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COMBINATORIAL PROOF OF IDENTITIES INVOLVING PARTITIONS WITH DISTINCT EVEN PARTS AND 4-REGULAR PARTITIONS

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Abstract. Recently Andrews and El Bachraoui proved identities relating certain restricted partitions into distinct even parts with restricted 4-regular partitions by the theory of basic hypergeometric series. They also posed a question regarding combinatorial proofs for these results. In this paper, we establish bijections to provide combinatorial proofs for these results.

Keywords*:* integer partitions, q-series.
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1. INTRODUCTION

In 2009, Andrews [1] denoted the ped(n) as the number of partitions of n with distinct even parts and found that

$$\sum_{n=0}^{\infty} \mathrm{ped}(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}, \quad \text{for } |q|<1, \tag{1}$$

where the q-shifted factorial is defined by [6]

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j),\ (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j) \text{ for } |q|<1.$$

From (1) we know that ped(n) equals the number of partitions of n whose parts are not divisible by 4. These partitions and their arithmetic properties have been studied extensively in recent years, see for instance [4,5,7].

Recently, Andrews and El Bachraoui [3] used the formula in [2]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{q(a;q)_\infty}{b(b;q)_\infty(1-aq/b)} + \frac{1-q/b}{1-aq/b},\ |q|<1,$$

to prove more identities related to restricted partitions with distinct even parts and restricted 4-regular partitions, where the l-regular partition is a partition whose summands are not divisible by l.

Motivated by the works of Andrews and El Bachraoui [3], we present a bijective proof of the following theorems.

THEOREM 1 ([3], Eq.(2.2)). *For* $|q|<1$*, there holds*

$$(1+q)\sum_{n\geq 0}\frac{(-q^2;q^2)_n q^{2n+1}}{(q;q^2)_{n+1}}=\frac{(q^4;q^4)_\infty}{(q;q)_\infty}-1. \tag{2}$$

THEOREM 2 ([3], Eq.(2.4)). *For* $|q|<1$*, there holds*

$$(1+q^3)\sum_{n\geq 0}\frac{(-q^2;q^2)_n q^{4n+2}}{(q;q^2)_{n+1}}=\frac{(q^4;q^4)_\infty}{(q^2;q)_\infty}-1. \tag{3}$$

THEOREM 3 ([3], Eq.(2.6)). *For* $|q|<1$*, there holds*

$$(1+q^3)\sum_{n\geq 0}\frac{(-q^2;q^2)_n q^{2n+1}}{(q;q^2)_n}=\frac{q^2(q^4;q^4)_\infty}{(q;q)_\infty}-q^2+q. \tag{4}$$

In Section 2, we establish bijections to provide proofs of the combinatorial for (2)– (4),respectively.

2. COMBINATORIAL PROOF OF THEOREM 1–3

A partition λ of a positive integer n is a finite non-increasing sequence of positive integers $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ such that $\sum_{i=1}^{k}\lambda_i=n$.

Definition 1 [3]. *Let DE1(n) denote the number of partitions of n in which no even part is repeated and the largest part is odd. Then*

$$\sum_{n=0}^{\infty} DE1(n)q^n=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{2n+1}}{(q;q^2)_{n+1}}. \tag{5}$$

Definition 2 [3]. *Let DE2(n) denote the number of partitions of n in which no even part is repeated and the largest part is odd and appears at least twice. Then*

$$\sum_{n=0}^{\infty} DE2(n)q^n=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{4n+2}}{(q;q^2)_{n+1}}. \tag{6}$$

Definition 3 [3]. *Let DE3(n) denote the number of partitions of n in which no even part is repeated and the largest part is odd and appears exactly once. Then*

$$\sum_{n=0}^{\infty} DE3(n)q^n=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{2n+1}}{(q;q^2)_n}. \tag{7}$$

We define $\pi_{ped}(n)$, $\pi_{DE1}(n)$, $\pi_{DE2}(n)$, $\pi_{DE3}(n)$ as the sets of ped partitions, DE1 partitions, DE2 partitions and DE3 partitions of n respectively. We also let $\text{ped}_{n>1}(n)$ denote the number of partitions of n in which each part is strictly larger than 1.

Now we provide proofs of the combinatorial for (2)–(4), respectively.

Proof of Theorem 1.1.

