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A CHARACTERIZATION OF THE LIPSCHITZ MAPPINGS ON CONVEX METRIC SPACES

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Abstract. In this paper we obtain a characterization of the Lipschitz continuous mappings on convex metric spaces with some weaker conditions. As an application of our result we obtain existence of fixed point of nonexpansive mappings on uniformly convex metric spaces.

Key words: fixed point, convex metric space, nonexpansive map, uniformly convex.

1. INTRODUCTION AND PRELIMINARIES

The notion of uniformly convex metric spaces was introduced by Takahashi et al [9, 10]. Afterward Beg [3, 4] further studied uniformly convexity in metric spaces and obtained several fixed point results. Recently, Abdelhakim [1], Berinde and Pacurar [5], Kumar and Tas [7] and several other researchers used these ideas to study fixed point problem on convex metric spaces. In this paper, first we obtain results characterizing the Lipschitz continuous mappings on convex metric spaces with some weaker conditions and then using these results we prove existence of fixed point of nonexpansive mappings on uniformly convex metric spaces.

Now we review some fundamental notions and few basic notations related to convex structure on metric spaces for use later on.

Definition 1 [10]. For a metric space (X,d) and a mapping $W : X \times X \times [0,1] \to X$ satisfying the property,

$$d(u,W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y). \tag{1}$$

for each $x,y,u \in X$ and $\lambda \in [0,1]$, mapping W is called convex structure on (X,d) and (X,d,W) is called convex metric space.

A subset $M \neq \phi$ of X is called convex if $W(x,y,\lambda) \in M$ for all $(x,y,\lambda) \in M \times M \times [0,1]$.

Remark 1 [1,3,10]. Obviously in a convex metric space (X,d,W), we have: i. $W(x,y,1) = x$, ii. $W(x,y,0) = y$, iii. $W(x,x,\lambda) = x$.

In [1, 2, 9, 10] several examples of convex metric spaces are discussed.

Definition 2 [9]. A convex metric space (X,d,W) is called uniformly convex if for $\varepsilon > 0$, there exists an $\eta = \eta(\varepsilon) > 0$ such that for any $\sigma > 0$ and x,y,z in (X,d,W) with $d(x,z) \leq \sigma, d(y,z) \leq \sigma$ and $d(x,y) \geq \sigma\varepsilon$, we have

$$d(W(x,y,\frac{1}{2}),z) \leq \sigma(1-\eta(\varepsilon)) < \sigma.$$

2. MAIN RESULTS

In this section, first we prove two propositions on characterization of the Lipschitz continuous mappings on convex metric spaces with some weaker conditions and then use these results to obtain almost fixed points and fixed points of nonexpansive mappings.

Definition 3. A convex metric space (X,d,W) have property (ß) if $x,y,z \in X$ and λ,μ in $[0,1]$, then

$$d(W(x,y,\lambda),W(x,y,\mu)) = |\lambda - \mu|\, d(x,y),$$

and

$$d(W(x,y,\lambda),W(x,z,\lambda)) \leq (1-\lambda)d(y,z).$$

Condition (I) of Guay et al. [6, Definition 3.2] and property (B) of Beg [2, Definition 1.2] are particular cases of Definition 3 when $\lambda = \mu$. If we define $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ then every normed space has property (ß).

Next we give a control function α, for subsequent use.

Let $\alpha : (0,\infty) \to [0,\infty)$ be a real valued function satisfying;

(i) $\overline{\lim\limits_{t\to 0^+}} \frac{\alpha(t)}{t} < +\infty$ and

(ii) $\underline{\lim\limits_{t\to 0^+}} \frac{\alpha(t)}{t} = L.$

PROPOSITION 1. *Let K be a convex subset of a convex metric space X having property (β) and $T : K \to X$. If for all distinct x,y in K,*

$$d(Tx,Ty) \leq \alpha(d(x,y)). \tag{2}$$

then

$$d(Tx,Ty) \leq Ld(x,y).$$

Proof. Property (i) of α and inequality (2) imply that T is a continuous map. In fact from property (i) of α there are some positive real β and δ such that $\alpha(t) \leq \beta t$ for every $t \in (0,\delta)$, and (2) implies that $d(Tx,Ty) \leq \beta d(x,y)$ for all x,y in K such that $d(x,y) < \delta$.

