$.

Proof. The condition in (4) is equivalent to

$$\mathfrak{Re}\left\{\frac{1+\frac{\alpha}{2}-\frac{zf'(z)}{f(z)}}{\frac{\alpha}{2}}\right\} > 0\,.$$

There exists an analytic function $\omega \in \mathcal{B}_0$ such that

$$\frac{1+\frac{\alpha}{2}-\frac{zf'(z)}{f(z)}}{\frac{\alpha}{2}} = \frac{1+\omega(z)}{1-\omega(z)}\,, \tag{19}$$

or equivalently, a function $w \in \mathcal{B}$ that

$$\frac{1+\frac{\alpha}{2}-\frac{zf'(z)}{f(z)}}{\frac{\alpha}{2}} = \frac{1+zw(z)}{1-zw(z)}\,. \tag{20}$$

From (20) we obtain

$$zf'(z)-f(z) = zw(z)\left[zf'(z)-(1+\alpha)f(z)\right]\,.$$

Using Theorem 1 with $F(z) = zf'(z)-f(z)$, $G(z) = z^2f'(z)-(1+\alpha)zf(z)$ and $\varphi(z) = z$, we get the inequality

$$\sum_{k=1}^{n}(k-1)^2|a_k|^2 \le \sum_{k=1}^{n-1}(k-1-\alpha)^2|a_k|^2\,, \quad a_1 = 1\,,\ n \ge 2$$

or equivalently,

$$\begin{aligned}(n-1)^2|a_n|^2 \le \alpha^2|a_1|^2 &+ \alpha(\alpha-2)|a_2|^2+\alpha(\alpha-4)|a_3|^2+\cdots+\alpha(\alpha-2(m-1))|a_m|^2 \\ &+\alpha(\alpha-2m)|a_{m+1}|^2+\cdots+\alpha(\alpha-2(n-2))|a_{n-1}|^2\,.\end{aligned} \tag{21}$$

We shall use the method of mathematical induction. From (21) we have: $|a_2| \le \alpha$ for $\alpha > 0$, $|a_3| \le \frac{1}{2}\alpha(\alpha-1)$ for $\alpha \ge 2$ and $|a_3| \le \frac{1}{2}\alpha$ for $\alpha \in (0,2]$, which means that (17) and (18) hold for $n=2$ and $n=3$. Let us assume that condition (18) is satisfied. We shall show that we have

$$|a_{n+1}| \le \frac{1}{n!}\prod_{k=0}^{n-1}(\alpha-k)\,, \quad \alpha > 2n-2\,. \tag{22}$$

From (21) for $|a_{n+1}|$ we get

$$n^2|a_{n+1}|^2 \le \alpha^2+\alpha(\alpha-2)|a_2|^2+\alpha(\alpha-4)|a_3|^2+\cdots+\alpha(\alpha-2n+2)|a_n|^2\,.$$

Thus, taking into account the induction assumption first and then Lemma 3, we obtain

$$n^2|a_{n+1}|^2 \le \alpha^2+\alpha(\alpha-2)\alpha^2+\alpha(\alpha-4)\left(\frac{\alpha(\alpha-1)}{2!}\right)^2+\dots+\alpha(\alpha-2n+2)\left(\frac{1}{(n-1)!}\prod_{k=0}^{n-2}(\alpha-k)\right)^2$$
$$=\left(\frac{1}{(n-1)!}\prod_{k=0}^{n-1}(\alpha-k)\right)^2 ,$$

which implies (22).

Let $\alpha\in(2m-2,2m]$, $m<n-2$. From (21) we have

$$(n-1)^2|a_n|^2 \le \alpha^2+\alpha(\alpha-2)|a_2|^2+\alpha(\alpha-4)|a_3|^2+\dots+\alpha(\alpha-2m+2)|a_m|^2 .$$

Hence, using (17) and then Lemma 3, we obtain

$$(n-1)^2|a_n|^2 \le \alpha^2+\alpha(\alpha-2)\alpha^2+\alpha(\alpha-4)\left(\frac{\alpha(\alpha-1)}{2!}\right)^2+\dots+\alpha(\alpha-2m+2)\left(\frac{1}{(m-1)!}\prod_{k=0}^{m-2}(\alpha-k)\right)^2$$
$$=\left(\frac{1}{(m-1)!}\prod_{k=0}^{m-1}(\alpha-k)\right)^2 ,$$

which implies (17). Inequality (17) can be applied since the interval $(2m-2,2m]$ is included in $(2m-4,\infty]$ for all considered m.