LEMMA 1. *There exists a bijection* $\phi_1:\pi_{DE1}(n)\cup\pi_{DE1}(n-1)\to\pi_{ped}(n)$ *such that* $DE1(n)+DE1(n-1)=$ *ped*(n)*, i.e.* $DE1(n)+DE1(n-1)$ *equals the number of* 4*-regular partitions of n.*

Proof. For $n\geq 1$, $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)\in\pi_{ped}(n)$, we divide λ into two cases.

Case 1: λ_1 is odd. Then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \pi_{DE1}(n)$.
Case 2: λ_1 is even. Then $\lambda' = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_k) \in \pi_{DE1}(n-1)$, since the even parts of ped(n) are distinct.
So that we have DE1(n)+DE1($n-1$)$\geq$ ped(n).

Conversely, from the generating function (5), we obtain that DE1($n-1$) equals the number of partitions of n in which no even is repeated and the largest part is even, while the largest part of DE1(n) is odd. Thus DE1(n)+DE1($n-1$)$\leq$ ped(n).

Combining two inequalities, we complete the proof of Lemma 1. □

Substituting the generating functions of DE1(n) and ped(n), (2) holds. □

Proof of Theorem 1.3.

LEMMA 2. *For* $n > 0$*, there exists a bijection* $\phi_3 : \pi_{DE3}(n+2) \cup \pi_{DE3}(n-1) \to \pi_{ped}(n)$ *such that* $DE3(n+2) + DE3(n-1) = ped(n)$*, i.e.* $DE3(n+2) + DE3(n-1)$ *equals the number of* 4*-regular partitions of n.*

Proof. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \pi_{ped}$(n), we divide λ into two cases.
Case 1: λ_1 is odd. Then $\lambda = (\lambda_1 + 2, \lambda_2, \ldots, \lambda_k) \in \pi_{DE3}(n+2)$.
Case 2: λ_1 is even. Now we have two conditions,
(i) $\lambda_2 = \lambda_1 - 1$. $\lambda = (\lambda_2 + 2, \lambda_1, \lambda_3, \ldots, \lambda_k) \in \pi_{DE3}(n+2)$ since λ_2 is odd.
(ii) $\lambda_2 < \lambda_1 - 1$. $\lambda = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_k) \in \pi_{DE3}(n-1)$ since $\lambda_1 - 1$ is unique.
So that we have DE3($n+2$) + DE3($n-1$) $\geq$ ped(n).

Conversely, from the generating function (7), we obtain that DE3($n+2$) equals the number of partitions of n in which no even is repeated, the largest part is odd or the largest part is even with the second largest part is 1 less than the largest part. Besides, DE3($n-1$) equals the number of partitions of n in which no even is repeated, the largest part is even, and the largest part is at least 2 more than the second largest part. Thus DE3($n+2$)+DE1($n-1$)$\leq$ ped(n).

Combining two inequalities, we complete the proof of Lemma 2. □

Substituting the generating functions of DE3(n) and ped(n), we complete the proof of (4). □

Proof of Theorem 1.2.

LEMMA 3. *For* $n > 0$, $DE2(n) + DE2(n-3) = ped_{n>1}(n)$*, i.e. DE2(n)+DE2(*$n-3$*) equals the number of 4-regular partitions of n into parts each* > 1.

Proof. From the definitions of DE1(n), DE2(n) and DE3(n), we deduce that $\mathrm{DE2}(n) = \mathrm{DE1}(n) - \mathrm{DE3}(n)$. By Lemmas 1 and 2, we have

$$\mathrm{DE1}(n) + \mathrm{DE1}(n-1) = \mathrm{DE3}(n+2) + \mathrm{DE3}(n-1).$$

Then

$$\begin{aligned} &\mathrm{DE2}(n) + \mathrm{DE2}(n-3) \\ &= \mathrm{DE1}(n) - \mathrm{DE3}(n) + \mathrm{DE1}(n-3) - \mathrm{DE3}(n-3) \\ &= \mathrm{DE1}(n) - \mathrm{DE1}(n-2) \\ &= \mathrm{DE1}(n) + \mathrm{DE1}(n-1) - (\mathrm{DE1}(n-1) + \mathrm{DE1}(n-2)) \\ &= \mathrm{ped}(n) - \mathrm{ped}(n-1) \\ &= \mathrm{ped}_{n>1}(n). \end{aligned}$$

where ped($n-1$) equals the number of partitions of n which 1 appears. □

By the generating functions of DE2(n) and ped(n), we immediately obtain (3). □

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