Using property (ii) of α, for any $\varepsilon > 0$ there is a $t(\varepsilon) > 0$ and a unique $n = n(\varepsilon) \in \mathbb{N}_0$ such that

$$\frac{\alpha(t(\varepsilon))}{t(\varepsilon)} \leq L + \varepsilon, \tag{3}$$

and for arbitrary distinct $x,y \in K$,

$$0 \leq d(x,y) - n(\varepsilon)t(\varepsilon) < t(\varepsilon). \tag{4}$$

Also

$$\lim_{\varepsilon\to 0} t(\varepsilon) = 0. \tag{5}$$

Assuming

$$z_k = W(y,x,\frac{kt(\varepsilon)}{d(y,x)}), \quad k = 0,1,...,n(\varepsilon).$$

Now by convexity of K, $z_k \in K$; and using property (ß), we get

$$d(z_k, z_{k+1}) = t(\varepsilon) \tag{6}$$

and by inequality (4)

$$d(z_{n(\varepsilon)}, y) = d(y,x) - n(\varepsilon)t(\varepsilon) < t(\varepsilon) \tag{7}$$

Using triangle inequality, inequalities (2), (3), (6), (4) respectively, we have

$$\begin{aligned}
d(Tx,Ty) &\leq \sum_{k=0}^{n(\varepsilon)-1} d(Tz_k,Tz_{k+1})+d(Tz_{n(\varepsilon)},Ty)\\
&\leq \sum_{k=0}^{n(\varepsilon)-1} \alpha(d(z_k,z_{k+1}))+d(Tz_{n(\varepsilon)},Ty)\\
&= \sum_{k=0}^{n(\varepsilon)-1} \alpha(t(\varepsilon))+d(Tz_{n(\varepsilon)},Ty)\\
&= n(\varepsilon)\alpha(t(\varepsilon))+d(Tz_{n(\varepsilon)},Ty)\\
&= \frac{\alpha(t(\varepsilon))}{t(\varepsilon)}n(\varepsilon)t(\varepsilon)+d(Tz_{n(\varepsilon)},Ty)\\
&\leq (L+\varepsilon)n(\varepsilon)t(\varepsilon)+d(Tz_{n(\varepsilon)},Ty)\\
&\leq (L+\varepsilon)d(x,y)+d(Tz_{n(\varepsilon)},Ty).
\end{aligned}$$

Thus

$$d(Tx,Ty)\leq (L+\varepsilon)d(x,y)+d(Tz_{n(\varepsilon)},Ty). \tag{8}$$

From continuity of T and inequalities (5), and (7), we further have

$$\lim_{\varepsilon\to 0} d(Tz_{n(\varepsilon)},Ty)=0.$$

Therefore as $\varepsilon \to 0$ in (8), we have

$$d(Tx,Ty)\leq Ld(x,y).$$

This completes the proof. □

Remark 2. If the control function α is subadditive then instead of Properties (i), (ii) of α, just $L<\infty$ is sufficient and necessary for a continuos map T to be Lipschitz.

PROPOSITION 2. *Let K be a bounded convex subset of a convex metric space X having property (ß) and $T: K\to X$ be a continuous map Let (t_n) and (c_n) be two convergent positive sequences and*

$$\lim_{n\to\infty} t_n=0, \quad \lim_{n\to\infty} c_n=L, \tag{9}$$

such that for all x,y in K and for every n ,

$$d(x,y)=t_n \quad \Rightarrow \quad d(T(x),T(y))\leq c_n t_n \tag{10}$$

then

$$d(T(x),T(y))\leq Ld(x,y).$$

Proof. Let x,y be any two distinct points in K. For all n in $\mathbb{N}$, there exist a unique $m_n\in\{0\}\cup\mathbb{N}$ such that

$$m_n t_n\leq d(x,y)<(m_n+1)t_n.$$

Assuming

$$z_k=W(y,x,\frac{kt_n}{d(y,x)}), \tag{11}$$

for $k = 0, 1, ..., m_n$. As

$$0 \leq \frac{m_n t_n}{d(y,x)} \leq 1. \tag{12}$$

Convexity of K further implies that, $z_k \in K$ for each $k = 0, 1, ..., m_n$.

By property (ß)(i), we have

$$d(z_k, z_{k+1}) = t_n, \tag{13}$$

for $k = 0, 1, ..., m_n - 1$, and for $k = m_n$, we get

$$d(z_{m_n}, y) = d(y,x) - m_n t_n < t_n. \tag{14}$$

From equalities (13) and (10), we obtain

$$d(Tz_k, Tz_{k+1}) \leq c_n t_n, \quad k = 0, 1, ..., m_n - 1.$$

Thus,

$$\begin{aligned} dTx, Ty) &\leq \sum_{k=0}^{m_n - 1} d(Tz_k, Tz_{k+1}) + d(Tz_{m_n}, Ty) \\ &\leq m_n c_n t_n + d(Tz_{m_n}, Ty) \\ &= c_n(m_n t_n) + d(Tz_{m_n}, Ty) \\ &\leq c_n d(x,y) + d(Tz_{m_n}, Ty), \quad \text{using (12)}. \end{aligned} \tag{15}$$

By inequality (14), $d(z_{m_n}, y) < t_n$, and we get $\lim_{n\to\infty} d(z_{m_n}, y) = 0$. Continuity of T further implies $\lim_{n\to\infty} d(Tz_{m_n}, Ty) = 0$.