Equality in (17), for every fixed n, $n\ge 2$, holds for the function

$$f_n(z)=z(1-z^{n-1})^{\alpha/(n-1)}=z-\frac{\alpha}{n-1}z^n-\dots . \tag{23}$$

Equalities for all $n\in\mathbb{N}$ in (18) are satisfied for the function

$$f_0(z)=z(1-z)^{\alpha}=z-\alpha z^2-\frac{1}{2!}\alpha(1-\alpha)z^3-\frac{1}{3!}\alpha(1-\alpha)(2-\alpha)z^4-\dots-\frac{1}{(n-1)!}\prod_{k=0}^{n-2}(\alpha-k)z^n-\dots . \tag{24}$$

Note that $f_0=f_2$. □

Finding the sharp bound of $|a_n|$ for other ranges of α is a difficult problem. We can solve it only for $|a_4|$.

THEOREM 3. *If $f\in\mathscr{M}(\alpha)$ is given by* (1) *and $\alpha\in[2,4]$, then*

$$|a_4|\le\begin{cases}\frac13\alpha , & \alpha\in[2,\alpha_1]\\ \frac13\left(\alpha-\frac23\right)^{3/2} , & \alpha\in[\alpha_1,\alpha_2]\\ \frac13(1-\alpha)(2-\alpha)\left(\frac32\alpha-4\right)\left(\frac{\frac32\alpha-4}{\alpha-3}\right)^{1/2} , & \alpha\in[\alpha_2,16/5]\\ \frac16\alpha(1-\alpha)(2-\alpha) , & \alpha\in[16/5,4] ,\end{cases} \tag{25}$$

where $\alpha_1=2.52\dots$ is the only solution of the equation $27\alpha^3-81\alpha^2+36\alpha-8=0$ and $\alpha_2=\frac16(13+\sqrt{33})=3.12\dots$. The result is sharp.

Proof. The coefficients of $f\in\mathscr{M}(\alpha)$ can be expressed by coefficients of a relative function h from the class $\mathscr{P}$. For f and h given by (1) and (12), respectively, we have

$$\frac{1+\frac{\alpha}{2}-\frac{zf'(z)}{f(z)}}{\frac{\alpha}{2}}=h(z) . \tag{26}$$

Hence, comparing the coefficients of functions in (26), we get

$$a_2 = -\tfrac{1}{2}\alpha p_1\,, \qquad a_3 = -\tfrac{1}{4}\alpha(p_2 - \tfrac{1}{2}\alpha p_1{}^2)\,, \qquad a_4 = -\tfrac{1}{6}\alpha\left(p_3 - \tfrac{3}{4}\alpha p_1 p_2 + \tfrac{1}{8}\alpha^2 p_1{}^3\right)\,. \tag{27}$$

From (14) and (27) we obtain

$$a_2 = -\alpha c_1\,,\ a_3 = -\tfrac{1}{2}\alpha\left(c_2 + (1-\alpha)c_1{}^2\right)\,,\ a_4 = -\tfrac{1}{3}\alpha\left(c_3 + (2-\tfrac{3}{2}\alpha)c_1c_2 + (1-\tfrac{3}{2}\alpha+\tfrac{1}{2}\alpha^2)c_1{}^3\right)\,. \tag{28}$$

Inequalities for $|a_4|$ can be obtained using Lemma 1 with $\mu = 2-\frac{3}{2}\alpha$ and $\nu = 1-\frac{3}{2}\alpha+\frac{1}{2}\alpha^2$.

I. After solving the system of inequalities $\frac{1}{2} \le |\mu| \le 2$ and $\frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \le \nu \le 1$, we get $\alpha \in [2,\alpha_1]$, where $\alpha_1 = 2.52\ldots$ is the only solution of the equation $27\alpha^3 - 81\alpha^2 + 36\alpha - 8 = 0$. Then $|c_3 + \mu c_1 c_2 + \nu c_1{}^3| \le 1$, so $|a_4| \le \frac{1}{3}\alpha$.

II. Inequalities $\frac{1}{2} \le |\mu| \le 2$ and $-\frac{2}{3}(|\mu|+1) \le \nu \le \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1)$ are satisfied for $\alpha \in [\alpha_1, 8/3]$. Hence, $|c_3 + \mu c_1 c_2 + \nu c_1{}^3| \le \frac{1}{\alpha}(\alpha - \frac{2}{3})^{3/2}$, so $|a_4| \le \frac{1}{3}\left(\alpha - \frac{2}{3}\right)^{3/2}$.

III. From $|\mu| \ge 2$ and $-\frac{2}{3}(|\mu|+1) \le \nu \le \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}$, we obtain $\alpha \in [8/3, \alpha_2]$, where $\alpha_2 = \frac{1}{6}(13+\sqrt{33}) = 3.12\ldots$ is the greater solution of the equation $9\alpha^2 - 39\alpha + 34 = 0$. Then we have again $|a_4| \le \frac{1}{3}\left(\alpha - \frac{2}{3}\right)^{3/2}$.

IV. Solving inequalities $2 \le |\mu| \le 4$ and $\frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \le \nu \le \frac{1}{12}(\mu^2+8)$ yields $\alpha \in [\alpha_2, 16/5]$. Thus, $|c_3 + \mu c_1 c_2 + \nu c_1{}^3| \le \frac{1}{\alpha}(1-\alpha)(2-\alpha)\left(\frac{3}{2}\alpha - 4\right)\left(\frac{\frac{3}{2}\alpha-4}{\alpha-3}\right)^{1/2}$.

V. From the system of inequalities $2 \le |\mu| \le 4$ and $\nu \ge \frac{1}{12}(\mu^2+8)$, we have $\alpha \in [16/5, 4]$. Then $|c_3 + \mu c_1 c_2 + \nu c_1{}^3| \le \frac{1}{2}(1-\alpha)(2-\alpha)$, so $|a_4| \le \frac{1}{6}\alpha(1-\alpha)(2-\alpha)$.

The sharpness of the result follows from the sharpness of Lemma 1. □

From Theorem 2, Theorem 3 and (8) we can obtain the following corollaries.

COROLLARY 1. *If $g \in \mathcal{N}(\alpha)$ is given by* (7), *then $|b_n| \le \frac{1}{n}|a_n|$, where a_n are given by* (18) *and* (17).

COROLLARY 2. *If $g \in \mathcal{N}(\alpha)$ is given by* (7), *then $|b_4| \le \frac{1}{4}|a_4|$, where a_4 is given by* (25).

Remark 1. Notice that

(i) Corollary 1 for $m = 1$ is a generalization of the result of Obradović (6),

(ii) formula (18) is similar to the result of Schild (3) for $\tilde{\mathcal{S}}^*(\alpha)$.

4. DIFFERENCE OF SUCCESSIVE COEFFICIENTS

THEOREM 4. *If $f \in \mathcal{M}(\alpha)$ is given by* (1) *and $n \ge 2$, then:*

$$|a_{n+1}| - |a_n| \le \frac{1}{n(m-1)!}\prod_{k=0}^{m-1}(\alpha - k) \quad \text{for } \alpha \in (2m-2, 2m],\ m = 1,2,\ldots,n; \tag{29}$$

for $m = n$ the interval $(2n-2, 2n]$ is extended to $(2n-2, \infty)$; and

$$|a_{n+1}| - |a_n| \ge -\frac{1}{(n-1)(m-1)!}\prod_{k=0}^{m-1}(\alpha - k) \quad \text{for } \alpha \in (2m-2, 2m],\ m = 1,2,\ldots,n-1; \tag{30}$$

for $m = n-1$ the interval $(2n-4, 2n-2]$ is extended to $(2n-4, \infty)$.

The result is sharp for $n \ge 3$ and $\alpha > 2n-4$.

Proof. From the obvious inequalities

$$-|a_n| \le |a_{n+1}| - |a_n| \le |a_{n+1}| \ , \quad n \ge 2$$

and Theorem 2 we obtain (29) and (30). For the sharpness of the result, observe that the extremal functions f_n in Theorem 2 given by (23), provided that $n \ge 3$, are n-fold symmetric. For the upper bound, it means that if $|a_{n+1}| = \frac{\alpha}{n}$, then $a_n = 0$. For the lower bound, if $|a_n| = \frac{\alpha}{n-1}$, then $a_{n+1} = 0$. □

From Theorem 4 and (8) we can obtain the following corollary.