Now taking $n \to \infty$ in (15) and using (9), we obtain

$$d(T(x), T(y)) \leq Ld(x,y).$$

□

Above propositions are convex metric spaces analogue of the Matkowski [8, Lemma 1& 2].

THEOREM 1. *Let K be a bounded closed convex subset of a convex complete metric space X having property (β) and $T : K \to K$ be a continuous map Let (t_n) and (c_n) be two convergent positive sequences and*

$$\lim_{n\to\infty} t_n = 0, \quad \lim_{n\to\infty} c_n = 1,$$

such that for all x, y in K and for every n ,

$$d(x,y) = t_n \quad \Rightarrow \quad d(T(x), T(y)) \leq c_n t_n$$

then

$$\inf\{d(x, T(x)) : x \in K\} = 0. \tag{16}$$

Proof. Using Proposition (2) with $L = 1$, we obtain

$$d(T(x), T(y)) \leq d(x,y).$$

and then (16) follows from Beg [2, Theorem 2.1]. □

In case K is also compact then T has a fixed point. This fixed point may not be unique. When K is not compact, we need to impose further conditions on X to guarantee the existence of fixed point. To overcome this situation, in our next theorem we obtain a fixed point result using uniformly convexity of the metric.

THEOREM 2. *Let K be a nonempty convex bounded and closed subset of a uniformly convex complete metric space X having property (β), and $T : K \to K$ a mapping satisfying (2) with $L = 1$, then T has a fixed point in K.*

Proof. Proposition (1) implies that T is nonexpansive and Beg [2, Theorem 2.3] yield the result. □

In a similar way we can prove the following Lemma using Proposition 2 and Beg [2, Theorem 2.3].

LEMMA 1. *Let K be a nonempty convex bounded and closed subset of a uniformly convex complete metric space X having property (β), and $T : K \to K$ a continuous map. Let (t_n) and (c_n) be two convergent positive sequences and*

$$\lim_{n\to\infty} t_n = 0, \quad \lim_{n\to\infty} c_n = 1,$$

such that for all x, y in K and for every n,

$$d(x,y) = t_n \quad \Rightarrow \quad d(T(x),T(y)) \le c_n t_n,$$

then there exists an x_0 such that $T(x_0) = x_0$.

THEOREM 3. *Let K be a nonempty convex bounded and closed subset of a uniformly convex complete metric space X having property (β), and $T : K \to K$ be a continuous map. If there exist a function $\alpha : (0,\infty) \to [0,\infty)$ and a positive real number sequence (s_n) with $\lim_{n\to\infty} s_n = 0$, satisfying the condition $\lim_{n\to\infty} \frac{\alpha(s_n)}{s_n} = 1$, such that for all x, y in K and for every natural number n,*

$$d(x,y) = s_n \quad \Rightarrow \quad d(Tx,Ty) \le \alpha(d(x,y)),$$

then there exists an x_0 such that $T(x_0) = x_0$.

Proof. Assuming $c_n = \frac{\alpha(s_n)}{s_n}$, we obtain $\lim_{n\to\infty} c_n = 1$. Now for all n in $\mathbb{N}$ and for every $x, y \in K$, if $d(x,y) = s_n$, then

$$d(Tx,Ty) \le \alpha(s_n) = s_n c_n.$$

Lemma(1) further implies that there exists an x_0 such that $T(x_0) = x_0$. □

Example 1. Let $X = \{(x,y) : x, y \in \mathbb{R}\}$ be a set, $a = (u_1, v_1)$ and $b = (u_2, v_2)$ be any two points in X. Consider the metric $d : X \times X \to \mathbb{R}$ given by

$$d(a,b) = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$$

and convex structure W on X is given by

$$W(a,b,\lambda) = (\lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2).$$

The space (X,d,W) is a uniformly convex complete metric space having property (ß). Let

$$K = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}.$$

Then the set K is a nonempty closed convex and bounded subset of the space (X,d,W). Define $T : K \to K$ by $T(a) = T(u_1,v_1) = (1-u_1, 1-v_1)$.Obviously mapping T is continuous. Let $\alpha : (0,\infty) \to [0,\infty)$ be defined by $\alpha(t) = t^2 + t$ and consider the sequence of positive real number $s_n = e^{-n}$. Now $\lim_{n\to\infty} s_n = 0$ and $\lim_{n\to\infty} \frac{\alpha(s_n)}{s_n} = 1$.

Also for all x, y in K, we have

$$d(x,y) = s_n \quad \Rightarrow \quad d(Tx,Ty) \le \alpha(d(x,y)),$$

All the hypothesis of theorem 3 are satisfied and the fixed point of the map T in K is $(0.5, 0.5)$.

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