COROLLARY 3. *If $g \in \mathscr{N}(\alpha)$ is given by* (7)*, then $-\frac{1}{n}|a_n| \le |b_{n+1}| - |b_n| \le \frac{1}{n+1}|a_{n+1}|$, where a_n are given by* (18) *and* (17)*.*

THEOREM 5. *If $f \in \mathscr{M}(\alpha)$ is given by* (1)*, then*

$$|a_3| - |a_2| \le \begin{cases} \frac{1}{2}\alpha \ , & \alpha \in (0,4] \\ \frac{1}{2}\alpha(\alpha-3) \ , & \alpha \ge 4 \end{cases} \quad \textit{and} \quad |a_3| - |a_2| \ge \begin{cases} \dfrac{-\alpha}{\sqrt{2-\alpha}} \ , & \alpha \in (0,1] \\ -\sqrt{\alpha} \ , & \alpha \ge 1 \ . \end{cases}$$

The bounds are sharp for all $\alpha > 0$.

Proof. I. From (28) we have

$$|a_3| - |a_2| = \tfrac{1}{2}\alpha\left(|c_2 + (1-\alpha){c_1}^2| - 2|c_1|\right) \ . \tag{31}$$

Putting (16) and $|c_1| = c \in [0,1]$ in (31), we get

$$|a_3| - |a_2| = \tfrac{1}{2}\alpha\left(|(1-c^2)x + (1-\alpha)c^2| - 2c\right) \ .$$

Hence, the upper bound for this coefficient difference is as follows

$$|a_3| - |a_2| \le \tfrac{1}{2}\alpha h_1(c) \ ,$$

where $h_1(c) = 1 - c^2 + |1-\alpha|c^2 - 2c$, $c \in [0,1]$. If $\alpha \in (0,4]$, then the function h_1 reaches its greatest value for $c = 0$, so $h_1(c) \le h_1(0) = 1$. If $\alpha > 4$, then the function h_1 attains its greatest value for $c = 1$, so $h_1(c) \le h_1(1) = \alpha - 3$.

The upper bound in both cases is sharp. Indeed, for $\alpha \in (0,4]$ the greatest value of $|a_3| - |a_2|$ is obtained for $\omega(z) = z^2 \in \mathscr{B}_0$ which means that f_3 given by (23), i.e., $f_3(z) = z(1-z^2)^{\alpha/2} = z - \frac{\alpha}{2}z^3 + \ldots$ is the extremal function in $\mathscr{M}(\alpha)$. Similarly, if $\alpha > 4$ this greatest value is reached by $\omega(z) = z$. Hence, for f_0 given by (24) we have $|a_3| - |a_2| = \frac{1}{2}\alpha(\alpha-3)$.

II. We shall now consider the lower bound for $|a_3| - |a_2|$. If $\alpha \in (0,1]$ and $(1-\alpha)c^2 - (1-c^2) \ge 0$, i.e., $c \in [\frac{1}{\sqrt{2-\alpha}}, 1]$, then we have

$$|a_3| - |a_2| \ge \tfrac{1}{2}\alpha\left(-(1-c^2) + (1-\alpha)c^2 - 2c\right) = \tfrac{1}{2}\alpha h_2(c) \ ,$$

where $h_2(c) = (2-\alpha)c^2 - 2c - 1$. Since the function h_2 is increasing in the interval $[\frac{1}{\sqrt{2-\alpha}}, 1]$, so it reaches its lowest value for $c = \frac{1}{\sqrt{2-\alpha}}$. Thus,

$$|a_3| - |a_2| \ge \tfrac{1}{2}\alpha h_2\left(\tfrac{1}{\sqrt{2-\alpha}}\right) = \frac{-\alpha}{\sqrt{2-\alpha}} \ .$$

If $\alpha \in (0,1]$ and $(1-\alpha)c^2 - (1-c^2) \le 0$, i.e., $c \in [0, \frac{1}{\sqrt{2-\alpha}}]$, then we get

$$|a_3| - |a_2| \ge \tfrac{1}{2}\alpha(-2c) = -\alpha c \ge \frac{-\alpha}{\sqrt{2-\alpha}} \ .$$

If $\alpha > 1$ and $(1-\alpha)c^2 + 1 - c^2 \le 0$, i.e., $c \in [\frac{1}{\sqrt{\alpha}}, 1]$, then we obtain

$$|a_3| - |a_2| \ge \tfrac{1}{2}\alpha\left(-(1-c^2) - (1-\alpha)c^2 - 2c\right) = \tfrac{1}{2}\alpha h_3(c)\ ,$$

where $h_3(c) = \alpha c^2 - 2c - 1$. Since the function h_3 is increasing in the interval $[\frac{1}{\sqrt{\alpha}}, 1]$, so it takes its lowest value for $c = \frac{1}{\sqrt{\alpha}}$. Hence,

$$|a_3| - |a_2| \ge \tfrac{1}{2}\alpha h_3\left(\tfrac{1}{\sqrt{\alpha}}\right) = -\sqrt{\alpha}\ .$$

If $\alpha > 1$ and $(1-\alpha)c^2 + 1 - c^2 \ge 0$, i.e., $c \in [0, \frac{1}{\sqrt{\alpha}}]$, then we have

$$|a_3| - |a_2| \ge \tfrac{1}{2}\alpha(-2c) = -\alpha c \ge -\sqrt{\alpha}\ .$$

For sharpness of the results observe that the equality in the lower bound holds for the Schwarz functions: $-\omega(z, \frac{1}{\sqrt{2-\alpha}})$ if $\alpha(0,1]$ and $\omega(-z, \frac{1}{\sqrt{\alpha}})$ if $\alpha > 1$, where $\omega(z) = \frac{z(c+z)}{1+cz}$. Applying the relation (19) we conclude the form of the extremal fuctions in $\mathscr{M}(\alpha)$: $f_*(z, \frac{1}{\sqrt{2-\alpha}})$ for $\alpha \in (0,1]$ and $f^*(z, \frac{1}{\sqrt{\alpha}})$ for $\alpha > 1$, where

$$f_*(z,c) = z\left(1 + 2cz + z^2\right)^{\alpha/2} \text{ and } f^*(z,c) = z\left[\left(1-z^2\right)\left(\frac{1+z}{1-z}\right)^c\right]^{\alpha/2}\ . \tag{32}$$

Consequently,

$$f_*\left(z, \frac{1}{\sqrt{2-\alpha}}\right) = z + \frac{\alpha}{\sqrt{2-\alpha}}z^2 - \frac{(1-\alpha)\alpha}{3\sqrt{2-\alpha}}z^4 + \dots \text{ and } f^*\left(z, \frac{1}{\sqrt{\alpha}}\right) = z + \sqrt{\alpha}z^2 - \frac{1}{3}\sqrt{\alpha}(\alpha-1)z^4 + \dots\ .$$

□

THEOREM 6. *If $f \in \mathscr{N}(\alpha)$ is given by* (7)*, then*

$$|b_3| - |b_2| \le \begin{cases} \frac{1}{6}\alpha\ , & \alpha \in (0,5] \\ \frac{1}{6}\alpha(\alpha-4)\ , & \alpha \ge 5 \end{cases} \quad \text{and} \quad |b_3| - |b_2| \ge \begin{cases} \dfrac{-\alpha(17-4\alpha)}{24(2-\alpha)}\ , & \alpha \in (0,1/2] \\ -\frac{1}{6}\alpha(2+\alpha)\ , & \alpha \in [1/2,1] \\ -\frac{1}{6}\alpha(4-\alpha)\ , & \alpha \in [1,3/2] \\ -\frac{1}{24}(9+4\alpha)\ , & \alpha \in [3/2,9/4] \\ -\frac{1}{2}\sqrt{\alpha}\ , & \alpha \ge 9/4\ . \end{cases}$$

The bounds are sharp for all $\alpha > 0$.

Proof. I. The coefficients of $f \in \mathscr{N}(\alpha)$ can be expressed in terms of the coefficients of $h \in \mathscr{P}$ or $w \in \mathscr{B}_0$. From the Alexander relation $f \in \mathscr{N}(\alpha)$, if and only if, $zf'(z) \in \mathscr{M}(\alpha)$. Hence, from (27) and (28) for $f \in \mathscr{N}(\alpha)$, we obtain

$$b_2 = -\tfrac{1}{4}\alpha p_1\ , \qquad b_3 = -\tfrac{1}{12}\alpha(p_2 - \tfrac{1}{2}\alpha {p_1}^2)$$

and

$$b_2 = -\tfrac{1}{2}\alpha c_1\ , \qquad b_3 = -\tfrac{1}{6}\alpha\left(c_2 + (1-\alpha){c_1}^2\right)\ . \tag{33}$$

Now, from (33) and (16) we have

$$|b_3| - |b_2| = \tfrac{1}{6}\alpha\left(|c_2 + (1-\alpha){c_1}^2| - 3|c_1|\right) = \tfrac{1}{6}\alpha\left(|(1-c^2)x + (1-\alpha)c^2| - 3c\right)\ .$$

Thus, the upper bound for this coefficient difference is as follows

$$|b_3| - |b_2| \le \tfrac{1}{6}\alpha h_1(c)\ ,$$

where $h_1(c) = 1-c^2+|1-\alpha|c^2-3c$, $c\in[0,1]$. If $\alpha\in(0,5]$, then the function h_1 reaches its greatest value for $c=0$, so $h_1(c)\le h_1(0)=1$. If $\alpha>5$, then the function h_1 reaches its greatest value for $c=1$, so $h_1(c)\le h_1(1)=\alpha-4$.

The same argument as given in the proof of Theorem 5 leads to the extremal functions in the upper bound of $|b_3|-|b_2|$, i.e.,

$$g_3(z)=\int_0^z \frac{f_3(\zeta)}{\zeta}\,\mathrm{d}\zeta = \mathrm{z}-\frac{\alpha}{6}\mathrm{z}^3+\dots$$

and

$$g_0(z)=\int_0^z \frac{f_0(\zeta)}{\zeta}\,\mathrm{d}\zeta = \frac{1-(1-\mathrm{z})^{\alpha+1}}{\alpha+1} = \mathrm{z}-\frac{1}{2}\alpha\mathrm{z}^2+\frac{1}{6}(\alpha-1)\alpha\mathrm{z}^3+\dots\ ,$$

where f_0 and f_3 are given by (24) and (23), respectively.

II. We shall now consider the lower bound for this coefficient difference. If $\alpha\in(0,1]$ and $(1-\alpha)c^2-(1-c^2)\ge 0$, i.e., $c\in[\frac{1}{\sqrt{2-\alpha}},1]$, then we get

$$|b_3|-|b_2|\ge \tfrac{1}{6}\alpha\left(-(1-c^2)+(1-\alpha)c^2-3c\right)=\tfrac{1}{6}\alpha h_2(c)\ ,$$

where $h_2(c)=(2-\alpha)c^2-3c-1$. For $\alpha\in(0,1/2]$, the function h_2 is decreasing in the interval $[\frac{1}{\sqrt{2-\alpha}},\frac{3}{4-2\alpha}]$ and increasing in $[\frac{3}{4-2\alpha},1]$, so it reaches its lowest value in $\frac{3}{4-2\alpha}$. Hence,

$$|b_3|-|b_2|\ge \tfrac{1}{6}\alpha h_2\left(\tfrac{3}{4-2\alpha}\right)=\frac{\alpha(4\alpha-17)}{24(2-\alpha)}\ . \tag{34}$$

For $\alpha\in[1/2,1]$, the function h_2 is decreasing in the interval $[\frac{1}{\sqrt{2-\alpha}},1]$, so it reaches its lowest value in 1. Thus,

$$|b_3|-|b_2|\ge \tfrac{1}{6}\alpha h_2(1)=-\tfrac{1}{6}\alpha(2+\alpha)\ . \tag{35}$$

If $\alpha\in(0,1]$ and $(1-\alpha)c^2-(1-c^2)\le 0$, i.e., $c\in[0,\frac{1}{\sqrt{2-\alpha}}]$, then

$$|b_3|-|b_2|\ge \tfrac{1}{6}\alpha(-3c)=-\tfrac{1}{2}\alpha c\ge \frac{-\alpha}{2\sqrt{2-\alpha}}\ . \tag{36}$$

Comparing the expressions from (34) and (35) with the expression from (36), we get the first two inequalities for the lower estimate of the expression $|b_3|-|b_2|$ from Theorem 6.

If $\alpha>1$ and $(1-\alpha)c^2+1-c^2\le 0$, i.e., $c\in[\frac{1}{\sqrt{\alpha}},1]$, then we obtain

$$|b_3|-|b_2|\ge \tfrac{1}{6}\alpha\left(-(1-c^2)-(1-\alpha)c^2-3c\right)=\tfrac{1}{6}\alpha h_3(c)\ ,$$

where $h(c)=\alpha c^2-3c-1$. For $\alpha\in(1,3/2]$, the function h_3 is decreasing in $[\frac{1}{\sqrt{\alpha}},1]$, so it reaches its lowest value in 1. Thus,

$$|b_3|-|b_2|\ge \tfrac{1}{6}\alpha h_3(1)=\tfrac{1}{6}\alpha(\alpha-4)\ . \tag{37}$$

For $\alpha\in[3/2,9/4]$, the function h_3 is decreasing in $[\frac{1}{\sqrt{\alpha}},\frac{3}{2\alpha}]$ and increasing in $[\frac{3}{2\alpha},1]$, so it reaches its lowest value in $\frac{3}{2\alpha}$. Then we have

$$|b_3|-|b_2|\ge \tfrac{1}{6}\alpha h_3\left(\tfrac{3}{2\alpha}\right)=-\tfrac{1}{24}(9+4\alpha)\ . \tag{38}$$

For $\alpha\ge 9/4$, the function h_3 is increasing in $[\frac{1}{\sqrt{\alpha}},1]$, so it reaches its lowest value in $\frac{1}{\sqrt{\alpha}}$. Hence,

$$|b_3|-|b_2|\ge \tfrac{1}{6}\alpha h_3\left(\tfrac{1}{\sqrt{\alpha}}\right)=-\tfrac{1}{2}\sqrt{\alpha}\ . \tag{39}$$

If $\alpha > 1$ and $(1-\alpha)c^2+1-c^2 \geq 0$, i.e., $c \in [0, \frac{1}{\sqrt{\alpha}}]$, then

$$|b_3|-|b_2| \geq \tfrac{1}{6}\alpha(-3c) = -\tfrac{1}{2}\alpha c \geq -\tfrac{1}{2}\sqrt{\alpha}\,. \tag{40}$$

Comparing the expressions from (37), (38) and (39) with the expression from (40), we get the last three inequalities for the lower estimate of the expression $|b_3|-|b_2|$ from Theorem 6.

It can be easily verified that the equality in the lower bound holds for:

$$\begin{aligned}
&g_*\left(z, \frac{3}{4-2\alpha}\right) = z + \frac{3\alpha}{4(2-\alpha)}z^2 - \frac{(1+4\alpha)\alpha}{24(2-\alpha)}z^3 + \ldots && \text{if } \alpha \in (0,1/2],\\
&g_0(z) && \text{if } \alpha \in [1/2,3/2],\\
&g^*\left(z, \frac{3}{2\alpha}\right) = z + \frac{3}{4}z^2 + \frac{9-4\alpha}{24}z^3 + \ldots && \text{if } \alpha \in [3/2,9/4],\\
&g^*\left(z, \frac{1}{\sqrt{\alpha}}\right) = z + \frac{\sqrt{\alpha}}{2}z^2 - \frac{\sqrt{\alpha}(\alpha-1)}{12}z^4 + \ldots && \text{if } \alpha > 9/4\,,
\end{aligned}$$

where $zg_*'(z,c) = f_*(z,c)$ and $zg^{*\prime}(z,c) = f^*(z,c)$ and $f_*(z,c)$, $f^*(z,c)$ are defined in (32). □

Remark 2. Notice that:

(i) The result obtained in Theorem 5 is similar in some ranges of α to the result obtained in [18] by Sim and Thomas, see (11).

(ii) In [19] Peng and Obradović proved that for the class $\mathcal{N}$ there is $|a_3-a_2| \leq \frac{1}{2}$, which implies $|a_3|-|a_2| \leq \frac{1}{2}$. From Theorem 6 for $\alpha = 1$ we conclude that

$$-\tfrac{1}{2} \leq |a_3|-|a_2| \leq \tfrac{1}{6}\,,$$

which is an improvement of the result obtained by Peng and Obradović.

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